

Problem sheet # 6
Gaussian processes

Exercise 6.1 Lower bounds on norms of Gaussian matrices

Let G be a $n \times n$ random matrix with i.i.d. $N(0, 1)$ entries.

1. Show that G has the same distribution as OAP , where O, A, P are independent random matrices, with O and P being Haar-distributed on $O(n)$ and A a bidiagonal matrix with independent entries with distributions $a_{i,i} \sim \chi^{(n+1-i)}$ and $a_{i,i+1} \sim \chi^{(n-i)}$ (the $\chi^{(k)}$ distribution is the distribution of $|G|$ where G is a standard Gaussian vector in \mathbf{R}^k).
2. Derive the inequality (κ_n being the expectation of a $\chi^{(n)}$ variable)

$$\mathbf{E}\|G\|_{\text{op}} \geq \left\| \left\| \begin{pmatrix} \kappa_n & \kappa_{n-1} & 0 & \cdots & 0 \\ 0 & \kappa_{n-1} & \kappa_{n-2} & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \kappa_2 & \kappa_1 \\ 0 & \cdots & \cdots & 0 & \kappa_1 \end{pmatrix} \right\|_{\text{op}} \right\|.$$

3. Conclude that $\mathbf{E}\|G\|_{\text{op}}$ is equivalent to $2\sqrt{n}$ as $n \rightarrow \infty$ (the upper bound appeared in the main course).

Exercise 6.2 Some inequalities for Gaussian measure

1. Let X_N be a random vector uniformly distributed in the Euclidean ball $\sqrt{N}B_2^N$, and $\pi_{N,n}$ be the orthogonal projection from \mathbf{R}^N onto its subspace \mathbf{R}^n . Show that as $N \rightarrow \infty$, $\pi_{N,n}(X_N)$ converges in distribution towards a standard Gaussian vector in \mathbf{R}^n .
2. Deduce the following inequality for the standard Gaussian measure γ_n : for any compact sets $K, L \subset \mathbf{R}^n$ and $\lambda \in [0, 1]$,

$$\gamma_n(\lambda K + (1 - \lambda)L) \geq \gamma_n(K)^\lambda \gamma_n(L)^{1-\lambda}.$$

3. Let $K \subset \mathbf{R}^n$ be a symmetric convex body. Define $f : \mathbf{R} \rightarrow \mathbf{R}$ by $f(t) = \gamma_{n-1}(\{y \in \mathbf{R}^{n-1} : (t, y) \in K\})$. Show that $\log f$ is a concave function. Deduce that

$$\int_0^t f \, d\gamma_1 \geq 2\gamma_1([0, t]) \int_0^\infty f \, d\gamma_1.$$

4. Show the following: if K a symmetric convex body and $L = \{x \in \mathbf{R}^n : |\langle x, u \rangle| \leq a\}$ for some $u \in \mathbf{R}^n$ and $a > 0$, then $\gamma_n(K \cap L) \geq \gamma_n(K)\gamma_n(L)$.

Exercise 6.3 Covering with centers outside

If K, L are symmetric convex bodies in \mathbf{R}^n , denote by $N(K, L, \varepsilon) = \inf\{\text{card}(A) : A \subset K \subset A + \varepsilon L\}$ and $N'(K, L, \varepsilon) = \inf\{\text{card}(A) : A \subset \mathbf{R}^n, K \subset A + \varepsilon L\}$. Show that $N(K, L, \varepsilon) \leq N'(K, L, \varepsilon/2)$. Can you give an example with $N(K, L, \varepsilon) \neq N'(K, L, \varepsilon)$?

Exercise 6.4 Exact duality for ellipsoids

Show that if \mathcal{E} and \mathcal{F} are ellipsoids in \mathbf{R}^n , then $N(\mathcal{E}, \mathcal{F}, \varepsilon) = N(\mathcal{F}^\circ, \mathcal{E}^\circ, \varepsilon)$ for every ε .

Exercise 6.5 Covering balls by cubes

Deduce from the dual Sudakov inequality that for every $\varepsilon > 0$, there is a polynomial P such that $N(B_2^n, B_\infty^n, \varepsilon) \leq P(n)$ for every n . Can you give a direct proof of this fact, say for $\varepsilon = 1/2$?