Problem sheet # 6

Gluskin's theorem: around tools used in the proof

Exercise 6.1 Operator norm unit ball

What are the extreme points of the convex body $K = \{A \in \mathsf{M}_n(\mathbf{R}) \,:\, \|A\|_{\mathrm{op}} \leqslant 1\}$?

What are the extreme points of $K \cap H$, where $H \subset \mathsf{M}_n(\mathbf{R})$ is the subspace of symmetric matrices ?

Exercise 6.2 Trace norm

For $A \in \mathsf{M}_n(R)$, consider the following norm (this is the dual norm to the operator norm)

$$\|A\|_{\mathrm{tr}} = \sup\{\mathrm{tr}(AB) \,:\, B \in \mathsf{M}_n(\mathbf{R}),\ \|B\|_{\mathrm{op}} \leqslant 1\}.$$

1. Show that $\|A\|_{\mathrm{tr}} = \mathrm{tr}\,|A|$, where $|A| = \sqrt{AA^t}$.
2. Identify the extreme points of the unit ball for $\|.\|_{\mathrm{tr}}$.

Exercise 6.3 Gaussian rectangular matrices

Let G be $N \times n$ random matrix with independent $N(0,1)$ entries, seen as a linear map from $\mathbf{R}^n$ to $\mathbf{R}^N$. Assume $n \leqslant N$. The goal of the exercise is to find a constant C such that, with high probability,

$$\forall x \in \mathbf{R}^n, \quad (\sqrt{N} - C\sqrt{n})|x| \leqslant |Gx| \leqslant (\sqrt{N} + C\sqrt{n})|x|.$$

1. Let g and h be two independent random vectors in $\mathbf{R}^N$ with distribution $N(0, \mathrm{Id}_N)$. Find constants C, $c > 0$ such that for any $t > 0$,
$$\mathbf{P}(\langle g, h\rangle > t\sqrt{N}) \leqslant C\left(\exp(-cN) + \exp(-ct^2)\right).$$
2. Fix x and $y \in S^{n-1}$ and consider the random variable $X = \langle Gx, Gy\rangle$. Show that $\mathbf{E}[X] = N\langle x, y\rangle$, and that for all $t > 0$,
$$\mathbf{P}(|X - N\langle x, y\rangle| \geqslant t) \leqslant C\left(\exp(-cN) + \exp(-ct^2)\right).$$
3. Let $A = G^tG - N\mathrm{Id} \in \mathsf{M}_n(\mathbf{R})$. Show that if $\mathcal{N}$ is a $\frac{1}{4}$-net in $(S^{n-1}, |\cdot|)$, then
$$\|A\|_{\mathrm{op}} \leqslant 2 \sup_{x,y \in \mathcal{N}} \langle Ax, y\rangle$$
4. Combine 2., 3. and estimates on cardinality of ε-nets to show that $\|A\|_{\mathrm{op}} \leqslant C'\sqrt{n}$ with high probability.
5. Conclude.

Exercise 6.4 Non-symmetric convex bodies

In this exercise we study a variant of the Banach–Mazur compactum, allowing non-symmetric convex bodies and using affine maps instead of linear maps.

If K and L are two convex bodies in $\mathbf{R}^n$, not necessarily symmetric, define (a and b being positive)

$$\tilde{d}_{BM}(K,L) = \inf\left\{\frac{b}{a} \ : \quad \exists T \in GL_n(\mathbf{R}),\ x,\ y \in \mathbf{R}^n \ : a(K-x) \subset T(L-y) \subset b(K-x)\right\}.$$

1. Show that if K and L are symmetric convex bodies, then $d_{BM}(K,L) = \tilde{d}_{BM}(K,L)$.
2. Show that if Δ and Δ' are simplices in $\mathbf{R}^n$, then $\tilde{d}_{BM}(\Delta, \Delta') = 1$. In what follows Δ_n denotes an arbitrary simplex in $\mathbf{R}^n$.
3. Show that $\tilde{d}_{BM}(\Delta_n, B_2^n) = n$. Show more generally that if K is a symmetric convex body, then $\tilde{d}_{BM}(\Delta_n, K) = n$ (for the upper bound, consider a maximal volume simplex inside K).
4. Define an *affine ellipsoid* to be the image of a (symmetric) ellipsoid under a translation. Show the non-symmetric version of John's theorem: if $K \subset \mathbf{R}^n$ is a convex body (not assumed to be symmetric) such that B_2^n is has maximal volume among affine ellipsoids contained in K, then there are contact points $(x_i) \in \partial K \cap S^{n-1}$ and a convex combination (λ_i) such that $\sum \lambda_i x_i = 0$ and $\sum \lambda_i |x_i\rangle\langle x_i| = \frac{\mathrm{Id}}{n}$.
5. Deduce that $\tilde{d}_{BM}(K, B_2^n) \leqslant n$ for every convex body $K \subset \mathbf{R}^n$.
6. Show that Dvoretzky's theorem extends to the setting of non-necessarily symmetric convex bodies.

Remark: It follows from 5. that if K, L are convex bodies in $\mathbf{R}^n$, then $\tilde{d}_{BM}(K,L) \leqslant n^2$. The order of magnitude of the diameter of the non-symmetric variant of the Banach–Mazur compactum is unknown. The best known estimate is

$$n \leqslant \sup_{K,L} \tilde{d}_{BM}(K,L) \leqslant Cn^{4/3}(\log n)^9.$$