Université Claude Bernard Mathématiques
M2 Quantum mechanics and quantum information theory

Partial Exam

Exercise 1 Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two Hilbert spaces and let ψ be a unit vector in $\mathcal{H}_A \otimes \mathcal{H}_B$. Using a Schmidt decomposition, show that the states

$$\mathrm{Tr}_{\mathcal{H}_A}|\psi\rangle\langle\psi| \quad \text{and} \quad \mathrm{Tr}_{\mathcal{H}_B}|\psi\rangle\langle\psi|$$

have the same nonzero eigenvalues.

Exercise 2 Let α and β in $(0, \pi/2)$. Consider the matrices

$$A = \begin{pmatrix} \cos\alpha & 0 \\ 0 & \cos\beta \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \sin\beta \\ \sin\alpha & 0 \end{pmatrix},$$

and the map $\Psi : M_2 \to M_2$ defined as $\Psi(X) = AXA^* + BXB^*$.

1. Show that Ψ is a quantum channel. Is it unital?
2. How many states ρ on $\mathbb{C}^2$ have the property that $\Psi(\rho)$ is a pure state?

Exercise 3 Let $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d$, $T : M_d \to M_d$ the transposition map $T(A) = A^t$ and $T_1 = T \otimes \mathrm{Id}_{M_d} : B(\mathcal{H}) \to B(\mathcal{H})$ the partial transposition.

1. Show that if ρ is separable, then $T_1(\rho)$ is a positive operator.
2. Deduce that the following state is entangled on $\mathbb{C}^2 \otimes \mathbb{C}^2$ for $0 \leqslant t < 2/3$

$$t\frac{\mathrm{Id}}{4} + (1-t)|\Phi^+\rangle\langle\Phi^+|,$$

where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Exercise 4 The first part of the exercise shows the following lemma

Lemma. If $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$ in M_n are such that the matrix $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ is Hermitian positive, then $\|B\|_1^2 \leqslant \|A\|_1 \cdot \|C\|_1$.

1. Using the singular value decomposition, show that it suffices to show the lemma in the case where B is diagonal with nonnegative real coefficients.
2. We assume that B is diagonal with nonnegative real coefficients. Show that for every i, $b_{ii}^2 \leqslant a_{ii}c_{ii}$.
3. Complete the proof of the lemma.

Let d, n be integers and $\Phi : M_d \to M_n$. We compare the following quantities

$$\|\Phi\|_a = \sup_{A \in M_d,\ \|A\|_1 \leqslant 1} \|\Phi(A)\|_1$$

$$\|\Phi\|_b = \sup_{A \in M_d,\ A = A^*,\ \|A\|_1 \leqslant 1} \|\Phi(A)\|_1$$

4. Show that

$$\|\Phi\|_a = \sup_{x,y \in \mathbb{C}^d,\ \|x\| = \|y\| = 1} \|\Phi(|x\rangle\langle y|)\|_1$$

$$\|\Phi\|_b = \sup_{x \in \mathbb{C}^d,\ \|x\| = 1} \|\Phi(|x\rangle\langle x|)\|_1$$

5. Show that if Φ is completely positive, then $\|\Phi\|_a = \|\Phi\|_b$.
Indication: consider $(\Phi \otimes \mathrm{Id}_{M_2})(|u\rangle\langle u|)$, where $|u\rangle = |x\rangle \otimes |0\rangle + |y\rangle \otimes |1\rangle$ and use the lemma.

6. Let $(|0\rangle, |1\rangle)$ be the canonical basis of $\mathbb{C}^2$ and $(|0\rangle, |1\rangle, |2\rangle, |3\rangle)$ be the canonical basis of $\mathbb{C}^4$. We consider the following vectors in $\mathbb{C}^2$

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

We define $\Phi_0, \Phi_1 : M_2 \to M_4$ by the formulas

$$\Phi_0(X) = \frac{1}{2}\left(|0\rangle\langle 0|X|0\rangle\langle 0| + |1\rangle\langle +|X|+\rangle\langle 1| + |2\rangle\langle 1|X|1\rangle\langle 2| + |3\rangle\langle -|X|-\rangle\langle 3|\right)$$

$$\Phi_1(X) = \frac{1}{2}\left(|0\rangle\langle 1|X|1\rangle\langle 0| + |1\rangle\langle -|X|-\rangle\langle 1| + |2\rangle\langle 0|X|0\rangle\langle 2| + |3\rangle\langle +|X|+\rangle\langle 3|\right).$$

Explain briefly why Φ_0 and Φ_1 are quantum channels, and then show that $\|\Phi_0 - \Phi_1\|_a = 2$ while $\|\Phi_0 - \Phi_1\|_b < 2$.