Exercise sheet # 1
Around the trace class operators

Exercise 1.1

Let $T \in B(\mathcal{H})$ and (e_n) be an orthonormal basis.

1. Show that if $\sum_n \|Te_n\| < \infty$ then T is trace-class.
2. Give an example where T is trace-class and $\sum_n \|Te_n\| = \infty$.

Exercise 1.2 Extreme points

What are the extreme points of the following sets ?

1. The set of trace-class operators of trace norm $\leqslant 1$,
2. The set of self-adjoint trace-class operators of trace norm $\leqslant 1$,
3. The set of positive trace-class operators of trace norm $\leqslant 1$,
4. The set of bounded operators of operator norm $\leqslant 1$,
5. The set of bounded self-adjoint operators of operator norm $\leqslant 1$,
6. The set of bounded positive operators of operator norm $\leqslant 1$.

For some questions the situation is much simpler in finite dimension.

Exercise 1.3 The antiderivation operator

Consider the operator T on $L^2(0,1)$ defined as

$$(Tf)(x) = \int_0^x f(t)\,\mathrm{d}t$$

1. Show that T is compact.
2. Is T trace-class ?

Exercise 1.4 Isometries

1. Show that the set of unitary operators is a closed and path-connected subset of $B(\mathcal{H})$.
2. Recall that an operator $T : \mathcal{H} \to \mathcal{H}$ is an isometry if it satisfies $\|Tx\| = \|x\|$ for every $x \in \mathcal{H}$. Show that the set of isometries is a closed subset of $B(\mathcal{H})$, but not connected if $\mathcal{H}$ is infinite-dimensional.
   Hint. Show that if a unitary operator U and an isometry V satisfy $\|U - V\| < 1$, then V is unitary.