Université Claude Bernard Mathématiques
M2 Quantum mechanics and quantum information theory

Exercise sheet # 2
Tensor products of operators

Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces.

Exercice 2.1. Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$. For each of the following properties, assuming A and B satisfy it, can we conclude that $A \otimes B$ satisfies it ?

(a) invertible. (b) self-adjoint. (c) normal. (d) unitary. (e) compact. (f) trace-class.

Exercice 2.2. It the set

$$\mathrm{Vect}\{A \otimes B \ : \ A \in B(\mathcal{H}), B \in B(\mathcal{K})\}$$

dense in $B(\mathcal{H} \otimes \mathcal{K})$ for the norm topology ?

Exercice 2.3. Assume that $\mathcal{H}$ and $\mathcal{K}$ are finite-dimensional. Is every self-adjoint operator on $\mathcal{H} \otimes \mathcal{K}$ of the form

$$\sum_{i=1}^{N} \lambda_i A_i \otimes B_i$$

with λ_i real numbers and A_i (resp. B_i) self-adjoint operators on $\mathcal{H}$ (resp. on $\mathcal{K}$) ?

Exercice 2.4. Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ be non zero operators such that $A \otimes B$ compact. Show that A and B are compact.

Exercice 2.5. Let $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$. We show that

$$\sigma(A \otimes B) = \sigma(A) \cdot \sigma(B) := \{\lambda\mu \ : \ \lambda \in \sigma(A), \mu \in \sigma(B)\}.$$

1. Show the result assuming that A and B are normal.
2. Show that if S, T are two bounded commuting operators on the same Hilbert space, then

   $$\sigma(ST) \subset \sigma(S) \cdot \sigma(T).$$

   (This requires to know about the Gelfand transform ! !) Deduce that the inclusion

   $$\sigma(A \otimes B) \subset \sigma(A) \cdot \sigma(B)$$

   always holds.
3. Let $S \in B(\mathcal{H})$. We denote by $\sigma_1(S)$ the set of $\lambda \in \mathbb{C}$ such that

   $$\inf\{\|(S - \lambda \mathrm{Id})x\| \ : \ \|x\| = 1\} = 0$$

   and $\sigma_2(S) = \sigma(S) \setminus \sigma_1(S)$. Show that $\sigma_1(S)$ is closed, that $\partial\sigma(S) \subset \sigma_1(S)$ and that $\lambda \in \sigma_2(S)$ implies $\overline{\lambda} \in \sigma_1(S^*)$.
4. Show that if $\lambda \in \sigma(A)$ and $\mu \in \sigma(B)$ then $\lambda\mu \in \sigma(A \otimes B)$ by considering the following cases
   (a) $\lambda \in \sigma_1(A)$ and $\mu \in \sigma_1(B)$,
   (b) $\lambda \in \sigma_2(A)$ and $\mu \in \sigma_2(B)$,
   (c) $\lambda = 0$ or $\mu = 0$,
   (d) the remaining cases.