Université Claude Bernard — Mathématiques

M2 Phenomena of High Dimension

**Problem sheet # 2**

The Banach–Mazur compactum

**Exercise 2.1 Convex bodies at distance 1**

Let $K$, $L$ be two convex bodies in $\mathbf{R}^n$ with $d_{BM}(K,L) = 1$. Show that there exists $T \in \mathsf{GL}_n(\mathbf{R})$ such that $K = TL$.

**Exercise 2.2 Around John's theorem**

1. Let $x \in \mathbf{R}^n$ with $|x| > \sqrt{n}$. Show that $\mathrm{conv}(B_2^n \cup \{\pm x\})$ contains an ellipsoid $\mathcal{E}$ with $\mathrm{vol}(\mathcal{E}) > \mathrm{vol}(B_2^n)$. Deduce another proof of the inclusion $K \subset \sqrt{n}\mathcal{E}_J(K)$ for any symmetric convex body $K$.
2. Let $x \in \mathbf{R}^n$ with $|x| > n$. Show that $\mathrm{conv}(B_2^n \cup \{x\})$ contains a translate of an ellipsoid $\mathcal{E}$ with $\mathrm{vol}(\mathcal{E}) > \mathrm{vol}(B_2^n)$. Deduce the following: for any convex body $K$, there exists an affine bijection $T$ such that $B_2^n \subset T(K) \subset nB_2^n$. Show that the constant $n$ is sharp (take $K$ to be a simplex).

**Exercise 2.3 Hadamard matrices**

A Hadamard matrix is a matrix $A \in \mathsf{M}_n(\mathbf{R})$ with entries $\pm 1$ and such that $\frac{1}{\sqrt{n}}A$ is an orthogonal matrix.

1. Show that if a Hadamard matrix of size $n > 2$ exists, then $n$ is a multiple of 4. The Hadamard conjecture postulates the existence of a Hadamard matrix of size $4k$ for every $k$.
2. Show that if the Hadamard conjecture is true, then $d_{BM}(B_1^n, B_\infty^n) \leqslant \sqrt{n} + 3$ for every $n$.
3. Show that there exists a Hadamard matrix of size 12, and then of size $2^k 12^l$ for every $k, l \in \mathbf{N}$.
4. Show that $d_{BM}(B_1^n, B_\infty^n) \leqslant \sqrt{n} + C$ for every $n$, with $C$ a universal constant.

**Exercise 2.4 Kadets–Snobar theorem**

Let $X$ be a normed space, and $Y \subset X$ an $n$-dimensional subspace. Show the existence of a projection $P : X \to X$ with range $Y$ satisfying $\|P\| \leqslant \sqrt{n}$.

**Indication.** Apply John's theorem to the unit ball of $Y$, deduce a decomposition of $\mathrm{Id} : Y \to Y$ in terms of contact points and use the Hahn–Banach extension theorem: any linear form $\ell : Y \to \mathbf{R}$ extends into a linear form $\tilde{\ell} : X \to \mathbf{R}$ with $\|\tilde{\ell}\| \leqslant \|\ell\|$.

**Exercise 2.5 Auerbach theorem**

A cross-polytope is a convex body in $\mathbf{R}^n$ of the form $\mathrm{conv}\{\pm x_i\}$, where $(x_i)$ is a basis of the vector space $\mathbf{R}^n$.

1. Show that every symmetric convex body contains a maximal volume cross-polytope. Is it unique?
2. Let $K \subset \mathbf{R}^n$ be a symmetric convex body. Show that if $B_1^n$ is a maximal volume cross-polytope inside $K$, then $K \subset B_\infty^n$.
3. Let $K \subset \mathbf{R}^n$ be a symmetric convex body. Show the existence of $T \in \mathsf{GL}_n(\mathbf{R})$ such that $B_1^n \subset T(K) \subset B_\infty^n$.

**Exercise 2.6 Sums of ellipsoids**

Let $\mathcal{E}$, $\mathcal{F}$ be two ellipsoids. Show that

$$\mathcal{E} + \mathcal{F} \text{ is an ellipsoid} \iff \exists \lambda > 0 \,:\, \mathcal{E} = \lambda \mathcal{F}.$$

**Exercise 2.7 Löwner ellipsoid**

Let $K \subset \mathbf{R}^n$ be a symmetric convex body. Show that there exists a unique ellipsoid (denoted $\mathcal{E}_L(K)$) of minimal volume containing $K$. Show a characterization of the equality $\mathcal{E}_L(K) = B_2^n$ in the spirit of John's theorem.

**Exercise 2.8 Khintchine inequalities via convex domination**

Given $X$, $Y$ two random variables with finite expectation, one says that $X \leqslant_{cvx} Y$ ($Y$ dominates $X$ in the convex ordering) if $\mathbf{E}\varphi(X) \leqslant \mathbf{E}\varphi(Y)$ for any convex function $\varphi : \mathbf{R} \to \mathbf{R}$.

1. Show that is $X_1$ is independent from $X_2$ and $Y_1$ is independent from $Y_2$ with $X_1 \leqslant_{cvx} Y_1$ and $X_2 \leqslant_{cvx} Y_2$, then $X_1 + X_2 \leqslant_{cvx} Y_1 + Y_2$.
2. Let $\varepsilon$ be a random sign (i.e. a random variable uniformly distributed on $\pm 1$) and $G_\sigma$ be a $N(0, \sigma^2)$ random variable. Find a value of $\sigma$ (or even the smallest possible value) for which $\varepsilon \leqslant_{cvx} G_\sigma$.
3. Show that $\|G_1\|_{L^p} \leqslant C\sqrt{p}$ for every $p \geqslant 1$, where $C$ is a universal constant.
4. Deduce the following: if $X = \sum_{i=1}^n \varepsilon_i a_i$, where $(\varepsilon_i)$ are i.i.d. random signs and $(a_i)$ real numbers, then $\|X\|_{L^p} \leqslant C'\sqrt{p}\|X\|_{L^2}$ for every $p \geqslant 2$, where $C'$ is a universal constant.
5. Show that for $1 \leqslant p \leqslant 2$, we have $\|X\|_{L^p} \geqslant c\|X\|_{L^2}$ for a universal constant $c > 0$.

**Exercise 2.9 $L^1$ Khintchine inequality with sharp constant**

Consider the probability space $\Omega = \{-1, 1\}^n$ equipped with uniform measure. Let $\varepsilon_i : \Omega \to \{-1, 1\}$ be the $i$th coordinate, so that the r.v. $(\varepsilon_i)$ are i.i.d.

1. For $A \subset \{1, \ldots, n\}$, denote $w_A = \prod_{i \in A} \varepsilon_i$ (and $w_\emptyset = 1$). Show that the family $(w_A)$ is an orthonormal basis of $L^2(\Omega)$ (called the Walsh–Fourier basis).
2. For $f : \Omega \to \mathbf{R}$ and $A \subset \{1, \ldots, n\}$, denote $\hat{f}_A = \mathbf{E}[f w_A]$. Show that $f = \sum_A \hat{f}_A w_A$.
3. Define an operator $L$ on $L^2(\Omega)$ by the formula $Lf = \sum_A \operatorname{card}(A) \hat{f}_A w_A$. Show that
$$Lf(x) = \sum_{i=1}^n \frac{f(x) - f(x^{\oplus i})}{2},$$
where $x^{\oplus i}$ denotes the vector obtain by flipping the sign of the $i$th coordinate of $x$.
4. Show that if $f : \Omega \to \mathbf{R}$ is an even function, then $\mathbf{Var}(f) \leqslant \frac{1}{2}\mathbf{E}\left[f \cdot Lf\right]$.
5. Fix real numbers $(a_i)$, and consider the function $f : \Omega \to \mathbf{R}$ defined as $f(x_1, \ldots, x_n) = |\sum_{i=1}^n a_i x_i|$. Show that $Lf \leqslant f$ pointwise.
6. Conclude that $\mathbf{E}f \geqslant \frac{1}{\sqrt{2}}(\mathbf{E}f^2)^{\frac{1}{2}}$.
7. Show that the constant $\frac{1}{\sqrt{2}}$ is optimal in the above inequality.