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THE MULTIPLICATIVE PROPERTY CHARACTERIZES ℓ_p AND L_p NORMS

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We show that ℓ_p norms are characterized as the unique norms which are both invariant under coordinate permutation and multiplicative with respect to tensor products. Similarly, the L_p norms are the unique rearrangement-invariant norms on a probability space such that $\|XY\| = \|X\| \cdot \|Y\|$ for every pair X, Y of independent random variables. Our proof combines the tensor power trick and Cramér's large deviation theorem.

Keywords: L_p norms; large deviation theory; tensor power trick.

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1. Introduction

The ℓ_p and L_p spaces are among the most important examples of Banach spaces and have been widely investigated (see e.g. [2] for a survey). In this note, we exhibit a characterization of the ℓ_p/L_p norms by a simple algebraic identity: the *multiplicative property*. In the case of ℓ_p norms, this property reads as $\|x \otimes y\| = \|x\| \cdot \|y\|$ for every (finite) sequences x, y. In the case of L_p norms, it becomes $\|XY\| = \|X\| \cdot \|Y\|$ whenever X, Y are independent (bounded) random variables.

There are many examples of theorems showing how special are ℓ_p/L_p spaces among Banach spaces. An early axiomatic characterization of ℓ_p/L_p spaces goes back to Bohnenblust [6]: among Banach lattices, they are the only spaces in which $\|x + y\|$ depends only on $\|x\|$ and $\|y\|$ whenever x, y are orthogonal. Let us also mention a deep theorem by Krivine [11]: every basic sequence, in any Banach space, admits a p such that ℓ_p is block finitely represented therein.

Although it did not appear explicitly in the literature, the main result of this note is not completely new. For example it can be derived from Krivine's aforementioned theorem (see Sec. 1.3). However, we put emphasis on our method of proof, which is original and — we believe — elegant, and on the simplicity of the statement. Shortly after this note was made public, our result was used by Tom Leinster [13] to provide a new characterization of power means.

Inspiration for the present note comes from quantum information theory, where the multiplicative property of the commutative and noncommutative ℓ_p norms play an important role; see [12, 3, 4] and references therein. Of great importance in classical and quantum information theory, *Rényi entropies* are tightly connected to ℓ_p norms:

$$H_p(x) = \frac{p}{1-p}\log\|x\|_p, \quad p \in [0,\infty].$$

The multiplicative property of the ℓ_p norms translates into additivity for the corresponding Rényi entropy.

The monograph [1] contains many axiomatic characterizations of such entropies (especially for $p = 1$, the Shannon entropy) and many questions remain open (such as the problem after Proposition (5.2.38)). Note that the results in this work do not apply directly to characterize nicely Rényi entropies, since the triangle inequality for norms does not have a natural expression in terms of entropies.

1.1. *Discrete case: Characterization of ℓ_p norms*

Let c_{00} be the space of finitely supported real sequences. The coordinates of an element $x \in c_{00}$ are denoted $(x_i)_{i\in\mathbf{N}^*}$.

If $x, y \in c_{00}$, we define $x \otimes y$ to be double-indexed sequence $(x_i y_j)_{(i,j)\in\mathbf{N}^*\times\mathbf{N}^*}$. Throughout the paper, we consider $x\otimes y$ as an element of c_{00} via some fixed bijective map between $\mathbf{N}^*$ and $\mathbf{N}^* \times \mathbf{N}^*$.

We consider a norm $\|\cdot\|$ on c_{00} satisfying the following conditions

(1) (*permutation-invariance*) If $x, y \in c_{00}$ are equal up to permutation of their coordinates, then $\|x\| = \|y\|$.
(2) (*multiplicativity*) If $x, y \in c_{00}$, then $\|x \otimes y\| = \|x\| \cdot \|y\|$.

Because of the invariance under permutation, the specific choice of a bijection between $\mathbf{N}^*$ and $\mathbf{N}^* \times \mathbf{N}^*$ is irrelevant. Examples of a norm satisfying both conditions are given by ℓ_p norms, defined by

$$\|x\|_p = \left(\sum_{i\in\mathbf{N}^*} |x_i|^p\right)^{1/p} \quad \text{if } 1 \le p < +\infty; \quad \|x\|_\infty = \sup_{i\in\mathbf{N}^*} |x_i|.$$

The next theorem shows that there are no other examples.

Theorem 1.1. *If a norm $\|\cdot\|$ on c_{00} is permutation-invariant and multiplicative, then it coincides with $\|\cdot\|_p$ for some $p \in [1,+\infty]$.*

A more invariant formulation of Theorem 1.1 can be stated without referring to a particular bijection between $\mathbf{N}$ and $\mathbf{N}^2$ (this was pointed to us by Tom Leinster). Assume that, for every finite set I, a norm is given on $\mathbf{R}^I$, in such a way that for any finite sets I, J

(1) If $f : I \to J$ is any injective map, then $\|f_*(x)\| = \|x\|$ for every $x \in \mathbf{R}^I$, where $f_* : \mathbf{R}^I \to \mathbf{R}^J$ is the map obtained by re-indexing coordinates according to f, and padding with zeros.
(2) For every $x \in \mathbf{R}^I, y \in \mathbf{R}^J$, we have $\|x \otimes y\| = \|x\| \cdot \|y\|$ for every $x \in \mathbf{R}^I, y \in \mathbf{R}^J$ (here $x \otimes y$ is considered as an element of $\mathbf{R}^{I \times J}$).

Then these norms are actually the usual ℓ_p norms for some $p \in [1, +\infty]$.

Our proof of Theorem 1.1 is simple and goes as follows. First, the value of p is retrieved by looking at $\|(1, 1)\|$. Then, for every $x \in c_{00}$, the quantity $\|x\|$ is shown to equal $\|x\|_p$ by examining the statistical distribution of large coordinates of the nth tensor power $x^{\otimes n}$ (n large) through Cramér's large deviations theorem. We defer the proof to Sec. 2.

1.2. *Continuous case: Characterization of L_p norms*

We now formulate a version of Theorem 1.1 in a continuous setting, in order to characterize L_p norms. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a *rich* probability space, which means that it is possible to define on it one continuous random variable. This implies that we can define on Ω an arbitrary number of independent random variables with arbitrary distributions; one can think of Ω as the interval $[0, 1]$ equipped with the Lebesgue measure. A random variable is said to be *simple* if it takes only finitely many values. For a random variable $X : \Omega \to \mathbf{R}$, the L_p norms are defined as

$$\|X\|_{L_p} = \begin{cases} (\mathbf{E}|X|^p)^{1/p} & \text{if } 1 \le p < +\infty, \\ \inf\{M \text{ s.t. } \mathbf{P}(|X| \le M) = 1\} & \text{if } p = \infty. \end{cases}$$

The L_p norms are rearrangement-invariant (i.e. the norm of a random variable depends only on its distribution) and satisfy the property $\|XY\| = \|X\| \cdot \|Y\|$ whenever X, Y are independent random variables. Note that product of independent random variables correspond to the tensor product in c_{00}. These properties characterize the L_p norms.

Theorem 1.2. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a rich probability space, and let $\mathcal{E}$ be the space of simple random variables. Let $\|\cdot\|$ be a norm on $\mathcal{E}$ with the following properties*:

(1) *If two random variables $X, Y \in \mathcal{E}$ have the same distribution, then $\|X\| = \|Y\|$.*
(2) *If two random variables $X, Y \in \mathcal{E}$ are independent, then $\|XY\| = \|X\| \cdot \|Y\|$.*

Then there exists $p \in [1, +\infty]$ such that $\|X\| = \|X\|_{L_p}$ for every $X \in \mathcal{E}$.

We prove Theorem 1.2 in Sec. 3. We will derive Theorem 1.2 as a consequence of Theorem 1.1. Alternatively one could prove it by mimicking the proof of Theorem 1.1.

1.3. *Connection to previous works*

As we mentioned in the introduction, our main result can be derived from much deeper structural results from Banach space theory, and the point of the present paper is mostly to provide a simple proof to an easy-to-understand notable fact.

A closely related result appears as [9, Theorem 3.1], where the authors use the Banach space machinery to study rearrangement invariant norms on tensor products. Alternatively, one can derive our theorem from a proof of Krivine's theorem (we thank an anonymous referee for pointing out this to us). Krivine's theorem states in particular that for any norm $\|\cdot\|$ on c_{00} which is permutation-invariant and unconditional, there is a $p \in [1,\infty]$ such that for any $\varepsilon > 0$ and $n \in \mathbf{N}$, there are n vectors $x_1, \ldots, x_n \in c_{00}$ with disjoint supports such that, for any scalars $(\alpha_1, \ldots, \alpha_n)$,

$$\|(\alpha_1, \ldots, \alpha_n)\|_p \leq \left\| \sum_{k=1}^{n} \alpha_k x_k \right\| \leq (1+\varepsilon)\|(\alpha_1, \ldots, \alpha_n)\|_p.$$

Moreover, the vectors (x_k) can be chosen to be equal up to permutation (this is clear if we follow the proof in [14, Sec. 12.3.1]). The multiplicativity property implies then that

$$\|(\alpha_1, \ldots, \alpha_n)\|_p \|x_1\| \leq \|(\alpha_1, \ldots, \alpha_n)\| \leq (1+\varepsilon)\|(\alpha_1, \ldots, \alpha_n)\|_p \|x_1\|.$$

It follows easily that $\|\cdot\|$ is equal to the ℓ_p norm.

2. The Case of ℓ_p Norms: Proof of Theorem 1.1

Let $\|\cdot\|$ be a norm on c_{00} which is permutation-invariant and multiplicative.

Step 1. We first show that the norm of an element of c_{00} depends only on the absolute values of its coordinates.

Lemma 2.1. *A norm on c_{00} which is permutation-invariant and multiplicative is also* unconditional: *if $x, y \in c_{00}$ have coordinates with equal absolute values ($|x_i| = |y_i|$ for every i), then $\|x\| = \|y\|$. As a consequence, if $a, b \in c_{00}$ and $0 \leq a \leq b$ (coordinate-wise), then $\|a\| \leq \|b\|$.*

Proof. If x, y have coordinates with equal absolute values, then the vectors $x \otimes (1,-1)$ and $y \otimes (1,-1)$ are equal up to permutation of their coordinates. Permutation-invariance and multiplicativity imply that $\|x\| = \|y\|$. For the second part of the lemma, note that $0 \leq a \leq b$ implies that a belongs to the convex hull of the vectors $\{(\varepsilon_i b_i); \varepsilon_i = \pm 1\}$ and use the triangle inequality to conclude. □

Remark 2.2. In the literature, unconditional and permutation-invariant norms are sometimes called *symmetric norms*.

Step 2. We now focus on sequences whose nonzero coefficients are equal to 1. We write $\mathbf{1}^n$ for the sequence formed with n 1's followed by infinitely many zeros and we put $u_n = \|\mathbf{1}^n\|$. By Lemma 2.1, the sequence $(u_n)_n$ is nondecreasing. Moreover, the multiplicativity property of the norm implies that the sequence $(u_n)_n$ itself is multiplicative: $u_{kn} = u_k u_n$. It is folklore that a nonzero nondecreasing sequence $(u_n)_n$ such that $u_{kn} = u_k u_n$ must equal $(n^\alpha)_n$ for some $\alpha \geq 0$ (for a proof see [10] or [1, Corollary 0.4.17]). We set $p = 1/\alpha$ ($p = +\infty$ if $\alpha = 0$). By the triangle inequality, $u_{n+k} \leq u_n + u_k$, which implies that $p \geq 1$. At this point we have proved that

$$\|\mathbf{1}^n\| = n^{1/p}.$$

To prove Theorem 1.1, we need to show that $\|x\| = \|x\|_p$ for every $x \in c_{00}$. The case $p = +\infty$ is easily handled, so we may assume that $1 \leq p < +\infty$. By Lemma 2.1, without loss of generality, we may also assume that the coordinates of x are nonnegative and in nonincreasing order. Let k be the number of nonzero coordinates of x; then $x_i = 0$ for $i > k$. We will separately show the inequalities $\|x\| \geq \|x\|_p$ and $\|x\| \leq \|x\|_p$. In both cases, we compare $x^{\otimes n}$ with simpler vectors and apply Cramér's theorem (which we now review) to estimate the number of "large" coordinates of $x^{\otimes n}$ when n goes to infinity.

Cramér's theorem. Fix $x \in c_{00}$ with nonnegative nonincreasing coordinates, and let k be the number of nonzero coordinates of x. For $a > 0$, let $N(x,a)$ be the number of coordinates of x which are larger than or equal to a. To estimate this number, we introduce the convex function $\Lambda_x : \mathbf{R} \to \mathbf{R}$

$$\Lambda_x(\lambda) = \ln\left(\sum_{i=1}^{k} x_i^\lambda\right)$$

and its convex conjugate $\Lambda_x^* : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$

$$\Lambda_x^*(t) = \sup_{\lambda \in \mathbf{R}} \lambda t - \Lambda_x(\lambda).$$

The Fenchel–Moreau theorem (see e.g. [7]) implies that convex conjugation is an involution: we have, for any $\lambda \in \mathbf{R}$,

$$\Lambda_x(\lambda) = \sup_{t \in \mathbf{R}} \lambda t - \Lambda_x^*(t).$$

Proposition 2.3. (Cramér's large deviation theorem) *Let $x \in c_{00}$ such that $x_i > 0$ for $1 \leq i \leq k$ and $x_i = 0$ for $i > k$. Let t be a real number such that* $\exp(t) \leq \|x\|_\infty$. *Then,*

$$\lim_{n\to\infty} \frac{1}{n} \ln N(x^{\otimes n}, \exp(tn)) = \left\{ \begin{matrix} \ln k & \text{if } \exp(t) \leq \left(\prod_{i=1}^k x_i\right)^{1/k} \\ -\Lambda_x^*(t) & \text{otherwise} \end{matrix} \right\} \geq -\Lambda_x^*(t).$$

Proof. To see how Proposition 2.3 follows from the standard formulation of Cramér's theorem, let (X_n) be independent random variables with common distribution given by

$$\frac{1}{k}\sum_{i=1}^{k}\delta_{\ln x_i}.$$

Then $\mathbf{P}(\frac{1}{n}(X_1+\cdots+X_n)\geq t)=\frac{1}{k^n}N(x^{\otimes n},\exp(tn))$. The usual Cramér theorem (see [8] for a short proof) asserts that

$$\lim_{n\to\infty}\frac{1}{n}\ln\mathbf{P}\left(\frac{1}{n}(X_1+\cdots+X_n)\geq t\right)=\begin{cases}0 & \text{if } t\leq\mathbf{E}X_1,\\ -\sup_{\lambda\in\mathbf{R}}(\lambda t-\ln\mathbf{E}e^{\lambda X_1}) & \text{otherwise.}\end{cases}$$

This is equivalent to the equality in Proposition 2.3. The last inequality follows easily since $\Lambda_x^*(t)\geq-\ln k$ for every real t. □

We now complete the proof of the main theorem by comparing $x^{\otimes n}$ with simpler vectors, as shown in Fig. 1.

Step 3. *The lower bound* $\|x\|\geq\|x\|_p$ For $t\in\mathbf{R}$, we have the lower bound

$$\|x\|=\|x^{\otimes n}\|^{1/n}\geq\|\exp(tn)\mathbf{1}^{N(x^{\otimes n},\exp(tn))}\|^{1/n}=\exp(t)N(x^{\otimes n},\exp(tn))^{1/np}.$$

Proposition 2.3 asserts that

$$\lim_{n\to\infty}N(x^{\otimes n},\exp(tn))^{1/n}\geq\exp(-\Lambda_x^*(t)).$$

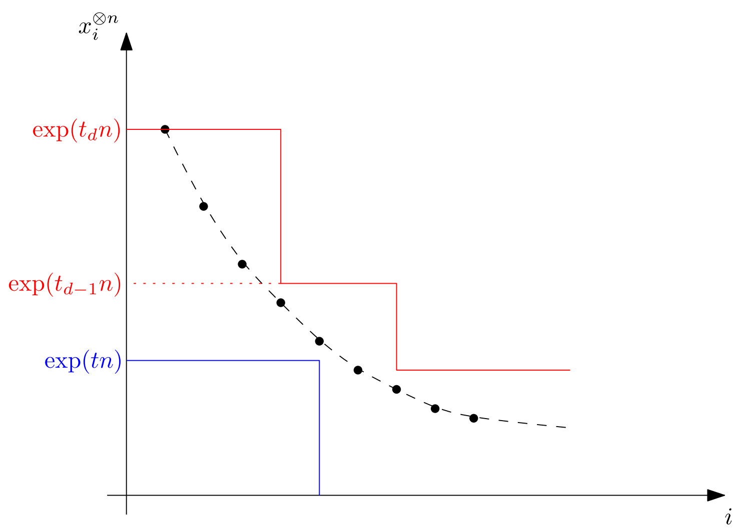

Fig. 1. Bounding the vector $x^{\otimes n}$ by vectors with simpler profiles. The coordinates of the tensor power $x^{\otimes n}$ are represented by dark circles, the vector used in for the lower bound has only one nonzero value $\exp(tn)$ and the upper-bounding vector has values $\exp(t_d n)\geq\cdots\geq\exp(t_1 n)\geq 0$.

We have therefore

$$\|x\| \geq \exp(t - \Lambda_x^*(t)/p) = \exp(pt - \Lambda_x^*(t))^{1/p}$$

for any $t \in \mathbf{R}$. Taking the supremum over t and using the Fenchel–Moreau theorem shows that

$$\|x\| \geq \exp(\Lambda_x(p))^{1/p} = \|x\|_p.$$

Step 4. *The upper bound* $\|x\| \leq \|x\|_p$. Fix $\varepsilon > 0$ and choose $t_0 < \cdots < t_d$ such that

$$\exp(t_0) = \min_{1 \leq i \leq k} x_k, \quad \exp(t_1) = \left(\prod_{i=1}^{k} x_i\right)^{1/k},$$

$$\exp(t_d) = \|x\|_\infty \quad \text{and} \quad \sup_{2 \leq i \leq d} |t_i - t_{i-1}| < \varepsilon.$$

For $n \in \mathbf{N}^*$, we define a vector $y_n \in c_{00}$ as follows: the coordinates of y_n belong to the set

$$\{0, \exp(nt_1), \exp(nt_2), \ldots, \exp(nt_d)\}$$

and are minimal possible such that the inequality $x^{\otimes n} \leq y_n$ holds coordinatewise. Lemma 2.1 implies that $\|x^{\otimes n}\| \leq \|y_n\|$. On the other hand, for $1 \leq i \leq d$, the number of coordinates of y_n which are equal to $\exp(nt_i)$ is less than $N(x^{\otimes n}, \exp(nt_{i-1}))$. The triangle inequality implies that

$$\begin{aligned}
\|y_n\| &\leq \sum_{i=1}^{d} \|\exp(t_i n)\mathbf{1}^{N(x^{\otimes n}, \exp(t_{i-1}n))}\| \\
&\leq \sum_{i=1}^{d} \exp(t_i n) N(x^{\otimes n}, \exp(t_{i-1}))^{1/p} \\
&\leq d \max_{1 \leq i \leq d} \{\exp(t_i n) N(x^{\otimes n}, \exp(t_{i-1}n))^{1/p}\}.
\end{aligned}$$

This gives an upper bound for $\|x\|$

$$\|x\| = \|x^{\otimes n}\|^{1/n} \leq \|y_n\|^{1/n} \leq d^{1/n} \max_{1 \leq i \leq d} \{\exp(t_i) N(x^{\otimes n}, \exp(t_{i-1}n))^{1/np}\}. \quad (2.1)$$

For $2 \leq i \leq d$, Proposition 2.3 implies that

$$\begin{aligned}
\lim_{n \to \infty} \exp(t_i) N(x^{\otimes n}, \exp(t_{i-1}n))^{1/np} &= \exp(t_i) \exp(-\Lambda_x^*(t_{i-1}))^{1/p} \\
&\leq \exp(t_i) \exp(-(pt_{i-1} - \Lambda_x(p)))^{1/p} \\
&\leq \exp(\varepsilon) \|x\|_p.
\end{aligned}$$

Similarly, for $i = 1$,

$$\exp(t_1) N(x^{\otimes n}, \exp(t_0 n))^{1/np} \leq \exp(t_1) k^{1/p} \leq \|x\|_p,$$

where the last inequality follows from the inequality of arithmetic and geometric means. Therefore, taking the limit $n \to \infty$ in inequality (2.1) implies that $\|x\| \leq \exp(\varepsilon)\|x\|_p$, and the result follows when ε goes to 0.

3. The Case of L_p Norms: Proof of Theorem 1.2

Let $\|\cdot\|$ be a norm on the space $\mathcal{E}$ of simple random variables which satisfies the hypotheses of Theorem 1.2. Throughout the proof, we denote by $B_n \in \mathcal{E}$ a Bernoulli random variable with parameter $1/n$, i.e. such that $\mathbf{P}(B_n = 1) = 1/n$ and $\mathbf{P}(B_n = 0) = 1 - 1/n$. Moreover, we assume that the random variables $(B_n)_{n\in\mathbf{N}}$ are independent.

We will define a norm $|||\cdot|||$ on c_{00} which will satisfy the hypotheses of Theorem 1.1. It is convenient to identify c_{00} with the union of an increasing sequence of subspaces

$$c_{00} = \bigcup_{n\in\mathbf{N}} \mathbf{R}^n. \tag{3.1}$$

For $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, we define

$$|||x||| = \frac{\|X\|}{\|B_n\|},$$

where $X \in \mathcal{E}$ is a random variable with distribution $\frac{1}{n}(\delta_{x_1} + \cdots + \delta_{x_n})$.

This defines a norm on c_{00} provided the construction is compatible with the union (3.1). To check this, consider x as an element of $\mathbf{R}^m$ for $m > n$, obtained by padding x with $m - n$ zeros. Let X' be a random variable with distribution $\frac{1}{m}(\delta_{x_1} + \cdots + \delta_{x_n} + (m-n)\delta_0)$. If we moreover assume that the random variables X, X', B_n, B_m are independent, it is easily checked that XB_m and $X'B_n$ both have the distribution $\frac{1}{nm}(\delta_{x_1} + \cdots + \delta_{x_n}) + (1 - \frac{1}{nm})\delta_0$. By the hypotheses on the norm, this implies that $\|X\| \cdot \|B_m\| = \|X'\| \cdot \|B_n\|$ and therefore

$$\frac{\|X\|}{\|B_n\|} = \frac{\|X'\|}{\|B_m\|}.$$

This shows that $|||x|||$ is properly defined for $x \in c_{00}$. It is easily checked that $|||\cdot|||$ is a norm on c_{00} which is both permutation-invariant and multiplicative (for the latter, use the fact that $B_n B_m$ and B_{nm} have the same distribution).

By Theorem 1.1, the norm $|||\cdot|||$ equals the norm of ℓ_p for some $p \in [1, +\infty]$. To compute $\|B_n\|$, consider the vector $x \in \mathbf{R}^{2n}$ given by n 1's followed by n 0's. We have

$$n^{1/p} = \|x\|_p = |||x||| = \frac{\|B_2\|}{\|B_{2n}\|} = \frac{1}{\|B_n\|},$$

where the last equality follows from the aforementioned property of Bernoulli random variables. This implies that the equation

$$\|X\| = \|X\|_{L_p}. \tag{3.2}$$

Holds for every $X \in \mathcal{E}$ with rational weights, i.e. with distribution $\frac{1}{n}(\delta_{x_1} + \cdots + \delta_{x_n})$ for some n. The extension to all random variables in $\mathcal{E}$ follows by an approximation argument. Indeed, for every positive random variable $X \in \mathcal{E}$, there exist sequences $(Y_n), (Z_n)$ of positive random variables, with rational weights, such that

$$Y_n \le X \le Z_n$$

and

$$\lim_{n\to\infty} \|Y_n\|_{L_p} = \lim_{n\to\infty} \|Z_n\|_{L_p} = \|X\|_{L_p}.$$

Therefore, we may use the following lemma (a continuous version of Lemma 2.1) to extend formula (3.2) to every $X \in \mathcal{E}$.

Lemma 3.1. *Let $\|\cdot\|$ be a norm on $\mathcal{E}$ which satisfies the hypotheses of Theorem* 1.2. *If $X \in \mathcal{E}$, then the random variables X and $|X|$ have the same norm. If $X, Y \in \mathcal{E}$ are two random variables such that $0 \le X \le Y$, then $\|X\| \le \|Y\|$.*

Proof. To prove the first part, note that if ε is a random variable which is independent from X and such that $\mathbf{P}(\varepsilon = 1) = \mathbf{P}(\varepsilon = -1) = 1/2$, then εX and $\varepsilon|X|$ are identically distributed. Assume now that $0 \le X \le Y$. There exists a finite measurable partition $(\Omega_1, \ldots, \Omega_n)$ of Ω such that X and Y are constant on each set Ω_i. Let x_i (respectively, y_i) be the value of X (respectively, Y) on Ω_i; then $x_i \le y_i$. For any $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{\pm 1\}^n$, one may define a random variable Z_ε by setting $Z_\varepsilon(\omega) = \varepsilon_i$ for $\omega \in \Omega_i$. The random variable X can be written as a convex combination of the random variables $\{Z_\varepsilon Y\}_{\varepsilon \in \{\pm 1\}^n}$ (this is a consequence of the fact that $(x_1, \ldots, x_n)$ is in the convex hull of $(\pm y_1, \ldots, \pm y_n)$ — a fact already used in the proof of Lemma 2.1). We now conclude by the triangle inequality and the fact that $\|Z_\varepsilon Y\| = \|Y\|$ since both variables are equal in absolute value. □

4. Extensions

4.1. *Extension to the complex case*

Theorems 1.1 and 1.2 extend easily to the complex case. We only state the discrete version. Up to a small detail, the proof is the same as in the real case.

Theorem 4.1. *Let $\|\cdot\|$ be a permutation-invariant and multiplicative norm on the space of finitely supported complex sequences. Then, there exists some $p \in [1, +\infty]$ such that $\|\cdot\| = \|\cdot\|_p$.*

Proof. We argue in the same way as we did for real sequences. The proof adapts *mutatis mutandis*, except for the first part of Lemma 2.1 whose proof requires a slight modification. Let ω be a primitive kth root of unity. If the coordinates of x and y differ only by a power of ω, then the vectors $x \otimes (1, \omega, \ldots, \omega^{k-1})$ and $y \otimes (1, \omega, \ldots, \omega^{k-1})$ are equal up to permutation of coordinates, and therefore $\|x\| = \|y\|$. The case of a general complex phase follows by continuity. □

4.2. *Noncommutative setting*

Theorem 1.1 can be formulated to characterize the Schatten p-norms.

Let H be an infinite-dimensional (real or complex) separable Hilbert space and $F(H)$ be the space of finite rank operators on H. Let $\|\cdot\|$ be a norm on $F(H)$ which is *unitarily invariant*: whenever U, V are unitary operators on H and $A \in F(H)$, we have $\|UAV\| = \|A\|$. Assume also that the norm is *multiplicative* in the following sense: for any $A, B \in F(H)$,

$$\|A \otimes B\| = \|A\| \cdot \|B\|.$$

As in the commutative case, we fix a isometry between H and the Hilbertian tensor product $H \otimes H$ to define $\|A \otimes B\|$ — the particular choice we make is irrelevant because of the unitary invariance. The next theorem asserts that the only norms which are unitarily invariant and multiplicative are the Schatten p-norms defined as $\|A\|_p = (\mathrm{tr}|A|^p)^{1/p}$ for $1 \leq p < +\infty$, while $p = \infty$ corresponds to the operator norm.

Theorem 4.2. *Let $\|\cdot\|$ be a norm on the space of finite-rank operators on an infinite-dimensional Hilbert space which is both multiplicative and unitarily invariant. Then, there exists some $p \in [1, +\infty]$ such that $\|\cdot\|$ is the Schatten p-norm.*

Proof. By a result of von Neumann (see [5, Theorem IV.2.1]), a norm N on $F(H)$ is unitarily invariant if and only if $N(\cdot) = \|s(\cdot)\|$ for some symmetric norm $\|\cdot\|$ on c_{00} — here $s(A) \in c_{00}$ denotes the list of singular values of an operator $A \in F(H)$. The result follows then from the commutative case (Theorem 1.1 or Theorem 4.1). □

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References

1. J. Aczél and Z. Daróczy, *On Measures of Information and their Characterizations,* Mathematics in Science and Engineering, Vol. 115 (Academic Press, 1975), xii+234 pp.
2. D. Alspach and E. Odell, L_p spaces, in *Handbook of the Geometry of Banach Spaces*, Vol. I (North-Holland, 2001), pp. 123–159.
3. G. Aubrun and I. Nechita, Catalytic majorization and l_p norms, *Comm. Math. Phys.* **278**(1) (2008) 133–144.
4. G. Aubrun and I. Nechita, Stochastic domination for iterated convolutions and catalytic majorization, *Ann. Inst. H. Poincaré Probab. Statist.* **45** (2009) 611–625.
5. R. Bhatia, *Matrix Analysis*, Graduate Texts in Mathematics, Vol. 169 (Springer-Verlag, 1997).
6. F. Bohnenblust, An axiomatic characterization of L_p-spaces, *Duke Math. J.* **6** (1940) 627–640.
7. H. Brézis, *Analyse Fonctionnelle*: *Théorie et Applications* (Masson, 1983) (in French).
8. R. Cerf and P. Petit, A short proof of Cramér's theorem, to appear in *Amer. Math. Monthly* (2011).
9. C. Fernández-González, C. Palazuelos and D. Pérez-García, The natural rearrangement invariant structure on tensor products, *J. Math. Anal. Appl.* **343** (2008) 40–47.

10. E. Howe, A new proof of Erdős' theorem on monotone multiplicative functions, *Amer. Math. Monthly* **93** (1986) 593–595.
11. J.-L. Krivine, Sous-espaces de dimension finie des espaces de Banach réticulés, *Ann. of Math. (2)* **104** (1976) 1–29.
12. G. Kuperberg, The capacity of hybrid quantum memory, *IEEE Trans. Inform. Th.* **49** (2003) 1465–1473.
13. T. Leinster, A multiplicative characterization of the power means, to appear in *Bull. London Math. Soc.* (2011).
14. V. D. Milman and G. Schechtman, *Asymptotic Theory of Finite-Dimensional Normed Spaces. With an Appendix by M. Gromov*, Lecture Notes in Mathematics, Vol. 1200 (Springer-Verlag, 1986).