Université Claude Bernard — Mathématiques
M2 Quantum mechanics and quantum information theory

Partial Exam

Exercise 1 Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two Hilbert spaces and let ψ be a unit vector in $\mathcal{H}_A \otimes \mathcal{H}_B$. Using a Schmidt decomposition, show that the states

$$\mathrm{Tr}_{\mathcal{H}_A}|\psi\rangle\langle\psi| \quad \text{and} \quad \mathrm{Tr}_{\mathcal{H}_B}|\psi\rangle\langle\psi|$$

have the same nonzero eigenvalues.

Solution. Consider a Schmidt decomposition $\psi = \sum s_i e_i \otimes f_i$ where $s_i > 0$, (e_i) is an orthonormal family in $\mathcal{H}_A$ and (f_i) is an orthonormal family in $\mathcal{H}_B$. We compute

$$\mathrm{Tr}_{\mathcal{H}_A}|\psi\rangle\langle\psi| = \sum_i s_i^2 |e_i\rangle\langle e_i|$$
$$\mathrm{Tr}_{\mathcal{H}_B}|\psi\rangle\langle\psi| = \sum_i s_i^2 |f_i\rangle\langle f_i|$$

showing that the nonzero eigenvalues of both states are the numbers s_i^2.

Exercise 2 Let α and β in $(0, \pi/2)$. Consider the matrices

$$A = \begin{pmatrix} \cos\alpha & 0 \\ 0 & \cos\beta \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \sin\beta \\ \sin\alpha & 0 \end{pmatrix},$$

and the map $\Psi : M_2 \to M_2$ defined as $\Psi(X) = AXA^* + BXB^*$.

1. Show that Ψ is a quantum channel. Is it unital? Solution. It is clear that Ψ is completely positive, from the easy direction of Choi's theorem. Since $A^*A + B^*B = \mathrm{Id}$, it is also trace-preserving. The quantum channel Ψ is unital if and only if $AA^* + BB^* = \mathrm{Id}$. Since $AA^* + BB^* = \begin{pmatrix} \cos^2\alpha + \sin^2\beta & 0 \\ 0 & \sin^2\alpha + \cos^2\beta \end{pmatrix}$, this happens iff $\cos^2\alpha + \sin^2\beta = 1$, i.e., iff $\alpha = \beta$.

2. How many states ρ on $\mathbb{C}^2$ have the property that $\Psi(\rho)$ is a pure state? Solution. There are precisely two such states. First, for every state ρ, we have $\mathrm{rank}\Phi(\rho) \geqslant \mathrm{rank}A\rho A^* = \mathrm{rank}\rho$, we see that if $\Psi(\rho)$ is pure, then ρ is pure. Consider a pure state $\rho = |\psi\rangle\langle\psi|$ for a unit vector $\psi = (\psi_1, \psi_2) \in \mathbb{C}^2$. The state $\Psi(\rho)$ is pure iff the operators $A\rho A* = |A\psi\rangle\langle A\psi|$ and $B\rho B^* = |B\psi\rangle\langle B\psi|$ are proportional, which happens iff the vectors $A\psi$ and $B\psi$ are proportional. Since

$$\det(A\psi, B\psi) = \det \begin{pmatrix} \psi_1 \cos\alpha & \psi_2 \sin\beta \\ \psi_2 \cos\beta & \psi_1 \sin\alpha \end{pmatrix} = \psi_1^2 \cos\alpha \sin\alpha - \psi_2^2 \cos\beta \sin\beta,$$

we see that the condition $\det(A\psi, B\psi) = 0$ is equivalent to $\psi_2 = \pm\frac{\cos\alpha \sin\alpha}{\cos\beta \sin\beta}\psi_1$. It follows that there are two states with a pure image (one for each sign choice).

Exercise 3 Let $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d$, $T : M_d \to M_d$ the transposition map $T(A) = A^t$ and $T_1 = T \otimes \mathrm{Id}_{M_d} : B(\mathcal{H}) \to B(\mathcal{H})$ the partial transposition.

1. Show that if ρ is separable, then $T_1(\rho)$ is a positive operator. Solution. This is easy: if ρ can be written as $\rho = \sum \lambda_i \sigma_i \otimes \tau_i$ for states σ_i, τ_i on $\mathbb{C}^d$ and a convex combination (λ_i), then

$$T_1(\rho) = (T \otimes \mathrm{Id})(\rho) = \sum \lambda_i \sigma_i \otimes T(\tau_i)$$

is positive ($T(\tau_i)$ is positive since it has the same spectrum as τ_i).

2. Deduce that the following state is entangled on $\mathbb{C}^2 \otimes \mathbb{C}^2$ for $0 \leqslant t < 2/3$

$$t\frac{\mathrm{Id}}{4} + (1-t)|\Phi^+\rangle\langle\Phi^+|,$$

where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Solution. Let

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right).$$

The operator

$$T_1(\rho) = \frac{1}{2}\left(|00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

has eigenvalues $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$. If we set $\rho_t = t\frac{\mathrm{Id}}{4} + (1-t)\rho$, it follows that ρ_t has an eigenvalue equal to $\frac{t}{4} - \frac{1-t}{2} = \frac{3t-2}{4}$. For $t < 2/3$, this number is < 0 and therefore the state ρ_t is entangled.

Exercise 4 The first part of the exercise shows the following lemma

Lemma. If $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ in M_n are such that the matrix $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is Hermitian positive, then $\|B\|_1^2 \leqslant \|A\|_1 \cdot \|C\|_1$.

1. Using the singular value decomposition, show that it suffices to show the lemma in the case where B is diagonal with nonnegative real coefficients. Solution. By the singular value decomposition, we may write $B = U\Delta V^*$ for unitary matrices U, V and a matrix Δ diagonal with nonnegative real coefficients. Define $A' = U^*AU$ and $C' = V^*CV$ and observe that $\|B\|_1 = \|\Delta\|_1$, $\|A\|_1 = \|A'\|_1$ and $\|C\|_1 = \|C'\|_1$. Since

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} = \begin{bmatrix} UA'U^* & U\Delta V^* \\ V\Delta U^* & VC'V^* \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \cdot \begin{bmatrix} A' & \Delta \\ \Delta & C' \end{bmatrix} \cdot \begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix},$$

the matrix $\begin{bmatrix} A' & \Delta \\ \Delta & C' \end{bmatrix}$ is positive under the assumptions from the lemma, and the lemma applied to that matrix implies the general case.

2. We assume that B is diagonal with nonnegative real coefficients. Show that for every i, $b_{ii}^2 \leqslant a_{ii}c_{ii}$. Solution. Since the 2×2 matrix $\begin{pmatrix} a_i & b_{ii} \\ b_{ii} & c_i \end{pmatrix}$ is a principal submatrix of $\begin{bmatrix} A & B \\ B & C \end{bmatrix}$, it is positive and therefore its determinant $a_ic_i - b_{ii}^2$ is $\geqslant 0$.

3. Complete the proof of the lemma. Since A and C are positive as submatrices of a positive matrix, we have $\|A\|_1 = \operatorname{Tr} A = \sum a_{ii}$ and $\|C\|_1 = \sum c_{ii}$. Thus, using the Cauchy-Schwartz inequality for sums,

$$\|B\|_1 = \sum b_{ii} \leqslant \sum \sqrt{a_ic_i} \leqslant \left(\sum a_i\right)^{1/2}\left(\sum c_i\right)^{1/2} = \|A\|_1^{1/2}\|C\|_1^{1/2}$$

Let d, n be integers and $\Phi : M_d \to M_n$. We compare the following quantities

$$\|\Phi\|_a = \sup_{A \in M_d,\ \|A\|_1 \leqslant 1} \|\Phi(A)\|_1$$

$$\|\Phi\|_b = \sup_{A \in M_d,\ A = A^*,\ \|A\|_1 \leqslant 1} \|\Phi(A)\|_1$$

4. Show that

$$\|\Phi\|_a = \sup_{x,y \in \mathbb{C}^d,\ \|x\| = \|y\| = 1} \|\Phi(|x\rangle\langle y|)\|_1$$

$$\|\Phi\|_b = \sup_{x \in \mathbb{C}^d,\ \|x\| = 1} \|\Phi(|x\rangle\langle x|)\|_1$$

Solution. By the singular value decomposition, any operator $A \in M_d$ with $\|A\|_1 = 1$ is a convex combination of operators of the form $|x\rangle\langle y|$ for unit vectors x, y in $\mathbb{C}^d$. By the spectral theorem, any self-adjoint operator $A \in M_d$ with $\|A\|_1 = 1$ is a convex combination of operators of the form $\pm|x\rangle\langle x|$ for a unit vector x in $\mathbb{C}^d$. In each case, the results follow from the convexity of $X \mapsto \|\Phi(X)\|_1$

5. Show that if Φ is completely positive, then $\|\Phi\|_a = \|\Phi\|_b$.
Indication: consider $(\Phi \otimes \mathrm{Id}_{M_2})(|u\rangle\langle u|)$, where $|u\rangle = |x\rangle \otimes |0\rangle + |y\rangle \otimes |1\rangle$ and use the lemma. Solution. It is obvious that $\|\Phi\|_b \leqslant \|\Phi\|_a$. To prove the converse, consider unit vectors x, y in $\mathbb{C}^d$ and $|u\rangle = |x\rangle \otimes |0\rangle + |y\rangle \otimes |1\rangle$. We have

$$|u\rangle\langle u| = \begin{pmatrix} |x\rangle\langle x| & |x\rangle\langle y| \\ |y\rangle\langle x| & |y\rangle\langle y| \end{pmatrix}$$

$$(\Phi \otimes \mathrm{Id})(|u\rangle\langle u|) = \begin{pmatrix} \Phi(|x\rangle\langle x|) & \Phi(|x\rangle\langle y|) \\ \Phi(|y\rangle\langle x|) & \Phi(|y\rangle\langle y|) \end{pmatrix}$$

Since Φ is completely positive, this matrix is self-adjoint and positive, and thus

$$\|\Phi(|x\rangle\langle y|)\|_1 \leqslant \|\Phi(|x\rangle\langle x|)\|_1^{1/2} \|\Phi(|y\rangle\langle y|)\|_1^{1/2} \leqslant \|\Phi\|_b$$

Taking supremum over x, y gives $\|\Phi\|_a \leqslant \|\Phi\|_b$.

6. Let $(|0\rangle, |1\rangle)$ be the canonical basis of $\mathbb{C}^2$ and $(|0\rangle, |1\rangle, |2\rangle, |3\rangle)$ be the canonical basis of $\mathbb{C}^4$. We consider the following vectors in $\mathbb{C}^2$

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

We define $\Phi_0, \Phi_1 : M_2 \to M_4$ by the formulas

$$\Phi_0(X) = \frac{1}{2}\left(|0\rangle\langle 0|X|0\rangle\langle 0| + |1\rangle\langle +|X|+\rangle\langle 1| + |2\rangle\langle 1|X|1\rangle\langle 2| + |3\rangle\langle -|X|-\rangle\langle 3|\right)$$

$$\Phi_1(X) = \frac{1}{2}\left(|0\rangle\langle 1|X|1\rangle\langle 0| + |1\rangle\langle -|X|-\rangle\langle 1| + |2\rangle\langle 0|X|0\rangle\langle 2| + |3\rangle\langle +|X|+\rangle\langle 3|\right).$$

Explain briefly why Φ_0 and Φ_1 are quantum channels, and then show that $\|\Phi_0 - \Phi_1\|_a = 2$ while $\|\Phi_0 - \Phi_1\|_b < 2$. Solution. The maps Φ_0 and Φ_1 are completely positive by the easy direction of Choi's theorem and are trace-preserving since

$$\frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1| + |+\rangle\langle +| + |-\rangle\langle -|\right) = \mathrm{Id}$$

If we denote $X = (X_{ij})$, then

$$\begin{aligned}\langle 0|X|0\rangle &= X_{00}\\ \langle 1|X|1\rangle &= X_{11}\\ \langle +|X|+\rangle &= \frac{X_{00}+X_{01}+X_{10}+X_{11}}{2}\\ \langle -|X|-\rangle &= \frac{X_{00}-X_{01}-X_{10}+X_{11}}{2}\end{aligned}$$

we have

$$\Phi_0(X) - \Phi_1(X) = \frac{X_{00}-X_{11}}{2}(|0\rangle\langle 0| - |2\rangle\langle 2|) + \frac{X_{01}+X_{10}}{2}(|1\rangle\langle 1| - |3\rangle\langle 3|)$$

and thus

$$\|\Phi_0(X) - \Phi_1(X)\|_1 = |X_{00} - X_{11}| + |X_{10} + X_{01}|$$

If $X = |x\rangle\langle x|$ for a unit vector $x = (x_1, x_2) \in \mathbb{C}^2$, then

$$\|\Phi_0(X) - \Phi_1(X)\|_1 = \left||x_1|^2 - |x_2|^2\right| + |2\Re(x_1 x_2)|$$

and this quantity cannot be equal to 2 (each term in the sum is $\leqslant 1$ and the equality conditions are not compatible). Since the supremum is achieved, this shows that $\|\Phi_0 - \Phi_1\|_b < 2$. The matrix $X = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ is a non-selfadjoint rank 1 matrix with $\|X\|_1 = 1$ which satisfies $|X_{00} - X_{11}| = 1$ and $|X_{01} + X_{10}| = 1$ and thus $\|(\Phi_0 - \Phi_1)(X)\|_1 = 2$, proving that $\|\Phi_0 - \Phi_1\|_a = 2$