Exercise sheet # 5
Entanglement

Exercice 5.1. Transpositions

Let T and T' be transpositions with respect to two orthonormal bases on $\mathbb{C}^d$. Show that there exists a unitary matrix U such that $T'(X) = UT(X)U^*$ for every $X \in \mathsf{M}_d$. Deduce that the property of being PPT does not depend on a choice of basis.

Exercice 5.2. Partial tranpose

What are the eigenvalues and eigenspaces of the partial transposition, $X \mapsto X^\Gamma$ as a map on $B^{\mathrm{sa}}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

Exercice 5.3. Isotropic states

Let (e_i) the canonical basis of $\mathbb{C}^d$ and consider the maximally entangled vector

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |e_i\rangle \otimes |e_i\rangle.$$

For $\frac{-1}{d^2-1} \leqslant \beta \leqslant 1$, consider the state ρ_β on $\mathbb{C}^d \otimes \mathbb{C}^d$ defined as

$$\rho_\beta = \beta|\psi\rangle\langle\psi| + (1-\beta)\frac{\mathrm{Id}}{d^2}.$$

1. For which values of β is the state ρ_β PPT ?
2. (harder, optional) For which values of β is the state ρ_β separable ?

Exercice 5.4. Realignment

The realignment $A^R \in B(\mathbb{C}^n \otimes \mathbb{C}^n, \mathbb{C}^d \otimes \mathbb{C}^d)$ of an operator $A \in B(\mathbb{C}^d \otimes \mathbb{C}^n)$ is defined as follows : the map $A \mapsto A^R$ in $\mathbb{C}$-linear, and $|e_i \otimes e_j\rangle\langle e_k \otimes e_l|^R = |e_i \otimes e_k\rangle\langle e_j \otimes e_l|$.

1. Show that if ρ is a separable state on $\mathbb{C}^d \otimes \mathbb{C}^n$, then $\|\rho^R\|_1 \leqslant 1$.
2. Show that if ρ is a pure state on $\mathbb{C}^d \otimes \mathbb{C}^n$, then ρ is separable if and only if $\|\rho^R\|_1 \leqslant 1$.

Exercice 5.5. Length of separable decomposition

1. Show that every separable state on $\mathbb{C}^d \otimes \mathbb{C}^d$ can be written as the convex combination of $\leqslant d^4$ pure product states.
2. Using a dimension-counting argument, show that there exist a constant $c > 0$, and for every d a separable state on $\mathbb{C}^d \otimes \mathbb{C}^d$ which cannot be written as the convex combination of $\leqslant cd^3$ pure product states.