Exercise sheet # 9
Fidelity, entropy

Exercice 9.1. Properties of fidelity

Let ρ and σ be states on $\mathcal{H}$, and τ be a state on $\mathcal{H}'$. Show the following properties of fidelity

1. We have $F(\rho, \sigma) = F(\sigma, \rho)$.
2. We have $F(\rho, \sigma) \geqslant 0$ with equality if and only if $\rho\sigma = 0$.
3. We have $F(\rho, \sigma) \leqslant 1$ with equality if and only if $\rho = \sigma$.
4. If $V : \mathcal{H} \to \mathcal{H}'$ is an isometric embedding, then $F(V\rho V^*, V\sigma V^*) = F(\rho, \sigma)$.
5. We have $F(\rho \otimes \tau, \sigma \otimes \tau) = F(\rho, \sigma)$.

Exercice 9.2. von Neumann entropy and majorisation

1. Show that if ρ and σ are states on $\mathbb{C}^d$ such that $\operatorname{spec} \rho \prec \operatorname{spec} \sigma$, then $H(\rho) \geqslant H(\sigma)$.
2. Show that for every state ρ on $\mathbb{C}^d$, we have $\rho \leqslant \log_2 d$.
3. Show that H is a concave function on the set of quantum states.
4. Show that $\Phi : \mathsf{M}_d \to \mathsf{M}_d$ is a unital quantum channel, then $H(\Phi(\rho)) \geqslant H(\rho)$ for any state ρ on $\mathbb{C}^d$.