Université Claude Bernard — Mathématiques

M2 Quantum mechanics and quantum information theory

**Exercise sheet # 1**

**Exercise 1.1 Extreme points**

Recall that if $K$ is a convex compact set, a point $x \in K$ is said to be extreme if $K \setminus \{x\}$ is convex. What are the extreme points of the following convex compact sets in $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a finite-dimensional Hilbert space ?

1. The set of operators of trace norm $\leqslant 1$,
2. The set of self-adjoint operators of trace norm $\leqslant 1$,
3. The set of positive operators of trace norm $\leqslant 1$,
4. The set of operators of operator norm $\leqslant 1$,
5. The set of self-adjoint operators of operator norm $\leqslant 1$,
6. The set of positive operators of operator norm $\leqslant 1$.

**Exercise 1.2 Isomorphic ?**

1. Let $\mathcal{H}, \mathcal{K}$ be complex finite-dimensional Hilbert spaces. Are the spaces $\mathcal{B}(\mathcal{H})_{sa} \otimes_{\mathbb{R}} \mathcal{B}(\mathcal{K})_{sa}$ and $\mathcal{B}(\mathcal{H} \otimes_{\mathbb{C}} \mathcal{K})_{sa}$ canonically isomorphic ?
2. Let $\mathcal{H}, \mathcal{K}$ be real finite-dimensional Hilbert spaces. Are the spaces $\mathcal{B}(\mathcal{H})_{sa} \otimes_{\mathbb{R}} \mathcal{B}(\mathcal{K})_{sa}$ and $\mathcal{B}(\mathcal{H} \otimes_{\mathbb{R}} \mathcal{K})_{sa}$ canonically isomorphic ?

**Exercise 1.3 Spectrum of a tensor product**

Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be any operators (not assumed to be Hermitian). If you know the spectrum of $A$ and $B$, can you deduce the spectrum of $A \otimes B$ ?

**Exercise 1.4 Every state is an equal average of pure states!**

Let $\mathcal{H}$ be a $d$-dimensional Hilbert space. Show that every mixed state $\rho$ on $\mathcal{H}$ can be written

$$\rho = \frac{1}{d} \left( |\psi_1\rangle\langle\psi_1| + \cdots + |\psi_d\rangle\langle\psi_d| \right)$$

for unit vectors $\psi_1, \ldots, \psi_d$ in $\mathcal{H}$.