Exercise sheet # 4
Quantum channels

Exercice 4.1. Examples of channels

Show that the following maps $\mathsf{M}_n^{\mathrm{sa}} \to \mathsf{M}_n^{\mathrm{sa}}$

$$R(X) = \mathrm{Tr}(X)\frac{\mathrm{Id}}{n}$$

$$D(X) = \mathrm{diag}(X)$$

(the diagonal part of the matrix X) are quantum channels and give explicit Kraus decomposition. Are they mixed-unitary ?

Exercice 4.2. Direct sum of channels

Let $\Phi_1 : B(\mathcal{H}_1) \to B(\mathcal{H}_1')$ and $\Phi_2 : B(\mathcal{H}_2) \to B(\mathcal{H}_2')$ be quantum channels. Show that the map $B(\mathcal{H}_1 \oplus \mathcal{H}_2) \to B(\mathcal{H}_1' \oplus \mathcal{H}_2')$ defined for $X_{ij} \in B(\mathcal{H}_i, \mathcal{H}_j')$ as

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \mapsto \begin{bmatrix} \Phi_1(X_{11}) & 0 \\ 0 & \Phi_2(X_{22}) \end{bmatrix}$$

is a quantum channel and describe its Kraus operators in terms of the Kraus operators of Φ_1 and Φ_2.

Exercice 4.3. k-positive but not $(k+1)$-positive

Let $k < n$ be integers. Show that the map $\Phi : \mathsf{M}_n^{\mathrm{sa}} \to \mathsf{M}_n^{\mathrm{sa}}$ defined by $\Phi(X) = k\,\mathrm{Tr}(X)\mathrm{Id} - X$ is k-positive but not $(k+1)$-positive.

Exercice 4.4. Channels decrease trace norm

Show that any positive and trace-preserving map $\Phi : \mathsf{M}_n^{\mathrm{sa}} \to \mathsf{M}_n^{\mathrm{sa}}$ satisfies the inequality

$$\|\Phi(X)\|_1 \leqslant \|X\|_1$$

for every $X \in \mathsf{M}_n^{\mathrm{sa}}$.

Exercice 4.5. Schur multipliers

Denote by $\odot$ the entrywise product of matrices, i.e., $(a_{ij}) \odot (b_{ij}) = (a_{ij}b_{ij})$. Given a matrix $A \in \mathsf{M}_n^{\mathrm{sa}}$, consider the map $\Phi : \mathsf{M}_n^{\mathrm{sa}} \to \mathsf{M}_n^{\mathrm{sa}}$ given as $X \mapsto A \odot X$. Show that the following are equivalent

1. The matrix A is positive semi-definite,
2. The map Φ is positive,
3. The map Φ is completely positive.

Exercice 4.6. Unital qubit channels

This exercise proves that every unital quantum channel $\Phi : \mathsf{M}_2^{\mathrm{sa}} \to \mathsf{M}_2^{\mathrm{sa}}$ is mixed-unitary.

1. Argue that it is enough to prove the result for channels which are diagonal in the Pauli basis, i.e., such that
$$\Phi(\mathrm{Id}) = \mathrm{Id},\ \Phi(\sigma_x) = a\sigma_x,\ \Phi(\sigma_y) = b\sigma_y,\ \Phi(\sigma_z) = c\sigma_z \tag{1}$$
for some real numbers a, b, c.
2. Show that the map satisfying (1) is completely positive if and only if $(a+b)^2 \leqslant (1+c)^2$ and $(a-b)^2 \leqslant (1-c)^2$.
3. Conclude by describing the region of $\mathbb{R}^3$ delimited by these inequalities.