Problem sheet # 5
Gluskin's theorem

Exercise 5.1 Polytopes

1. Let $K \subset \mathbf{R}^n$ be a convex body. Show that K is the convex hull of a finite set iff it is the intersection of finitely many half-spaces.

2. Let $P \subset \mathbf{R}^n$ be polytope with v vertices and f facets, with 0 in the interior of P. Show that P° is a polytope with v facets and f vertices.

Exercise 5.2 Octahedron as a section of the cube

Fix a integer n. Show that there exist N and a n-dimensional subspace $E \subset \mathbf{R}^N$ such that $d_{BM}(B_1^n, E \cap B_\infty^N) = 1$. What is the minimal such N?

Exercise 5.3 Zonotopes

A zonotope in $\mathbf{R}^n$ is a polytope which can be written as the Minkoswki addition of finitely many segments. For example the cube $[-1,1]^n$ is a zonotope since

$$[-1,1]^n = [-e_1, e_1] + \cdots + [-e_n, e_n].$$

1. Show that a convex body is a zonotope iff it is of the form $A(B_\infty^N)$ for some integer N and some linear map $A : \mathbf{R}^N \to \mathbf{R}^n$.

2. Show that every centrally symmetric polygon (i.e. planar polytope) is a zonotope.

3. Show that the octahedron B_1^3 is not a zonotope.

Exercise 5.4 Mean width = perimeter for planar sets

Show that the perimeter of any planar convex body is equal to 2π times its mean width.
Therefore, the Urysohn inequality in dimension 2 is equivalent to the isoperimetric inequality.

Exercise 5.5 Another proof of Urysohn inequality

Show that the mean width of a convex body does not increase under Steiner symmetrisations. Deduce an alternative proof of Urysohn inequality.

Exercise 5.6 Simplex

Show that there exists a simplex $\Delta_n \in \mathbf{R}^n$ with the property that $\Delta^\circ = -\Delta$

Exercise 5.7 Mean width and volume radius for standard sets

Compute an equivalent as $n \to \infty$ of both the mean width and the volume radius of the following convex bodies: B_2^n, B_1^n, B_∞^n and Δ_n (from the previous exercise).

Exercise 5.8 Mean width in real life

An airline defines a suitcase (identified as a parallepiped) to be *admissible* in the sum of its dimensions (length+width+height) does not exceed 115 centimeters. Is is possible to hide a non-admissible suitcase inside an admissible suitcase ?

Exercise 5.9 Operator norm unit ball

What are the extreme points of the convex body $K = \{A \in \mathsf{M}_n(\mathbf{R}) \,:\, \|A\|_{\mathrm{op}} \leqslant 1\}$?
What are the extreme points of $K \cap H$, where $H \subset \mathsf{M}_n(\mathbf{R})$ is the subspace of symmetric matrices ?

Exercise 5.10 Trace norm

For $A \in \mathsf{M}_n(R)$, consider the following norm (this is the dual norm to the operator norm)

$$\|A\|_{\mathrm{tr}} = \sup\{\mathrm{tr}(AB) \,:\, B \in \mathsf{M}_n(\mathbf{R}),\ \|B\|_{\mathrm{op}} \leqslant 1\}.$$

1. Show that $\|A\|_{\mathrm{tr}} = \mathrm{tr}\,|A|$, where $|A| = \sqrt{AA^t}$.

2. Identify the extreme points of the unit ball for $\|.\|_{\mathrm{tr}}$.

Exercise 5.11 A challenging one

What is the value of the constant in Gluskin's theorem which is given by the proof?