A naive look at Schur–Weyl duality

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The goal of this note is to provide an elementary proof of the (most elementary version of) Schur–Weyl duality, without using anything about representation theory.

1 Bicommutant theorem

If V is a finite-dimensional vector space over $\mathbf{C}$, we denote the *commutant* of a subset $\mathscr{C} \subset \mathsf{End}(V)$ by

$$\mathscr{C}' = \{S \in \mathsf{End}(V) \;:\; \forall T \in \mathscr{C},\, ST = TS\}.$$

Note that $\mathscr{C}'$ is a sub-algbera[1] of $\mathsf{End}(V)$, and also that $\mathscr{C} \subset \mathscr{C}''$. There are several different results known as *bicommutant* or *double centralizer* theorems which give sufficient conditions on a sub-algebra $\mathscr{A} \subset \mathsf{End}(V)$ to ensure that $\mathscr{A} = \mathscr{A}''$: this holds if $\mathscr{A}$ is «semi-simple», or if $\mathscr{A}$ is generated by a single operator, or if $\mathscr{A}$ is a sub-$*$-algebra (i.e. such that $A \in \mathscr{A} \implies A^\dagger \in \mathscr{A}$). For completeness, here is an example with $\mathscr{A} \subsetneq \mathscr{A}''$.

Example 1. *Consider the algebra*

$$\mathscr{A} = \left\{ \begin{pmatrix} \lambda & x & y \\ 0 & \lambda & z \\ 0 & 0 & \lambda \end{pmatrix} \;:\; \lambda,\, x,\, y,\, z \in \mathbf{C} \right\} \subset \mathsf{End}(\mathbf{C}^3).$$

One checks that $\mathscr{A}'$ is the algebra generated by $|e_1\rangle\langle e_3|$, and therefore that $|e_2\rangle\langle e_2| \in \mathscr{A}'' \backslash \mathscr{A}$.

Theorem 1 is the finite-dimensional version of von Neumann's bicommutant theorem, which plays an important role in the study of von Neumann algebras.

Theorem 1 (Bicommutant theorem for $*$-algebras)**.** *Let $\mathscr{A} \subset \mathsf{End}(\mathbf{C}^n)$ be a sub-$*$-algebra (containing* Id*). Then $\mathscr{A}'' = \mathscr{A}$.*

Proof. Consider $\mathscr{B} := \mathscr{A} \otimes \mathrm{Id}_n \subset \mathsf{End}(\mathbf{C}^n \otimes \mathbf{C}^n)$. One checks that its commutant and bicommutant are $\mathscr{B}' = \mathscr{A}' \otimes \mathsf{End}(\mathbf{C}^n)$ and $\mathscr{B}'' = \mathscr{A}'' \otimes \mathrm{Id}_n$. Consider a tensor $\psi \in \mathbf{C}^n \otimes \mathbf{C}^n$ with full rank (e.g. maximally entangled), the subspace $E = \mathscr{B}\psi \subset \mathbf{C}^n \otimes \mathbf{C}^n$ and P_E the orthogonal projection onto E. Since E and $E^\perp$ are $\mathscr{B}$-invariant, we have $P_E \in \mathscr{B}'$.

[1] all sub-algebras are assumed to contain Id

Consider now any element $X \in \mathscr{A}''$. Since $(X \otimes \mathrm{Id}) \in \mathscr{B}''$, we have $P_E(X \otimes \mathrm{Id}) = (X \otimes \mathrm{Id})P_E$, and in particular $(X \otimes \mathrm{Id})\psi = (X \otimes \mathrm{Id})P_E\psi = P_E(X \otimes \mathrm{Id})\psi$, so that $(X \otimes \mathrm{Id})\psi \in E$, which means $(X \otimes \mathrm{Id})\psi = (Y \otimes \mathrm{Id})\psi$ for some $Y \in \mathscr{A}$. Since the map $A \mapsto (A \otimes \mathrm{Id})\psi$ is bijective, it follows that $X \in \mathscr{A}$. □

2 Schur–Weyl duality

This is the simplest version of Schur–Weyl duality.

Theorem 2 (Schur–Weyl duality). *Let n, k be positive integers, and consider the following subalgebras of $\mathsf{End}((\mathbf{C}^n)^{\otimes k})$*

- $\mathscr{A} := \mathrm{span}\{A^{\otimes k} \ : \ A \in \mathsf{End}(\mathbf{C}^n)\}$,
- $\mathscr{B} := \mathrm{span}\{V_\pi \ : \ \pi \in \mathfrak{S}_k\}$, *where V_π is defined by the formula*

$$V_\pi(x_1 \otimes \cdots \otimes x_k) = x_{\pi(1)} \otimes \cdots \otimes x_{\pi(k)}.$$

Then $\mathscr{A}$ and $\mathscr{B}$ are equal to the commutant of each other.

Proof. Since $V_\pi^\dagger = V_{\pi^{-1}}$, $\mathscr{B}$ is a sub-$*$-algebra and Theorem 1 applies. In particular, it suffices to prove that $\mathscr{A} = \mathscr{B}'$ and the identity $\mathscr{A}' = \mathscr{B}'' = \mathscr{B}$ follows.

The inclusion $\mathscr{B} \subset \mathscr{A}'$ is obvious. In order to prove $\mathscr{B}' \subset \mathscr{A}$, consider $X \in \mathscr{B}'$. In particular we have $X = \frac{1}{k!}\sum_{\pi\in\mathfrak{S}_k} V_\pi X V_{\pi^{-1}}$. Since $\mathsf{End}((\mathbf{C}^n)^{\otimes k})$ is generated as a vector space by elements of the form $X_1 \otimes \cdots \otimes X_k$ with $X_i \in \mathsf{End}(\mathbf{C}^n)$, it suffices to show that for any such k-tuple,

$$\sum_{\pi\in\mathfrak{S}_k} V_\pi(X_1 \otimes \cdots \otimes X_k)V_{\pi^{-1}} \in \mathscr{A}.$$

This in turn is a consequence of the identity

$$\begin{aligned}\sum_{\pi\in\mathfrak{S}_k} V_\pi(X_1 \otimes \cdots \otimes X_k)V_{\pi^{-1}} &= \sum_{\pi\in\mathfrak{S}_k} X_{\pi(1)} \otimes \cdots \otimes X_{\pi(k)} \\ &= \mathbf{E}\left[\left(\prod_{i=1}^k \varepsilon_i\right)\left(\sum_{j=1}^k \varepsilon_j X_j\right)^{\otimes k}\right],\end{aligned}$$

where (ε_i) are independent unbiased ± 1 random variables. (To prove the last equality, expand the right-hand side and use independence). □

3 Schur–Weyl duality for the unitary group

A more sophisticated version, often used in quantum information theory, is exactly similar to Theorem 2, but with $\mathsf{End}(\mathbf{C}^n)$ replaced by $\mathsf{U}(n)$.

Corollary 3 (Schur–Weyl duality, unitary group)**.** *Let n, k be positive integers, and consider the following subalgebras of* $\mathsf{End}((\mathbf{C}^n)^{\otimes k})$

- $\mathscr{C} := \operatorname{span}\{U^{\otimes k} \ : \ U \in \mathsf{U}(n)\}$,
- $\mathscr{B} := \operatorname{span}\{V_\pi \ : \ \pi \in \mathfrak{S}_k\}$.

Then $\mathscr{C}$ and $\mathscr{B}$ are equal to the commutant of each other.

Proof. With Theorem 2 already known, it suffices to show that $\mathscr{A} \subset \mathscr{C}$, the reverse inclusion being obvious. Let $A \in \mathsf{End}(\mathbf{C}^n)$; we show that $A^{\otimes k} \in \mathscr{C}$ by producing an explicit decomposition as a linear combination of unitary tensor powers. Without loss of generality, assume $\|A\|_{\mathrm{op}} < 1$. Consider the singular value decomposition

$$A = \sum_{i=1}^{n} s_i |e_i\rangle\langle f_i|$$

with (e_i), (f_i) orthonormal bases and $s_i \in [0,1)$. Denote by $\mathbf{T} \subset \mathbf{C}$ the unit circle; for any $(z_1, \dots, z_n) \in \mathbf{T}^n$, consider the unitary matrix

$$U_{z_1,\dots,z_n} = \sum_{i=1}^{n} z_i |e_i\rangle\langle f_i|.$$

We use Cauchy's formula from complex analysis: whenever $|s| < 1$, we have (contour integral)

$$\frac{1}{2i\pi} \int_{\mathbf{T}} z^k \, \frac{\mathrm{d}z}{z-s} = s^k$$

for any $k \in \mathbf{N}$. Using Fubini's theorem, we obtain as a consequence a multivariate version: whenever $|s_i| < 1$, for any choice of indices $i_1, \dots, i_k \in \{1, \dots, n\}$,

$$\frac{1}{(2i\pi)^n} \int_{\mathbf{T}^n} \left(\prod_{j=1}^{k} z_{i_j} \right) \frac{\mathrm{d}z_1}{z_1 - s_1} \cdots \frac{\mathrm{d}z_n}{z_n - s_n} = \prod_{j=1}^{k} s_{i_j}.$$

It remains to compute

$$
\begin{aligned}
&\frac{1}{(2i\pi)^n}\int_{\mathbf{T}^n} U^{\otimes k}_{z_1,\dots,z_n}\,\frac{\mathrm{d}z_1}{z_1-s_1}\cdots\frac{\mathrm{d}z_n}{z_n-s_n}\\
&=\sum_{i_1,\dots,i_k=1}^{n}\frac{1}{(2i\pi)^n}\int_{\mathbf{T}^n} z_{i_1}\dots z_{i_k}\,\frac{\mathrm{d}z_1}{z_1-s_1}\cdots\frac{\mathrm{d}z_n}{z_n-s_n}\,|e_{i_1}\otimes\cdots\otimes e_{i_k}\rangle\langle f_{i_1}\otimes\cdots\otimes f_{i_k}|\\
&=\sum_{i_1,\dots,i_k=1}^{n} s_{i_1}\dots s_{i_k}|e_{i_1}\otimes\cdots\otimes e_{i_k}\rangle\langle f_{i_1}\otimes\cdots\otimes f_{i_k}|\\
&=\left(\sum_{i=1}^{n} s_i|e_i\rangle\langle f_i|\right)^{\otimes k}=A^{\otimes k}.
\end{aligned}
$$

This show that $A^{\otimes k}\in\mathscr{C}$. $\square$