**Exercise sheet # 6**
Quantum nonlocality

**Exercise 6.1. Convexity of the set of quantum correlations**

Show directly that the set $\mathsf{Q}_{m,n}$ of quantum correlations is convex.
(Interestingly, this set is not closed in general).

**Exercise 6.2. Maximal violation of CHSH on 2 qubits**

Give an explicit family of observables on $\mathbb{C}^2$, as well as a state on $\mathbb{C}^2 \otimes \mathbb{C}^2$, which show that the matrix $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is a quantum correlation matrix.

**Exercise 6.3. Optimality of the CHSH violation**

1. Show that if $X_1$, $X_2$, $Y_1$, $Y_2$ are self-adjoint operators satisfying $X_1^2 = X_2^2 = \mathrm{Id}_{\mathcal{H}_A}$ and $Y_1^2 = Y_2^2 = \mathrm{Id}_{\mathcal{H}_B}$ then the operator $B = X_1 \otimes Y_1 + X_1 \otimes Y_2 + X_2 \otimes Y_1 - X_2 \otimes Y_2$ satisfies $\|B\|_{op} \leqslant 2\sqrt{2}$.
   **Hint**. Compute $B^2 + [X_1, X_2] \otimes [Y_1, Y_2]$.

2. Deduce that any matrix $(a_{ij})$ in $\mathsf{Q}_{2,2}$ satisfies the inequality $a_{11} + a_{12} + a_{21} - a_{22} \leqslant 2\sqrt{2}$.

**Exercise 6.4. A lemma used in the proof of Grothendieck inequality**

This exercise proves the following formula: let $u, v$ be unit vectors in $\mathbb{R}^N$ and $G$ a standard Gaussian vector in $\mathbb{R}^N$ (i.e., its coordinates are independent random variables with a $N(0,1)$ distribution). Then

$$\mathbf{E}[\mathrm{sign}(\langle u, G\rangle)\mathrm{sign}(\langle v, G\rangle)] = \frac{2}{\pi} \arcsin\langle u, v\rangle$$

1. Explain why it suffices to consider the case $N = 2$.

2. For $N = 2$, explain why it suffices to prove the analogous formula with $G$ replaced by a uniformly distributed random unit vector.

3. Prove the lemma.
   **Hint**. Draw a picture.