Université Claude Bernard Mathématiques
M2 Quantum mechanics and quantum information theory

Exercise sheet # 3
A bit of physics

Exercice 3.1. (Stern–Gerlach experiment)
The spin of an electron is modelled by the state space $\mathcal{H} = \mathbb{C}^2$. For every unit vector $u = (x, y, z) \in \mathbb{R}^3$, one can « measure the spin in the direction u ». This is described by the observable

$$S_{(x,y,z)} = \begin{pmatrix} z & x - iy \\ x + iy & z \end{pmatrix}.$$

Check that the eigenvalues of $S_{(x,y,z)}$ are ± 1. They correspond to «up» and «down» spins.
Suppose that we measure the spin in a direction u, observe the spin «up», and subsequently measure the spin in the direction v. Show that the probability of observing the spin «up» again equals $\cos^2(\theta/2)$, where θ is the angle between u and v.

Exercice 3.2. (Heisenberg incertaintly principle)
On a Hilbert space $\mathcal{H}$, consider a pure state $|\psi\rangle$ and two bounded observables A and B. Let X (resp. Y) be the outcome of the measure of the state $|\psi\rangle$ via the observable A (resp. B).

1. Show that
$$\mathbf{E}[X] = \langle\psi|A\psi\rangle \text{ and } \mathbf{Var}[X] = \langle\psi|A'\psi\rangle$$
with $A' = (A - \mathbf{E}[X]\mathrm{Id})^2$.
2. Show the inequality
$$\mathbf{Var}[X] \cdot \mathbf{Var}[Y] \geqslant \frac{1}{4}|\langle\psi|[A, B]\psi\rangle|^2 \tag{1}$$
where $[A, B] = AB - BA$. (Hint : reduce to the case where $\mathbf{E}[X] = \mathbf{E}[Y] = 0$).
3. On the Hilbert space $L^2(\mathbb{R})$, consider the unbounded self-adjoint operators A ("position") defined as $Af(x) = xf(x)$ and B ("momentum") defined as $Bf = -i\hbar f'$. Show that the relation
$$[A, B] = i\hbar\mathrm{Id}$$
holds on a dense subspace of $L^2(\mathbb{R})$. It follows that for these operators the product of standard deviations («uncertainties») when measuring any state is always $\geqslant \hbar/2$.
4. Can bounded operators A, B on a Hilbert space satisfy the relation
$$[A, B] = \mathrm{Id} \ \ ?$$