Exercise sheet # 6
Entanglement (2)

Exercice 6.1. Strictly positive maps

Let C be a proper cone in a finite-dimensional real vector space V. A linear map $\Phi : V \to V$ is said positive if $\Phi(C) \subset C$. Show that the set P of positive maps is a proper cone in the space $L(V)$ of endomorphisms of V, and that a map Φ belongs to the interior of P if and only if

$$\forall x \in C \setminus \{0\}, \quad \Phi(x) \in \mathrm{int}(C).$$

Exercice 6.2. Decomposable maps

Show that the cone of decomposable maps from $\mathbb{C}^d$ to $\mathbb{C}^n$ is closed.

Exercice 6.3. Størmer theorem, quantitative

Størmer's theorem shows that every positive map $M_2 \to M_2$ has the form

$$X \mapsto \sum_{i=1}^{k} A_i X A_i^* + \sum_{j=1}^{l} B_j X^T B_j^*.$$

Can you give a bound on the number $k + l$ of terms appearing in this expression ?

Exercice 6.4. An entangled PPT state

A set of vectors (v_i) in $\mathcal{H}_A \otimes \mathcal{H}_B$ is an unextendible product set if it satisfies the following properties

(i) it is an orthonormal family,

(ii) each vector is a product vector,

(iii) it cannot be exteded to a larger set satisfying (i) and (ii),

(iv) it does not span $\mathcal{H}_A \otimes \mathcal{H}_B$.

1. Show that there is no unextendible product set in $\mathbb{C}^2 \otimes \mathbb{C}^2$.
2. Let $v_1, \dots, v_5$ be the following vectors in $\mathbb{C}^3$

$$e_1, \ \frac{e_1 - e_2}{\sqrt{2}}, \ \frac{e_2 - e_3}{\sqrt{2}}, e_3, \frac{e_1 + e_2 + e_3}{\sqrt{3}}$$

   Find a permutation $\pi \in \mathfrak{S}_5$ such that the family $(v_i \otimes v_{\pi(i)})_{1 \leqslant i \leqslant 5}$ is an unextendible product set in $\mathbb{C}^3 \otimes \mathbb{C}^3$.
3. Let $(v_i)_{1 \leqslant i \leqslant n}$ be an unextendible product set in $\mathbb{C}^d \otimes \mathbb{C}^d$. Show that the formula

$$\frac{1}{d^2 - n} \left(\mathrm{Id} - \sum_{i=1}^{n} |v_i\rangle\langle v_i| \right)$$

   defines a state which is entangled (argue that its range contains no product vector) and PPT.