Exerise sheet # 1
Convexity : the Brunn–Minkowski theory

Exercise 1.1 Norm associated to a symmetric convex body

Let $K \subset \mathbf{R}^n$ be a symmetric convex body. Show that the formula

$$\|x\| = \inf\{t \geqslant 0 \,:\, x \in tK\}$$

defines a norm on $\mathbf{R}^n$ for which K is the unit ball.

Exercise 1.2 Hahn–Banach separation theorem.

1. Show that if K and L are two disjoint convex compact subsets of $\mathbf{R}^n$, there is $x \in \mathbf{R}^n$ such that

   $$\sup_{y \in K} \langle x|y\rangle < \inf_{z \in L} \langle x|z\rangle.$$

   Using the axiom of choice is not allowed.
2. Let $K \subset \mathbf{R}^n$ be a convex body, and $z \in \partial K$. Show that there exists a nonzero $x \in \mathbf{R}^n$ such that

   $$\sup_{y \in K} \langle x|y\rangle = \langle x|z\rangle$$

Exercise 1.3 Bipolar.

1. If K is a symmetric convex body, show that $K = K^{\circ\circ}$.
2. If A is any subset of $\mathbf{R}^n$, show that $(A^\circ)^\circ = \overline{\mathrm{conv}}(A \cup \{0\})$.

Exercise 1.4 Minkowski sum.

Let K, $L \subset \mathbf{R}^n$. Decide if each assertion is true of false.

1. If K and L are open, then $K + L$ is open.
2. If K and L are closed, then $K + L$ is closed.
3. If K and L are compact, then $K + L$ is compact.

Exercise 1.5 Polarity.

1. Compute the polar of the following subsets of $\mathbf{R}^n$: a singleton, a (vector) subspace.
2. Show that $(K \cup L)^\circ = K^\circ \cap L^\circ$.
3. Show that if K and L are closed convex subsets containing 0, then $(K \cap L)^\circ = \overline{\mathrm{conv}}(K^\circ \cup L^\circ)$.
4. Let E be a vector subspace of $\mathbf{R}^n$, and P_E the orthogonal projection onto E. Show that for every convex $K \subset \mathbf{R}^n$ such that $0 \in \mathrm{int}(K)$, we have

   $$(P_E K)^\circ = K^\circ \cap E \;\text{ et }\; (K \cap E)^\circ = P_E(K^\circ),$$

   where polarity in the left-hand sides is taken inside E.

Exercise 1.6 Support function.

If $K \subset \mathbf{R}^n$, we define $h_K(x) = \sup_{y \in K} \langle x|y\rangle$ for $x \in \mathbf{R}^n$.

1. Show that if K and L are convex bodies, we have the equivalences $K \subset L \iff h_K \leqslant h_L$ and $K \subset \mathrm{int}\, L \iff h_K < h_L$.
2. Show that $\delta(K, L) = \|h_K - h_L\|_\infty$, où $\|f\|_\infty = \sup\{|f(u)| \,:\, u \in S^{n-1}\}$.

Exercise 1.7 Parallel sections of a symmetric convex body.

Let K be a symmetric convex body, and E a k-dimensional subspace. Show that among sections of K parallel to E, the section through the origin has the largest k-dimensional volume.

Exercise 1.8 Isodiametric inequality.

The diameter of a subset $K \subset \mathbf{R}^n$ is defined as $\mathrm{diam}(K) = \sup\{|x-y| \,:\, x,\ y \in K\}$. Show that if B is a Euclidean ball with the same volume as K, we have $\mathrm{diam}(K) \geqslant \mathrm{diam}(B)$.

Exercise 1.9 Steiner symmetrization.

We denote by S_u the Steiner symmetrization in direction $u \in S^{n-1}$. Let K, L be convex bodies. Show the following

1. $S_u(\lambda K) = \lambda S_u(K)$,
2. $S_u(K) + S_u(L) \subset S_u(K+L)$,
3. S_u is continuous with respect to Hausdorff distance,
4. $a(S_u(K)) \leqslant a(K)$.

Exercise 1.10 Carathéodory theorem.

Let $A \subset \mathbf{R}^n$. Show that any element in $\mathrm{conv}\, A$ can be written as a convex combination of at most $n+1$ elements from A.

Exercise 1.11 Extreme points.

Let $K \subset \mathbf{R}^n$ a convex body. A point $x \in K$ is extreme is the identity $x = \lambda y + (1-\lambda)z$ for $0 < \lambda < 1$ and y, $z \in K$ implies $x = y = z$.

1. Show that K has at least one extreme point.
2. Show that K is the convex hull of its extreme points (use induction on the dimension, and the previous question).

Exercise 1.12 Harmonic mean.

Show that for K, L convex bodies in $\mathbf{R}^n$, we have

$$\left(\frac{K^\circ + L^\circ}{2}\right)^\circ \subset \frac{K+L}{2}.$$