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Exercice 1 - Triangles non-monochromatiques

Let G be a 3-colorable graph with n vertices. Consider the following algorithm which begins with an arbitrary 2-coloring of G. While there are any monochromatic triangles in G, the algorithm chooses one such triangle and changes the color of a randomly chosen vertex of that triangle.
By the problem statement, there exists (an unknown) 3-coloring Red, Blue, Yellow of G. Denote R (resp. B, Y) the set of red (resp. blue, yellow) vertices if this coloring. Consider now a 2-coloring c of G (say, red and blue) and let $m(c) = \#\{v \in R | c(v) = \text{ blue }\} + \#\{v \in B | c(v) = \text{ red }\}$.

1.1 Let h_j be the expected number of recoloring needed to be performed for a coloring c s.t. $m(c) = j$. Express h_j as a function of h_{j-1} and h_{j+1} for $j = 1, \dots, n-1$. Determine h_0, h_n.

1.2 Show that $h_j = h_{j+1} + f(j)$ for a certain function f to be determined s.t. $f(0) = -h_1$.

1.3 Show that $h_{\frac{n}{2}} = \mathcal{O}(n^2)$ and conclude on the running time of the algorithm. (One can use the symmetry relation between h_1 and h_{n-1}.)

Exercice 2 - Jouons avec les définitions

Consider Markov chains defined by the following transition matrices :

$$A = \begin{pmatrix} 2/3 & 0 & 1/3 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \\ 1/4 & 0 & 0 & 3/4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1/2 & 1/4 & 0 & 1/4 \\ 0 & 0 & 1/2 & 1/2 \\ 1/4 & 0 & 0 & 3/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For each case :

- — Give a graph representation of the chain
- — Give its irreducible components
- — Give its transient and recurrent states
- — Decided if the chain is periodic or aperiodic
- — Give the stationary distribution
- — For each state i, compute $h_{i,i}$ (the expected number of steps to come back to i)

Exercice 3 - Trucs utiles

The goal of this exercise is to prove the properties of Markov chains

3.1 Show that if a state i is periodic with period d and it is contained in the same communicating class as j (i.e., if i and j are accessible from each other), then j also has period d. In other words, period is a property of a class.

3.2 Show that recurrence is also a property of a class : if i and j communicate and i is recurrent (i.e., $\sum_{t\geq 1} p_{i,i}^{(t)} = \infty$, where $p_{i,i}^{(t)}$ is the probability of returning to i after t steps ; or the probability that we even re-enter i is 1), then j is also recurrent. Hence, the same is true for transient states.

3.3 A class of states C is *closed* if for all $i \in C$ and for all $j \notin C, p_{i,j} = 0$. In other words, there is no outgoing edge from this class. Show that

— A non-closed class is transient

— A closed *finite* class is recurrent

In particular, for a finite-state Markov chains, recurrent states form closed classes and transient states form non-closed classes.

3.4 Show that if π is a stationary distribution and if i is a transient state, then $\pi(i) = 0$.

Exercice 4 - Absorbée !

Consider a Markov chain given by the following transition matrix :

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/4 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 0 & 1/4 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 3/4 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

4.1 Draw a graph that corresponds to this chain

4.2 Describe communicating classes of this chain. Determine recurrent and transient classes and their periodicity.

Notice that there are two recurrent classes in this chain. Denote them A and B. For every transient state i, let a_i (resp. b_i) denote the probability of the absorption of the state i by class A (resp. B). That is, the probability of staying in A (resp. B) after one or more steps starting from i.

4.3 Which system of equations should a_i (resp. b_i) satisfy ?

Exercice 5 - Cat and mouse

A cat and mouse each independently take a random walk on a connected, undirected, non-bipartite graph G. They start at the same time on different nodes, and each makes one transition at each time step. The cat eats the mouse if they are ever at the same node at some time step. Let n and m denote, respectively, the number of vertices and edges of G. Show an upper bound of $\mathcal{O}(m^2 n)$ on the expected time before the cat eats the mouse. (Hint : Consider a Markov chain whose states are the ordered pairs (a, b), where a is the position of the cat and b is the position of the mouse.)