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Exercice 1 - Absorbée !

Consider a Markov chain given by the following transition matrix :

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/4 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 0 & 1/4 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 3/4 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

1.1 Draw a graph that corresponds to this chain

1.2 Describe communicating classes of this chain. Determine recurrent and transient classes and their periodicity.

Notice that there are two recurrent classes in this chain. Denote them A and B. For every transient state i, let a_i (resp. b_i) denote the probability of the absorption of the state i by class A (resp. B). That is, the probability of staying in A (resp. B) after one or more steps starting from i.

1.3 Which system of equations should a_i (resp. b_i) satisfy ?

Exercice 2 - Cat and mouse

A cat and mouse each independently take a random walk on a connected, undirected, non-bipartite graph G. They start at the same time on different nodes, and each makes one transition at each time step. The cat eats the mouse if they are ever at the same node at some time step. Let n and m denote, respectively, the number of vertices and edges of G. Show an upper bound of $\mathcal{O}(m^2 n)$ on the expected time before the cat eats the mouse. (Hint : Consider a Markov chain whose states are the ordered pairs (a, b), where a is the position of the cat and b is the position of the mouse.)

Exercice 3 - Confidence interval

You are given $n = 100$ iid samples from a normal distribution of standard deviation $\sigma = 3$. The observed mean is equal to 16 and the observed standard deviation is equal to 2.9. Compute a 95% confidence interval on the mean μ.
You can use the following quantities of the normal law $Z \approx \mathcal{N}(0,1)$:

$$\mathbf{P}\{Z < 1.96\} = 0.975 \quad \mathbf{P}\{Z < 1.65\} = 0.95 \quad \mathbf{P}\{Z < 1.29\} = 0.9$$

Exercice 4 - Maximum Likelihood Estimator — MLE

4.1 Suppose that $X_1, \ldots, X_n$ form a random sample from a uniform distribution on the interval $(0, \theta)$ where the parameter $\theta > 0$ but is unknown. Find MLE of θ.

4.2 Suppose again that $X_1, \ldots, X_n$ form a random sample from a uniform distribution on the interval $(0, \theta)$ where the parameter $\theta > 0$ but is unknown. However, suppose now we write the density function as

$$f(x|\theta) = \begin{cases} 1/\theta & \text{for } 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}$$

Proof that the MLE for θ does not exist in this case.

Exercice 5 - Hypothesis testing

Suppose that we observe a random sample of size n from a normally distributed population. If we are able to reject $H_0 : \mu = \mu_0$ in favor of $H_a : \mu \neq \mu_0$ at 5% significance level, is it true that we can definitely reject H_0 in favor of the appropriate one-tailed alternative at the 2.5% significance level ? Why or why not ?