TD n°02

7–8 February 2016

Exercice 1 - En noir et blanc

Consider the following balls-and-bin game. We start with one black ball and one white ball in a bin. We repeatedly do the following : choose one ball from the bin uniformly at random, and then put the ball back in the bin with another ball of the same color. We repeat until there are n balls in the bin. Show that the number of white balls is equally likely to be any number between 1 and $n-1$.

Exercice 2 - Inclusion

Let X and Y be independently and uniformly chosen subsets of $\{1,\ldots,n\}$. Compute $\mathbf{P}\{X \subseteq Y\}$.

Exercice 3 - Le problème des rencontres

We are given a bin that contains n balls numbered from 1 to n. We draw the balls from the bin without replacement until the bin is empty (sampling without replacement). We are interested in the events $E_i =$ 'the i-th drawn ball has number i'.

3.1 Propose a probability space that models this experiment.

3.2 Compute the probability of the following events : E_i, $E_i \cap E_j$ for $i < j$, and $\bigcap_{j=1}^{r} E_{i_j}$ for $1 \leq i_1 < \cdots < i_r \leq n$

3.3 Compute the probability that the event E_i occurs for at least 1 i. Compute the limit of the resulting probability for $n \to \infty$.

3.4 Compute the number of ways to place eight rooks (*tours* in French) on a chess board such that non of them can attack the others. Recompute the number when the main diagonal is required to be empty.

Exercice 4 - Moi d'abord !

Alan and Beth throw a fair coin in turn. If the coin turns up head, the person who threw it wins.

4.1 Alan throws the coin first. What is the probability that he wins ?

4.2 Can we tweak a coin in a way that allows a fair game ?

4.3 Is there a chance that the game runs forever ?

4.4 This time the players throw two dice. Alan wins if he gets 7 as the sum, Beth wins if she gets 6. Beth starts the game. Is the game fair ?

Exercice 5 - Le singe savant

A monkey types on a 26-letter keyboard that has lowercase letters only. Each letter is chosen independently and uniformly at random from the alphabet. If the monkey types $1,000,000$ letters, what is the expected number of times the sequence "preuve" appears ?

Exercice 6 - Test de dépistage

We want to perform a medical test on a large number n of blood samples. This can be done in two ways :

1. The test is performed on each sample separately. In this case, n tests are performed.
2. Samples are grouped into (disjoint) sets of size k and tests are performed on groups. If there is a group with a positive test-result, each sample in this group is then tested individually.

Let p be the probability that a sample is positive. We assume that samples are independent and identically distributed and that k divides n.

6.1 What is the probability p_k that a set of k samples produces positive result ?

6.2 Consider a random variable X that represents the total number of test performed when we follow the second method. Which values can X take and what are the corresponding probabilities ?

6.3 What is the expected value of the number of test performed when the second method is used, i.e., what is $\mathbb{E}[X]$?