Exercise sheet # 8
Manipulation of entanglement

Exercice 8.1. Extreme points

Show that a convex compact set in a finite-dimensional space is the convex hull of the set of its extreme points. (*This was used in class but not proved*).

Hint. Use induction on dimension; start by showing that a nonempty convex compact set has at least one extreme point.

Exercice 8.2. Majorisation

Let $p \geqslant 1$. This exercise uses majorisation to prove that $\|A\|_p := (\operatorname{Tr}|A|^p)^{1/p}$ defines a norm of M_n.

1. Let $M \in \mathsf{M}_n$ be a self-adjoint matrix, $d \in \mathbb{R}^n$ its diagonal (i.e., the vector $d_i = M_{ii}$) and $\sigma \in \mathbb{R}^n$ its spectrum. Show that $d \prec \sigma$.
2. Show that if vectors x, y in $\mathbb{R}^n$ satisfy $x \prec y$, then $\sum |x_i|^p \leqslant \sum |y_i|^p$.
3. Show that if A, B are self-adjoint matrices such that $A + B$ is diagonal, then
$$\|A + B\|_p \leqslant \|A\|_p + \|B\|_p.$$
4. Show that $\|\cdot\|_p$ is a norm on M_n.

Exercice 8.3. Discrimination problem for states

Let ρ and σ be two states on $\mathcal{H}$. Suppose that you are given an unknown state X which is equal to ρ or σ with probability 1/2.

1. Show that if you measure the unknown state X using a POVM $(M_i)_{1\leqslant i\leqslant N}$ and try to infer whether $X = \rho$ or $X = \sigma$ from the measurement outcome, the best strategy gives a correct guess with probability
$$\frac{1}{2} + \frac{1}{4} \sum_{1\leqslant i\leqslant N} |\operatorname{Tr}(M_i\rho) - \operatorname{Tr}(M_i\sigma)|$$
2. Show that if we optimize over all POVMs, the optimal strategy gives a correct guess with probability $p_{\text{opt}} = \frac{1}{2} + \frac{1}{4}\|\rho - \sigma\|_1$.
3. Show that $p_{\text{opt}} = 1$ if and only if the range of ρ is orthogonal to the range of σ.

Exercice 8.4. Equal partial traces (partially solved in class)

Let ψ and χ be two unit vectors in $\mathcal{H}_A \otimes \mathcal{H}_B$. Show the equivalence between

1. We have $\operatorname{Tr}_{\mathcal{H}_B} |\psi\rangle\langle\psi| = \operatorname{Tr}_{\mathcal{H}_B} |\chi\rangle\langle\chi|$.
2. There is a unitary transformation U on $\mathcal{H}_B$ such that $\chi = (\mathrm{Id} \otimes U)(\psi)$.

Exercice 8.5.

1. Show that if $A \in \mathsf{M}_n$ is a matrix such that $\operatorname{Tr} A = 0$, then there exists an orthonormal basis (e_i) such that $\langle e_i, Ae_i\rangle = 0$ for every i.
2. Consider two unit vectors ψ and ϕ in $\mathcal{H}_A \otimes \mathcal{H}_B$ such that $\psi \perp \phi$. Using the previous question, show that there exists an orthonormal basis (e_i) in $\mathcal{H}_A$ such that, for every i, we have
$$(\langle e_i| \otimes \mathrm{Id}_{\mathcal{H}_B})|\psi\rangle \perp (\langle e_i| \otimes \mathrm{Id}_{\mathcal{H}_B})|\phi\rangle$$
3. Explain why this means that the pure states $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$ can be perfectly distinguished by an LOCC protocol.

Exercice 8.6. Nondistillability of PPT states

Let ρ be a state on $\mathcal{H}_A \otimes \mathcal{H}_B$ with positive partial transpose. Show that if Φ is an LOCC channel, then $\Phi(\rho)$ also has positive partial transpose. Deduce that $E_D(\rho) = 0$.

Exercice 8.7. Mixtures with equal weights

Show that for every state ρ on $\mathbb{C}^d$, there exist unit vectors $\psi_1, \ldots, \psi_d$ such that

$$\rho = \frac{1}{d} \sum_{i=1}^{d} |\psi_i\rangle\langle\psi_i|$$