Problem sheet # 4
Dvoretzky's theorem

Exercise 4.1 Haar measure on $\mathsf{O}(n)$ from Gaussian matrices
Let G be a $n \times n$ matrix with independent $N(0,1)$ entries.

1. Check that for any $O \in \mathsf{O}(n)$, the matrices GO and OG have the same distribution as G.
2. Deduce that the matrix $\sqrt{G^t G} G^{-1}$ is distributed according to the Haar measure on $\mathsf{O}(n)$.

Exercise 4.2 Uniform measure on the sphere
Prove that the uniform measure σ on S^{n-1} is the only Borel probability measure which is invariant under rotations.

Exercise 4.3 Random sets are nets
Let $\mathcal{N} = \{x_i : 1 \leqslant i \leqslant N\}$ be a set of N i.i.d. points uniformly distributed on S^{n-1}. Fix $\varepsilon > 0$ and show that provided $N \geqslant \exp(Cn)$ (for some constant $C = C(\varepsilon)$), the set $\mathcal{N}$ is a ε-net in S^{n-1} with high probability.

Exercise 4.4 Dvoretzky's theorem with $\varepsilon = 0$
Which of the following statements are true ?

1. For any k, there is $n \in \mathbf{N}$ and a k-dimensional subspace E of $\ell_\infty^n = (\mathbf{R}^n, \|\cdot\|_\infty)$ such that $d_{BM}(\ell_2^k, E) = 1$.
2. For any k, there is a k-dimensional subspace E of ℓ_∞ (the Banach space of bounded sequences, equipped with the sup norm) such that $d_{BM}(\ell_2^k, E) = 1$.
3. For any k, there is a k-dimensional subspace E of c_0 (the Banach space of sequences tending to 0, equipped with the sup norm) such that $d_{BM}(\ell_2^k, E) = 1$.

Exercise 4.5 A variant on Dvoretzky's theorem

1. Show that if $\mathcal{E}$ is an ellipsoid in $\mathbf{R}^n$, there is a subspace F of dimension $\lceil n/2 \rceil$ such that $\mathcal{E} \cap F$ is a Euclidean ball.
2. Deduce the following variant of Dvoretzky's theorem: for any $\varepsilon > 0$, there is a constant $c(\varepsilon) > 0$ such that for any symmetric convex body $K \subset \mathbf{R}^n$, there is a subspace $E \subset \mathbf{R}^n$ with $\dim E \geqslant c(\varepsilon) \log n$ and a number $r > 0$ such that
$$rB_2^n \cap E \subset K \cap E \subset r(1+\varepsilon) B_2^n \cap E.$$

Exercise 4.6 Dvoretzky's theorem in ℓ_p^n
What is the dimension of almost Euclidean subspaces given by the proof we showed, for the space $\ell_p^n = (\mathbf{R}^n, \|\cdot\|_p)$?

Exercise 4.7 Optimality of Dvoretzky's theorem
Let K a convex body in John position, and $M = \int_{S^{n-1}} \|x\|_K \, \mathrm{d}\sigma(x)$. Let E be a random subspace of dimension k distributed according to $\mu_{n,k}$. Assume (for simplicity) that $k|n$ and that
$$\mathbf{P}\left(\forall x \in S^{n-1} \cap E, \quad \|x\|_K \leqslant 2M\right) \geqslant 1 - 1/n. \tag{1}$$

1. Show the existence of n/k pairwise orthogonal k-dimensional subspaces (E_i), such that each E_i satisfies the condition in (1).
2. Conclude that for every $x \in \mathbf{R}^n$, $\|x\|_K \leqslant 2M\sqrt{n/k}|x|$ and therefore $k \leqslant 4M^2 n$.

We show in the course the existence of almost Euclidean sections of dimension of order $M^2 n$; the exercise shows that this result is sharp.