Exercise sheet # 5
POVMs and symmetries

Exercise 5.1. SIC-POVMs

A POVM $(A_i)_{i\in I}$ on $\mathbb{C}^d$ is said to be a SIC-POVM (SIC=symmetric and informationnally complete) if the set I has cardinal d^2 and if there exist constants λ and K such that

(a) for every $i \in I$, we have $A_i = \lambda|\psi_i\rangle\langle\psi_i|$ for a unit vector $\psi_i \in \mathbb{C}^d$,

(b) for every $i \neq j$ in I, we have $\mathrm{Tr}(A_iA_j) = K$.

1. Determine, as a function of d, the unique possible value for λ and K.
2. Show that if (A_i) is a SIC-POVM, then the operators (A_i) are linearly independent
3. Show that if $d = 2$, we obtain a SIC-POVM by considering vectors ψ_i which are the vertices of a regular tetrahedron inscribed in the Bloch sphere.
4. Check that the family of 9 unit vectors in $\mathbb{C}^3$

$$(0,-1,\alpha);\quad (\alpha,0,-1);\quad (-1,\alpha,0)$$

where α ranges over third roots of unit, gives a SIC-POCM in $\mathbb{C}^3$.

Zauner's conjecture postulates that SIC-POVM exist in every dimension.

Exercise 5.2. Mutually unbiased bases

We say that two bases $(e_i)_{1\leqslant i\leqslant d}$ and $(f_j)_{1\leqslant j\leqslant d}$ are *mutually unbiased* if there exists a constant α such that

$$|\langle e_i, f_j\rangle| = \lambda$$

for every indices i, j in $\{1,\dots,d\}$.

1. What is the value of λ ?
2. How does this concept interpret in terms of measurements?
3. Show that there exists at most $d-1$ pairwise mutually unbiased basis in $\mathbb{C}^d$.
   Hint. If $\mathcal{B}_1,\dots,\mathcal{B}_k$ are pairwise mutually unbiased, what can be said about the subspaces

$$E_i = \mathrm{span}\Big\{|\psi\rangle\langle\psi| - \frac{1}{d}\mathrm{Id}\ :\ \psi\in\mathcal{B}_i\Big\}$$

4. Suppose that d is a prime number and let $\omega = \exp(2i\pi/d)$. Let (e_i) be the canonical basis of $\mathbb{C}^d$ (with the convention that $e_{d+1} = e_1$). Consider the unitary matrices

$$X = \sum_{i=1}^{d} |e_{i+1}\rangle\langle e_i|,\quad Z = \sum_{i=1}^{d}\omega^i|e_i\rangle\langle e_i|$$

Show that the $d+1$ bases obtained as the eigenbases of

$$X, Z, XZ, XZ^2, \dots, XZ^{d-1}$$

are mutually unbiased.

Exercise 5.3. A proof of the de Finetti theorem

The goal of this exercise is to prove the de Finetti theorem as used in the course. For simplicity we work with $F = \{0, 1\}$ but the general case is similar. For $p \in [0, 1]$, let β_p the probability distribution $p\delta_1 + (1-p)\delta_0$. We denote by $\mathrm{Prob}([0, 1])$ the set of Borel probability measures on $[0, 1]$. The de Finetti theorem asserts that if $(X_n)_{n\in\mathbb{N}}$ is an exchangeable sequence of $\{0, 1\}$-valued random variables, then there exists a unique $\pi \in \mathrm{Prob}([0, 1])$ such that, for every k

$$\mathbf{P}(X_1 = \cdots = X_k = 1) = \int_0^1 t^k \, \mathrm{d}\pi(t) \tag{1}$$

Let $C([0, 1])$ the Banach space of continuous functions from $[0, 1]$ to $\mathbb{R}$. We identify $\mathrm{Prob}([0, 1])$ with a subset of $C([0, 1])^*$ to define the weak-$*$ topology on the dual of a Banach space.

1. Show that (1) implies that for every k and $x_1, \ldots, x_k$ in $\{0, 1\}$,

$$\mathbf{P}(X_1 = x_1, \cdots, X_k = x_k) = \int_0^1 (1-t)^{k-\ell} t^\ell \, \mathrm{d}\pi(t) \tag{2}$$

   where $\ell = x_1 + \cdots + x_k$.

2. Consider for each n the random variable

$$S_n = \frac{X_1 + \cdots + X_n}{n}$$

   Show that, for every $k \geqslant 1$,

$$\lim_{n\to\infty} \mathbf{E}[S_n^k] = \mathbb{P}(X_1 = \cdots = X_k = 1).$$

   Hint. Expand the kth power and distinguish tuples of distinct indices from tuples with repetitions.

3. Show that the set $\mathrm{Prob}([0, 1])$ is compact for the weak-$*$ topology. Let π_n be the distribution of S_n. Show that there is a subsequence (π_{n_j}) which converges weakly to $\pi \in \mathrm{Prob}([0, 1])$.

4. Show that this probability measure π satisfies (1).

5. Using the Weierstrass approximation theorem, show that if $\pi, \pi' \in \mathrm{Prob}([0, 1])$ satisfy (1), then $\pi = \pi'$.