L^p SELF-IMPROVEMENT OF GENERALIZED POINCARÉ INEQUALITIES IN SPACES OF HOMOGENEOUS TYPE

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Abstract. In this paper we study self-improving properties in the scale of Lebesgue spaces of generalized Poincaré inequalities in spaces of homogeneous type. In contrast with the classical situation, the oscillations involve approximation of the identities or semigroups whose kernels decay fast enough and the resulting estimates take into account their lack of localization. The techniques used do not involve any classical Poincaré or Sobolev-Poincaré inequalities and therefore they can be used in general settings where these estimates do not hold or are unknown. We apply our results to the case of Riemannian manifolds with doubling volume form and assuming Gaussian upper bounds for the heat kernel of the semigroup $e^{-t\,\Delta}$ with Δ being the Laplace-Beltrami operator. We obtain generalized Poincaré inequalities with oscillations that involve the semigroup $e^{-t\,\Delta}$ and with right hand sides containing either ∇ or $\Delta^{1/2}$.

1. Introduction

In analysis and PDEs we can find various estimates that encode self-improving properties of the integrability of the functions involved. For instance, the John-Nirenberg inequality establishes that a function in BMO, which a priory is in $L^1_{\rm loc}(\mathbb{R}^n)$, is indeed exponentially integrable which in turn implies that is in $L^p_{\rm loc}(\mathbb{R}^n)$ for any $1 \le p < \infty$. Another situation where functions self-improve their integrability comes from the classical (p,p)-Poincaré inequality in $\mathbb{R}^n$, $n \ge 2$, $1 \le p < n$,

$$\fint_Q |f - f_Q|^p \, dx \le C\, \ell(Q) \fint_Q |\nabla f|^p \, dx.$$

It is well-known that this estimate yields that for any function $f \in L^p_{\rm loc}(\mathbb{R}^n)$ with $\nabla f \in L^p_{\rm loc}(\mathbb{R}^n)$,

$$\left(\fint_Q |f - f_Q|^{p^*} \, dx\right)^{1/p^*} \le C\, \ell(Q) \left(\fint_Q |\nabla f|^p \, dx\right)^{1/p}$$

where $p^* = \frac{p\,n}{n-p}$. Again f gains integrability properties, since the previous inequality gives $f \in L^{p^*}_{\rm loc}(\mathbb{R}^n)$. Both situations have something in common: they involve the

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oscillation of the functions on some cube Q via $f - f_Q$. In [FPW], general versions of these estimates are considered. They start with inequalities of the form

$$\fint_Q |f - f_Q|\, dx \le a(Q, f), \tag{1.1}$$

where a is a functional depending on the cube Q, and sometimes on the function f. There, the authors present a general method based on the Calderón-Zygmund theory and the good-λ inequalities introduced by Burkholder and Gundy [BG] that allows them to establish that under mild geometric conditions on the functional a, inequality (1.1) encodes an intrinsic self-improvement on L^p type for $p > 1$.

On the other hand, in [Ma1] a new sharp maximal operator associated with an approximation of the identity $\{S_t\}_{t>0}$ is introduced:

$$M_S^{\#} f(x) = \sup_{Q \ni x} \fint_Q |f - S_{t_Q} f|\, dy,$$

where t_Q is a parameter depending on the side-length of the cube Q. This operator allows one to define the space BMO_S, for which the John-Nirenberg inequality also holds (see [DY]). In this way, starting with an estimate as (1.1) where the oscillation $f - f_Q$ is replaced by $f - S_{t_Q} f$, and $a(Q, f) = C$ a self-improving property is obtained. This new way of measuring the oscillation allows one to define new function spaces as the just mentioned BMO_S of [DY] and the Morrey-Campanato associated with an approximation of the identity of [DDY], [Tan].

In [Jim] and [JM] self-improving properties related to this new way of measuring oscillation are under study. The starting estimate is as follows

$$\fint_Q |f - S_{t_Q} f|\, dx \le a(Q, f), \tag{1.2}$$

with S_t being a family of operators (e.g., semigroup) with fast decay kernel. By analogy to (1.1), we will refer to these estimates as generalized Poincaré inequalities. The case a increasing, considered in [Jim] both in the Euclidean setting an also in spaces of homogeneous type, yields local exponential integrability of the new oscillation $f - S_t f$. In [JM] functionals satisfying a weaker ℓ^r-summability condition (see D_r below) are studied in the Euclidean setting. In this case $L^{r,\infty}$ local integrability of the oscillation is obtained.

In this paper we continue the study in [Jim], [JM] considering (1.1) in the setting of the spaces of homogeneous type for functionals satisfying some summability conditions. We obtain estimates in weak Lebesgue spaces with the oscillation $f - S_t f$ in the left hand side and an expansion of a over dilations of balls on the right hand side. These expansions, that already appeared in [Jim], [JM], have fast decay coefficients and are natural due to the lack of localization of the operators S_t. The proofs are more technically involved since the setting is less friendly. However, we are able to obtain applications in settings where one may lack of Poincaré inequalities. That is the case of some Riemannian manifolds assuming only doubling volume form and Gaussian upper bounds for the heat kernel associated to the semigroup generated by the Laplace-Beltrami operator. In order to present these applications, which are the main motivation of the general results presented here, we need to introduce some notation, see Section 4.3 for more details.

Let M be a complete non-compact connected Riemannian manifold with d its geodesic distance. Assume that volume form μ is doubling and let n be its doubling order (see (2.1) below). Then M equipped with the geodesic distance and the volume form μ is a space of homogeneous type. Let Δ be the positive Laplace-Beltrami operator on M given by

$$\langle \Delta f, g\rangle = \int_M \nabla f \cdot \nabla g \, d\mu$$

where ∇ is the Riemannian gradient on M and $\cdot$ is an inner product on TM. We assume that the heat kernel $p_t(x,y)$ of the semigroup $e^{-t\Delta}$ has Gaussian upper bounds if for some constants $c, C > 0$ and all $t > 0, x, y \in M$,

$$p_t(x,y) \le \frac{C}{\mu(B(x,\sqrt{t}))} \, e^{-c\frac{d^2(x,y)}{t}}. \tag{UE}$$

We define $\tilde{q}_+$ as the supremum of those $p \in (1,\infty)$ such that for all $t > 0$,

$$\big\| \, |\nabla e^{-t\Delta} f| \, \big\|_{L^p} \le C\, t^{-1/2} \|f\|_{L^p}. \tag{G_p}$$

If the Riesz transform $|\nabla \Delta^{-1/2}|$ is bounded in L^p, by analyticity of the heat semigroup, then (G_p) holds. Therefore, $\tilde{q}_+$ is greater than the supremum on the exponents p for which the Riesz transform is bounded on L^p. In particular $q_+ \ge 2$ by [CD].

As a consequence of our main results and in the absence of Poincaré inequalities we obtain the following (see Corollary 4.5 below for the precise statement):

Theorem 1.1. *Let M be complete non-compact connected Riemannian manifold satisfying the doubling volume property and (UE). Given $1 \le p < \infty$ we set $p^* = n\,p/(n-p)$ if $1 \le p < n$ and $p^* = \infty$ otherwise.*

(a) Given $m \ge 1$ (m is taken large enough when $1 < p < n$), let $S_t^m = I - (I - e^{-t\Delta})^m$ and $1 < q < p^*$. Then, for any smooth function with compact support f we have

$$\left(\fint_B |f - S_{t_B}^m f|^q \, d\mu \right)^{1/q} \le C \sum_{k \ge 1} \phi(k)\, r(\sigma^k B) \left(\fint_{\sigma^k B} |\Delta^{1/2} f|^p \, d\mu \right)^{1/p},$$

where $\phi(k) = \sigma^{-k\theta}$ and θ depends on m, n and p.

(b) For any $p \in ((\tilde{q}_+)', \infty) \cup [2, \infty)$, any $1 < q < p^*$ and any smooth function with compact support f we have

$$\left(\fint_B |f - e^{-t_B\Delta} f|^q \, d\mu \right)^{1/q} \le C \sum_{k \ge 1} e^{-c\sigma^k}\, r(\sigma^k B) \left(\fint_{\sigma^k B} |\nabla f|^p \, d\mu \right)^{1/p}.$$

In this result σ is a large constant depending on the doubling condition (see below).

The plan of the paper is as follows. In Section 2 we give some preliminaries and definitions. The main result and its different extensions are in Section 3. Applications are considered in Section 4. In particular, we devote Sections 4.1 and 4.2 to study various Poincaré type inequalities in general spaces of homogeneous type. In the former we start from an estimate whose right hand side is localized to the given ball B, in the latter we take into account the lack of localization of the approximation of the identity or the semigroup and the right hand side contains a series of terms as in the applications to manifolds stated above. As a consequence, in Section 4.1 we obtain global pseudo-Poincaré inequalities. In Section 4.3 we consider the application above and obtain generalized Poincaré inequalities in Riemannian manifolds. The subsequent sections contain the proofs of our results.

2. Preliminaries

2.1. **Spaces of homogeneous type.** For full details and references we refer the reader to [CW] and [Chr]. Let (X, d, μ) be a space of homogeneous type: X is a set equipped with a quasi-metric d and a non-negative Borel measure μ satisfying the doubling condition

$$\mu\big(B(x, 2\,r)\big) \le c_\mu\, \mu\big(B(x,r)\big) < \infty,$$

for some $c_\mu \ge 1$, uniformly for all $x \in X$ and $r > 0$, and where $B(x,r) = \{y \in X : d(y,x) < r\}$. We note that, in general, different centers and radii can define the same ball. Therefore, given a ball B we implicitly assume that a center and a radius are specified: $B = B(x_B, r(B))$ where x_B is the center and $r(B)$ is the radius. The doubling property implies

$$\mu\big(B(x, \lambda\, r)\big) \le c_\mu\, \lambda^n\, \mu\big(B(x,r)\big) \qquad \text{and} \qquad \frac{\mu(B_2)}{\mu(B_1)} \le c_\mu \left(\frac{r(B_2)}{r(B_1)}\right)^n, \tag{2.1}$$

for some $c_\mu, n > 0$ and for all $x, y \in X$, $r > 0$ and $\lambda \ge 1$, and for all balls B_1 and B_2 with $B_1 \subsetneq B_2$.

Let us recall that d being a quasi-metric on X means that d is a function from $X \times X$ to $[0, +\infty)$ satisfying the same conditions as a metric, except for the triangle inequality that is weakened to

$$d(x,y) \le D_0\, \big(d(x,z) + d(z,y)\big),$$

for all $x, y, z \in X$ and where $1 \le D_0 < \infty$ is a constant independent of x, y, z. Unfortunately, when $D_0 > 1$ it does not follow, in general, that the balls are open. However, Macías and Segovia [MS] proved that given any quasi-metric d, there exists another quasi-metric d' equivalent to d such that the metric balls defined with respect to d' are open. Thus, without loss of generality, from now on we assume that the metric balls are open sets. Also, in order to simply the computations, we assume that X is unbounded and therefore $\mu(X) = \infty$, see for instance [Ma2].

We make some conventions: $A \lesssim B$ means that the ratio A/B is bounded by a constant that does not depend on the relevant variables in A and B. Throughout this paper, the letter C denotes a constant that is independent of the essential variables and that may vary from line to line. Given a ball $B = B(x_B, r(B))$ and $\lambda > 0$, we write $\lambda\, B = B(x_B, \lambda\, r(B))$. For any set E we write $\operatorname{diam}(E) = \sup_{x,y\in E} d(x,y)$. The average of $f \in L^1_{\rm loc}$ in B is denoted by

$$f_B = \fint_B f(x)\, d\mu(x) = \frac{1}{\mu(B)} \int_B f(x)\, d\mu(x)$$

and the localized and normalized norms of a Banach or a quasi-Banach function space $\mathbb{A}$ by

$$\|f\|_{\mathbb{A},B} = \|f\|_{\mathbb{A}(B, \frac{d\mu}{\mu(B)})} \qquad \text{and} \qquad \|f\|_{\mathbb{A}(w),B} = \|f\|_{\mathbb{A}(B, \frac{w}{w(B)})}.$$

Examples of spaces $\mathbb{A}$ are $L^{p,\infty}$, L^p or more general Marcinkiewicz and Orlicz spaces.

2.2. **Dyadic sets.** We take the dyadic structure given in [Chr] (here we use the notation in [Jim]).

Theorem 2.1 ([Chr]). *There exist $\sigma > 4\,D_0^3 > 1$ large enough, $0 < c_1, C_1, C_2 < \infty$ and $\mathcal{D} = \cup_{k\in\mathbb{Z}}\mathcal{D}_k$ a countable collection of open sets Q with the following properties:*

(i) *$\mathcal{D}_k$ is a countable collection of disjoint sets such that $X = \cup_{Q\in\mathcal{D}_k} Q$ μ-a.e.*

(ii) *If $Q \in \mathcal{D}_k$, then* $\operatorname{diam}(Q) \le C_1\,\sigma^k$.

(iii) *If $Q \in \mathcal{D}_k$, then there exist $x_Q \in Q$ and balls $B_Q = B(x_Q, c_1\,\sigma^k)$ and $\hat{B}_Q = B(x_Q, C_1\,\sigma^k)$ such that $B_Q \subset Q \subset \hat{B}_Q$.*

(iv) *If $Q_1 \in \mathcal{D}_{k_1}$ and $Q_2 \in \mathcal{D}_{k_2}$ with $k_1 \le k_2$, then either $Q_1 \cap Q_2 = \emptyset$ or $Q_1 \subseteq Q_2$.*

We will refer to Q as dyadic cubes and to $\mathcal{D}_k$ as the k-th generation of $\mathcal{D}$.

In what follows, we fix $\sigma > 4\,D_0^3$ large enough and consider the dyadic structure given by Theorem 2.1. We will use the following decomposition of X in dyadic annuli: given $Q \in \mathcal{D}$, we write $X = \cup_{k\ge 1} C_k(Q)$ with $C_1(Q) = \sigma\,\hat{B}_Q$ and $C_k(Q) = \sigma^k\,\hat{B}_Q \setminus \sigma^{k-1}\,\hat{B}_Q$, $k \ge 2$. Also, given a ball B, we write $X = \cup_{k\ge 1} C_k(B)$ with $C_1(B) = \sigma\,B$ and $C_k(B) = \sigma^k\,B \setminus \sigma^{k-1}\,B$, $k \ge 2$.

2.3. **Muckenhoupt weights.** A weight w is a non-negative locally integrable function. For any measurable set E, we write $w(E) = \int_E w(x)\,d\mu(x)$. Also, we set

$$\fint_B f\,dw = \fint_B f(x)\,dw(x) = \frac{1}{w(B)}\int_B f(x)\,w(x)\,d\mu(x).$$

We say that a weight $w \in A_p(\mu)$, $1 < p < \infty$, if there exists a positive constant C such that for every ball B

$$\left(\fint_B w\,d\mu\right)\left(\fint_B w^{1-p'}\,d\mu\right)^{p-1} \le C.$$

For $p = 1$, we say that $w \in A_1(\mu)$ if there is a positive constant C such that for every ball B,

$$\fint_B w\,d\mu \le C\,w(y), \qquad \text{for } \mu\text{-a.e. } y \in B.$$

We write $A_\infty(\mu) = \cup_{p\ge 1} A_p(\mu)$. See [ST] for more details and properties.

2.4. **Functionals.** Let $a : \mathcal{B}\times\mathcal{F} \longrightarrow [0, +\infty)$, where $\mathcal{B}$ is the family of all balls in X and $\mathcal{F}$ is some family of functions. When the dependence on the functions is not of our interest, we simply write $a(B)$. We say that a is doubling if there exists some constant $C_a > 0$ such that for every ball B,

$$a(\sigma\,B) \le C_a\,a(B).$$

In [FPW] the classes D_r are introduced: given a Borel measure ν and $1 \le r < \infty$, a satisfies the $D_r(\nu)$ condition (we simply write $a \in D_r(\nu)$), if there exists $1 \le C_a < \infty$ such that for each ball B and any family of pairwise disjoint balls $\{B_i\}_i \subset B$, the following holds

$$\sum_i a(B_i)^r\,\nu(B_i) \le C_a^r\,a(B)^r\,\nu(B).$$

We write $\|a\|_{D_r(\nu)}$ for the infimum of the constants C_a. By simplicity, we write D_r or $D_r(w)$, when $\nu = \mu$ or w is a weight. Note that, by Hölder's inequality, the $D_r(\nu)$ conditions are decreasing: $D_r(\nu) \subset D_s(\nu)$ and $\|a\|_{D_s(\nu)} \leq \|a\|_{D_r(\nu)}$, for $1 \leq s < r < \infty$. On the other hand, if a is quasi-increasing (that is, $a(B_1) \leq C_a\, a(B_2)$, for all $B_1 \subset B_2$) then, $a \in D_r(\nu)$ for any Borel measure ν and $1 \leq r < \infty$.

2.5. **Approximations of the identity and semigroups.** We work with families of linear operators $\{S_t\}_{t>0}$ that play the role of generalized approximations of the identity. The reader may find convenient to think of $\{S_t\}_{t>0}$ as being a semigroup since this is our main motivation. We assume from now on that these operators commute (that is, $S_t \circ S_s = S_s \circ S_t$ for every $s, t > 0$). Families of operators that form a semigroup (that is, $S_s\, S_t = S_{s+t}$ for all $s, t > 0$) satisfy this property. We assume that these operators admit an integral representation:

$$S_t f(x) = \int_X s_t(x,y)\, f(y)\, d\mu(y),$$

where $s_t(x,y)$ is a measurable function such that

$$|s_t(x,y)| \leq \frac{1}{\mu\big(B(x, t^{1/m})\big)}\, g\bigg(\frac{d(x,y)^m}{t}\bigg) \tag{2.2}$$

for some positive constant m and a positive, bounded and non-increasing function g. Observe that (2.2) leads to a rescaling between the parameter t and the space variables. Thus, given a ball B, we write $t_B = r(B)^m$ in such a way that the parameter t and S_t are "adapted" or "scaled" to B.

We also assume that for all $N \geq 0$,

$$\lim_{r\to\infty} r^N\, g(r) = 0.$$

We can relax the decay on g by fixing $N > 0$ large enough (such that the estimates obtained below are not trivial). Further details are left to the reader. Let us note that the decay of g yields that the integral representation of S_t makes sense for all functions $f \in L^p(X)$ and that the operators S_t are uniformly bounded on $L^p(X)$ for all $1 \leq p \leq \infty$. As in [DY], we consider a wider class of functions for which S_t is well defined: $\mathcal{M} = \bigcup_{x\in X} \bigcup_{\beta>0} \mathcal{M}_{(x,\beta)}$, where $\mathcal{M}_{(x,\beta)}$ is the set of measurable functions f such that

$$\|f\|_{\mathcal{M}_{(x,\beta)}} = \int_X \frac{|f(y)|}{\big(1 + d(x,y)\big)^{2n+\beta}\, \mu\big(B(x, 1 + d(x,y))\big)}\, d\mu(y) < \infty.$$

It is shown in [DY] that $\big(\mathcal{M}_{(x,\beta)}, \|\cdot\|_{\mathcal{M}_{(x,\beta)}}\big)$ is a Banach space, and if $f \in \mathcal{M}$ then, $S_t f$ and $S_s\,(S_t f)$ are well defined and finite almost everywhere, for all $t, s > 0$.

As examples of semigroups we can consider second order elliptic form operators in $\mathbb{R}^n$, $Lf = -\operatorname{div}(A\,\nabla f)$, with A being an elliptic $n\times n$ matrix with complex L^∞-valued coefficients. The operator $-L$ generates a C^0-semigroup $\{e^{-t\,L}\}_{t>0}$ of contractions on $L^2(\mathbb{R}^n)$. Under further assumptions (for instance, real A in any dimension; complex A in dimensions $n = 1$ or $n = 2$, etc.) the heat kernel has Gaussian bounds, that is, the above estimates hold with $m = 2$ and $g(t) = c\, e^{-c\,t^2}$. In this way we can take $S_t = e^{-t\,L}$ or $S_t = I - (I - e^{-t\,L})^N$ for some fixed $N \geq 1$. Note that for the latter we lose the semigroup property, however, we still have the commutation rule and the

Gaussian decay. Thus we can apply our results to that families. In some applications it is interesting to have N large enough so that one obtains extra decay in the resulting estimates (see [HM], [Aus], [AM2] and the references therein). Similar examples could be considered in smooth domains of $\mathbb{R}^n$ since these are spaces of homogenous type.

Another examples of interest are the Riemannian manifolds X with the doubling property. In such situation we can consider the Laplace-Beltrami operator Δ. We assume that the heat kernel $p_t(x,y)$ of the semigroup $e^{-t\Delta}$ has Gaussian upper bounds (UE). As before, this allows us to use our results both for $S_t = e^{-t\Delta}$ or $S_t = I - (I - e^{-t\Delta})^N$ for some fixed $N \geq 1$. Note that the Gaussian upper bounds imply (2.2) with $m = 2$ and $g(t) = c\,e^{-ct^2}$. See Section 4.3 for applications of our main results to this setting.

3. Main results

Theorem 3.1. *Let $\{S_t\}_{t>0}$ be as above, $1 < r < \infty$ and $a \in D_r(\mu)$. Let $f \in \mathcal{M}$ be such that*

$$\fint_B |f - S_{t_B} f|\, d\mu \leq a(B), \tag{3.1}$$

for all balls B and where $t_B = r(B)^m$. Then for any ball B, we have

$$\|f - S_{t_B} f\|_{L^{r,\infty},B} \leq C \sum_{k\geq 0} \sigma^{2nk}\, g(c\,\sigma^{mk})\, a(\sigma^k B) \tag{3.2}$$

with $C \geq 1$ and $0 < c < 1$. Furthermore, if a is doubling, then

$$\|f - S_{t_B} f\|_{L^{r,\infty},B} \lesssim a(B).$$

The previous theorem can be extended to spaces with $A_\infty(\mu)$ weights as follows:

Theorem 3.2. *Let $\{S_t\}_{t>0}$ be as above, $w \in A_\infty(\mu)$, $1 \leq r < \infty$ and $a \in D_r(w) \cap D_1(\mu)$. If $f \in \mathcal{M}$ satisfies (3.1) then,*

$$\|f - S_{t_B} f\|_{L^{r,\infty}(w),B} \leq C \sum_{k\geq 0} \sigma^{2nk}\, g(c\,\sigma^{mk})\, a(\sigma^k B)$$

for all balls B with $C \geq 1$ and $0 < c < 1$. Further, if a is doubling, we can write $C\,a(B)$ in the right hand side.

Remark 3.3. We would like to call attention to the fact that (3.1) is an unweighted estimate and that from it we obtain a weighted estimate for the oscillation $f - S_{t_B} f$.

Remark 3.4. We notice that we have imposed the mild condition $D_1(\mu)$, since in the proof we are going to use Lemma 5.1 and (f) in Theorem 5.2 below. Observe that if we assume $w \in A_r(\mu)$, then $a \in D_r(w)$ implies $a \in D_1(\mu)$, see [JM].

We would like to point out that one could have removed the condition $a \in D_1(\mu)$ in the particular case where S_t is a semigroup. The argument of the proof is somehow different and more technical as one needs an alternative proof for Lemma 5.1 and (f) in Theorem 5.2. We leave the details to the reader.

As in [FPW], [JM], we extend Theorems 3.1 and 3.2. We change the hypothesis on the functional a so that the $D_r(\mu)$ condition allows a different functional in the right hand side.

Theorem 3.5. *Let $\{S_t\}_{t>0}$ be as above and $f \in \mathcal{M}$ be such that* (3.1) *holds. Given $1 < r < \infty$, and functionals a and $\bar{a}$ we assume the following $D_r(\mu)$ type condition:*

$$\sum_i a(B_i)^r \, \mu(B_i) \le \bar{a}(B)^r \, \mu(B), \tag{3.3}$$

for each ball B and any family of pairwise disjoint balls $\{B_i\}_i \subset B$. Then, we have

$$\|f - S_{t_B} f\|_{L^{r,\infty},B} \le C \sum_{k\ge 0} \sigma^{2\,n\,k} \, g(c\,\sigma^{m\,k}) \, \bar{a}(\sigma^k \, B). \tag{3.4}$$

for all balls B with $C \ge 1$ and $0 < c < 1$. Furthermore, if $\bar{a}$ is doubling, we can write $C\,\bar{a}(B)$ in the right hand side.

Remark 3.6. Given two functionals a and $\bar{a}$, abusing the notation, we say that $(a, \bar{a}) \in D_r(\mu)$ if (3.3) holds. As in Theorem 3.2 we can consider a weighted extension of the previous result: we assume that $(a, \bar{a}) \in D_r(w) \cap D_1(\mu)$ and obtain the corresponding $L^{r,\infty}(w)$ estimate. Details are left to the reader.

4. Applications

We recall that Kolmogorov's inequality implies that for any $0 < q < r < \infty$

$$\|f\|_{L^q,B} \le \left(\frac{r}{r-q}\right)^{1/q} \|f\|_{L^{r,\infty},B}. \tag{4.1}$$

This means that whenever we apply the previous results, we can replace $L^{r,\infty}$ by L^q for every $0 < q < r$. Note that the same occurs in the weighted situations.

Example 1 (**BMO and Morrey-Campanato spaces**). Let $\alpha \ge 0$ and $\{S_t\}_{t>0}$ be as above, the space of Morrey-Campanato $L_S(\alpha)$ is defined as follows

$$L_S(\alpha) = \Big\{ f \in \mathcal{M} : \sup_B \frac{1}{\mu(B)^\alpha} \fint_B |f - S_{t_B} f| \, d\mu < \infty \Big\}.$$

When $\alpha = 0$ this space coincides with BMO_S. These spaces are defined in [DY] and [Tan] under the additional assumption that $\{S_t\}_{t>0}$ is a semigroup (see also [DDY], [Jim], [JM]).

Take $a(B) = \mu(B)^\alpha$, we intentionally drop the constant as it is harmless. Note that a is increasing ($a(B_1) \le a(B_2)$, for every $B_1 \subset B_2$) and doubling, therefore $a \in D_r(\mu)$ for every $1 \le r < \infty$. Thus, applying Theorem 3.1 and Kolmogorov's inequality (4.1), for any $f \in L_S(\alpha)$, $\alpha \ge 0$,

$$\|f - S_{t_B} f\|_{L^r,B} \lesssim \mu(B)^\alpha,$$

for every $1 < r < \infty$ and for all balls B. Also all these estimates hold in $L^r(w)$ with $w \in A_\infty(\mu)$.

We would like to call the reader's attention to the fact that in these examples [Jim] obtains a better self-improvement in the scale of Orlicz taking $\exp L$ in the left hand side (which clearly implies the previous estimates).

On the other hand, self-improving results for $f \in BMO_{\varphi,S}(\mu)$ can be also obtained. The spaces $BMO_{\varphi,S}(\mu)$ generalize those defined by S. Spanne [Spa] in $\mathbb{R}^n$ (see [JM] and [Jim] for further details.)

For the following examples we assume that all annuli are non-empty, i.e., $B(x,R)\setminus B(x,r)\neq\emptyset$ for all $0<r<R<\infty$. This implies that $r(B)\approx\operatorname{diam}(B)$ and also that $B_1\subset B_2$ yields $r(B_1)\lesssim r(B_2)$ —we notice that these two properties fail to hold in general. In particular,

$$\frac{\mu(B_2)}{\mu(B_1)}\le c_\mu\left(\frac{r(B_2)}{r(B_1)}\right)^n, \tag{4.2}$$

for every $B_1\subset B_2$. Also, in the examples below, $r(B)$ can be replaced by $\operatorname{diam}(B)$ which is univocally determined (we however keep $r(B)$ to emphasize the analogy with the Euclidean case). The non-empty annuli property implies that μ satisfies the reverse doubling condition (see [Whe]): there exist $\bar n>0$ and $\bar c_\mu>0$ such that

$$\frac{\mu(B_1)}{\mu(B_2)}\le \bar c_\mu\left(\frac{r(B_1)}{r(B_2)}\right)^{\bar n}, \tag{4.3}$$

for all balls B_1 and B_2 with $B_1\subset B_2$.

Example 2 (**Fractional averages**). These are related to the concept of higher gradient introduced by J. Heinonen and P. Koskela in [HeK1], [HeK2]. Given $\lambda\ge 1$, $0<\alpha<n$, $1\le p<n/\alpha$ and a weight u, we set

$$a(B)=r(B)^\alpha\left(\frac{u(\lambda\,B)}{\mu(B)}\right)^{1/p}.$$

Note that if $p\ge n/\alpha$, by (2.1) a is increasing; therefore, $a\in D_r(\mu)\cap D_r(w)$, for every $r\ge 1$ and $w\in A_\infty(\mu)$. Thus, Theorem 3.1 together with (4.1) give self-improvement in all the range $1\le r<\infty$ for $L^r(\mu)$ and $L^r(w)$ with $w\in A_\infty(\mu)$.

As in [FPW] (see also [JM]), we have that $a\in D_r(\mu)$ for $1<r<p\,n/(n-\alpha p)$. Thus, if $f\in\mathcal{M}$ satisfies

$$⨍_B|f-S_{t_B}f|\,d\mu\lesssim r(B)^\alpha\left(\frac{u(\lambda\,B)}{\mu(B)}\right)^{1/p}$$

for all balls B, then

$$\left(⨍_B|f-S_{t_B}f|^r\,dx\right)^{1/r}\lesssim\sum_{k\ge 0}\sigma^{2\,n\,k}\,g(c\,\sigma^{m\,k})\,r(\sigma^k\,B)^\alpha\left(\frac{u(\sigma^k\,\lambda\,B)}{\mu(\sigma^k\,B)}\right)^{1/p},$$

for every $1<r<p\,n/(n-\alpha p)$. If in addition we assume that $u\in A_\infty(\mu)$, as in [FPW], we have $a\in D_{\frac{p\,n}{n-\alpha\,p}+\epsilon}(\mu)$ for some $\epsilon>0$ depending on the $A_\infty(\mu)$ constant of the weight u. In this case u is doubling (thus, one can simply take $\lambda=1$) and consequently so is a. We can apply Theorem 3.1 to obtain an estimate of the generalized oscillation in $L^{\frac{p\,n}{n-\alpha\,p}+\epsilon,\infty}$, which in turns implies by (4.1),

$$\left(⨍_B|f-S_{t_B}f|^{\frac{p\,n}{n-\alpha\,p}}\,d\mu\right)^{\frac{n-\alpha\,p}{p\,n}}\lesssim r(B)^\alpha\left(\frac{u(B)}{\mu(B)}\right)^{1/p}.$$

A particular case of this is the following: let X be a differential operator such that for some function $f\in\mathcal{M}$ and for all balls B,

$$⨍_B|f-S_{t_B}f|\,d\mu\lesssim r(B)^\alpha\left(\frac{1}{\mu(B)}\int_{\lambda\,B}|X\,f|^p\,d\mu\right)^{1/p}$$

with $\lambda \geq 1$, $0 < \alpha < n$ and $1 \leq p < n/\alpha$. Then, we have proved the following self-improvement: for every $1 < r < p\,n/(n - \alpha\,p)$

$$\left(\fint_B |f - S_{t_B} f|^r \, d\mu\right)^{1/r} \lesssim \sum_{k\geq 0} \sigma^{2\,n\,k}\, g(c\,\sigma^{m\,k})\, r(\sigma^k\, B)^\alpha \left(\frac{1}{\mu(\sigma^k\, B)} \int_{\lambda\,\sigma^k\, B} |X f|^p \, d\mu\right)^{1/p}.$$

If we further assume that $|X f|^p \in A_\infty(\mu)$ then,

$$\left(\fint_B |f - S_{t_B} f|^{\frac{p\,n}{n-\alpha\,p}} \, d\mu\right)^{\frac{n-\alpha\,p}{p\,n}} \lesssim r(B)^\alpha \left(\fint_B |X f|^p \, d\mu\right)^{1/p}.$$

4.1. **Reduced Poincaré type inequalities.** As in the previous examples and motivated by the classical $(1,1)$-Poincaré inequality, one could consider estimates as follows: Let $f \in \mathcal{M}$ be such that

$$\fint_B |f - S_{t_B} f| \, d\mu \leq r(B) \fint_B h \, d\mu, \tag{4.4}$$

for all balls B and where h is some non-negative measurable function: Typically h depends on f. For instance, in $\mathbb{R}^n$ one can take $h = C\,|\nabla f|$. However, in the computations below we can work with any given function h. We call this estimate a reduced Poincaré type inequality, in contrast with the expanded estimates (4.17) that we consider in Section 4.2 below. In this context it is more natural to relax (4.4) and take as an initial estimate

$$\fint_B |f - S_{t_B} f| \, d\mu \leq r(B) \left(\fint_B h^p \, d\mu\right)^{1/p}, \tag{4.5}$$

with $1 \leq p < \infty$.

We would like to apply our results to obtain self-improvement from (4.5).

Example 3 (**Poincaré-Sobolev inequality**). We show that (4.5) yields

$$\|f - S_{t_B} f\|_{L^{p^*,\infty},B} \leq \sum_{k\geq 0} \phi(k)\, r(\sigma^k\, B) \left(\fint_{\sigma^k\, B} h^p \, d\mu\right)^{1/p}, \tag{4.6}$$

for all balls B, for some sequence $\{\phi(k)\}_{k\geq 0}$ and where $p^* = \frac{n\,p}{n-p}$. Hence, applying Kolmogorov's inequality (4.1), we get

$$\left(\fint_B |f - S_{t_B} f|^r \, d\mu\right)^{1/r} \leq \sum_{k\geq 0} \phi(k)\, r(\sigma^k\, B) \left(\fint_{\sigma^k\, B} h^p \, d\mu\right)^{1/p}, \tag{4.7}$$

for every $1 < r < p^*$.

We take $a(B) = r(B)\big(\fint_B h^p \, d\mu\big)^{1/p}$. Note that when $p \geq n$, $a \in D_r(\mu)$ for every $1 \leq r < \infty$ (since a is increasing). Therefore, (4.7) holds for every $1 < r < \infty$. This case is studied in [Jim] where an exponential type self-improvement is obtained:

$$\|f - S_{t_B} f\|_{\exp L,B} \leq \sum_{k\geq 0} \phi(k)\, r(\sigma^k\, B) \left(\fint_{\sigma^k\, B} h^p \, d\mu\right)^{1/p}.$$

Otherwise, if $1 < p < n$, we have $a \in D_{p^*}(\mu)$: Let B be a ball and $\{B_i\}_i$ a family of pairwise disjoint subballs of B. From (2.1) and the fact that

$$\sum_i \left(\int_{B_i} h^p \, d\mu \right)^{p^*/p} \leq \left(\sum_i \int_{B_i} h^p \, d\mu \right)^{p^*/p} \leq \left(\int_B h^p \, d\mu \right)^{p^*/p},$$

we get $a \in D_{p^*}(\mu)$. Hence, we can apply Theorem 3.1 and this readily leads to (4.6) as desired.

Example 4 (**Poincaré-Sobolev inequality for $A_1(\mu)$ weights**). Given $w \in A_1(\mu)$ and $1 \leq p < n$, (4.5) implies the following: for all balls B and for some sequence $\{\phi(k)\}_{k \geq 0}$

$$\|f - S_{t_B} f\|_{L^{p^*,\infty}(w),B} \leq \sum_{k \geq 0} \phi(k) \, r(\sigma^k B) \left(\fint_{\sigma^k B} h^p \, dw \right)^{1/p}. \tag{4.8}$$

As a consequence of the previous inequality and the weighted version of Kolmogorov's inequality, we get the strong norm $L^r(w, B)$ for every $1 < r < p^*$.

In order to show (4.8) we use Theorem 3.2. First, using that $w \in A_1(\mu)$, we have that (4.5) gives

$$\fint_B |f - S_{t_B} f| \, d\mu \lesssim r(B) \left(\fint_B h^p \, dw \right)^{1/p} = a(B).$$

We claim that $a \in D_{p^*}(w)$. Indeed, take a ball B and a family $\{B_i\}_i \subset B$ of pairwise disjoint balls. First, note that (2.1) and $w \in A_1(\mu)$ imply $w(B)/w(B_i) \lesssim \big(r(B)/r(B_i)\big)^n$ where we have used (5.15) below. Then, as $p^* > p$, we have

$$\begin{aligned} \sum_i a(B_i)^{p^*} w(B_i) &= \sum_i r(B_i)^{p^*} w(B_i)^{1-p^*/p} \left(\int_{B_i} h^p \, dw \right)^{p^*/p} \\ &\lesssim r(B)^{p^*} w(B)^{1-p^*/p} \sum_i \left(\int_{B_i} h^p \, dw \right)^{p^*/p} \leq a(B)^{p^*} w(B). \end{aligned}$$

Notice that $w \in A_1(\mu) \subset A_{p^*}(\mu)$ and therefore $a \in D_1(\mu)$ (see Remark 3.4). Thus, applying Theorem 3.2, we obtain (4.8).

As before, when $p \geq n$, we can obtain exponential type self-improvement since the functional is increasing (see [Jim]).

Example 5 (**Poincaré-Sobolev inequality for $A_r(\mu)$ weights, $r > 1$**). We show that (4.5) with $1 \leq p < n$ implies that for every $r > 1$ and $w \in A_r(\mu)$, there exists $q > \frac{n\,r\,p}{n-p}$ (depending on p, n, w) such that the following holds

$$\|f - S_{t_B} f\|_{L^q(w),B} \leq \sum_{k \geq 0} \phi(k) \, r(\sigma^k B) \left(\fint_{\sigma^k B} h^{r\,p} \, dw \right)^{1/(r\,p)}. \tag{4.9}$$

To check (4.9), we first see that (4.5) and $w \in A_r(\mu)$ give

$$\fint_B |f - S_{t_B} f| \, d\mu \lesssim r(B) \left(\fint_B h^{r\,p} \, dw \right)^{1/(r\,p)} = a(B).$$

The openness property of the $A_r(\mu)$ class gives that $w \in A_{\tau\,r}(\mu)$ for some $0 < \tau < 1$. Without loss of generality, τ can be chosen so that $\frac{p}{n} < \tau < 1$. Hence, for any ball B and any measurable set $E \subset B$ we have, by (5.15) below,

$$\frac{w(B)}{w(E)} \lesssim \left(\frac{\mu(B)}{\mu(E)}\right)^{\tau\,r}.$$

We pick $q_0 = (n\,\tau\,r\,p)/(n\,\tau - p)$ and observe that $q_0 > \frac{n\,r\,p}{n-p}$. Using this and proceeding as in the previous one can easily see that $a \in D_{q_0}(w)$ which by using Theorem 3.2 and Remark 3.4 (since $q_0 > r$) lead to an estimate in $L^{q_0,\infty}(w)$. Next taking $\frac{n\,r\,p}{n-p} < q < q_0$, Kolmogorov inequality gives to (4.9).

Example 6 (**Two-weight Poincaré inequality**). Given $1 \le p \le q \le r < \infty$, let (w, v) be a pair of weights with $w \in A_r(\mu)$, $v \in A_{q/p}(\mu)$ such that the following balance condition holds

$$\frac{r(B_1)}{r(B_2)}\left(\frac{w(B_1)}{w(B_2)}\right)^{1/r} \lesssim \left(\frac{v(B_1)}{v(B_2)}\right)^{1/q}, \qquad \text{for all } B_1, B_2 \text{ with } B_1 \subset B_2. \tag{4.10}$$

Then, (4.5) allows us to obtain

$$\|f - S_{t_B} f\|_{L^{r,\infty}(w),B} \le \sum_{k\ge 0} \phi(k)\, r(\sigma^k\,B) \left(\fint_{\sigma^k\,B} h^q\,dv\right)^{1/q}. \tag{4.11}$$

Consequently by Kolmogorov's inequality, we obtain strong type estimates in the range $1 < s < r$.

In order to obtain (4.11), note that by (4.5) and using that $v \in A_{q/p}(\mu)$, we get

$$\fint_B |f - S_{t_B} f|\,d\mu \lesssim r(B) \left(\fint_B h^q\,dv\right)^{1/q} = a(B).$$

Using the balance condition together with $r/q \ge 1$, it is not difficult to see that $a \in D_r(w)$. Hence, applying Remark 3.4 and Theorem 3.2, we obtain the desired inequality.

Example 7 (**Generalized Hardy inequality**). We take $1 < p < \bar{n}$ (where $\bar{n}$ is the exponent given in (4.3)) and fix $x_0 \in X$. Let us consider $w_{x_0}(x) = d(x, x_0)^{-p}$. Then from (4.5) we obtain

$$\|f - S_{t_B} f\|_{L^{p,\infty}(w_{x_0},B)} \le \sum_{k\ge 0} \phi(k) \left(\frac{1}{w_{x_0}(\sigma^k\,B)} \int_{\sigma^k\,B} h^p\,d\mu\right)^{1/p}. \tag{4.12}$$

As a consequence of (4.1), we automatically obtain strong type estimates in the range $1 < r < p$. Note that the claimed estimate implies

$$\sup_{\lambda>0} \lambda\, w_{x_0}\{x \in B : |f(x) - S_{t_B} f(x)| > \lambda\}^{1/p} \le \sum_{k\ge 0} \tilde{\phi}(k) \left(\int_{\sigma^k\,B} h^p\,d\mu\right)^{1/p}.$$

and this should be compared with the classical Hardy inequality

$$\int_B |f(x) - f_B|^2 \frac{dx}{|x|^2} \lesssim \int_B |\nabla f(x)|^2\,dx$$

To obtain (4.12) it is easy to see that for every ball $B = B(x_B, r(B))$

$$\fint_B d(x,x_0)^\alpha \, d\mu(x) \approx d(x_0,x_B)^\alpha, \qquad x_0 \notin 2\,D_0\,B, \quad \alpha \in \mathbb{R}, \tag{4.13}$$

and

$$\fint_B d(x,x_0)^\alpha \, d\mu(x) \approx r(B)^\alpha, \qquad x_0 \in 2\,D_0\,B, \quad \alpha > -\bar{n}. \tag{4.14}$$

Using these estimates it follows that $w_{x_0} \in A_1(\mu)$ and $r(B)\,(w_{x_0}(B)/\mu(B))^{1/p} \lesssim 1$. Then we readily obtain that (4.5) yields

$$\fint_B |f - S_{t_B} f| \, d\mu \lesssim \left(\frac{1}{w_{x_0}(B)} \int_B h^p \, d\mu \right)^{1/p} = a(B). \tag{4.15}$$

It is trivial to show that $a \in D_p(w_{x_0})$ and also that $a \in D_1(\mu)$ by Remark 3.4 and the fact that $w_{x_0} \in A_p(\mu)$. Thus Theorem 3.2 gives as desired (4.12).

Example 8 (**Generalized two weights Hardy inequality**). We take $1 < p < \bar{n}$ and $0 \le q \le p$. Fixed $x_0 \in X$ we set $w_{x_0}(x) = d(x,x_0)^{-p}$ and $\bar{w}_{x_0}(x) = d(x,x_0)^{-q}$. Then from (4.5) we obtain

$$\|f - S_{t_B} f\|_{L^{p,\infty}(\bar{w}_{x_0}),B} \le \sum_{k \ge 0} \phi(k) \left(\frac{1}{w_{x_0}(\sigma^k\,B)} \int_{\sigma^k\,B} h^p \, d\mu \right)^{1/p}. \tag{4.16}$$

As a consequence of the weighted version of (4.1), we automatically obtain estimates in $L^r(\bar{w}_{x_0})$ for every $1 \le r < p$.

Taking the functional from the previous example, we have already shown (4.15) and $a \in D_1(\mu)$. Using (4.13) and (4.14) we obtain the following balance condition

$$\frac{\bar{w}_{x_0}(B_1)}{\bar{w}_{x_0}(B_2)} \, \frac{w_{x_0}(B_2)}{w_{x_0}(B_1)} \lesssim 1, \qquad B_1 \subset B_2.$$

This easily gives $a \in D_p(\bar{w}_{x_0})$. Note also that $\bar{w}_{x_0} \in A_1(\mu)$. Thus, Theorem 3.2 yields (4.16).

Global pseudo-Poincaré inequalities. As a consequence of our results and arguing as in [JM], we are going to obtain the following generalized global pseudo-Poincaré inequalities, see [Sa2]. These are of interest to obtain interpolation and Gagliardo-Nirenberg inequalities, see [Sa2], [BCLS], [Led], [MM]. Assume that $f \in \mathcal{M}$ satisfies (4.5) with $1 \le p < n$. Then for all $t > 0$:

- *Global pseudo-Poincaré inequalities*:
$$\|f - S_t f\|_{L^p(X)} \lesssim t^{1/m}\, \|h\|_{L^p(X)}.$$
- *Global weighted pseudo-Poincaré inequalities*: for every $w \in A_r(\mu)$, $1 \le r < \infty$
$$\|f - S_t f\|_{L^{p\,r}(w)} \lesssim t^{1/m}\, \|h\|_{L^{p\,r}(w)}.$$
- *Global pseudo-Hardy inequalities*: Let $1 < p < \bar{n}$ and take $w_{x_0}(x) = d(x,x_0)^{-p}$, $x_0 \in X$, then
$$\|f - S_t f\|_{L^{p,\infty}(w_{x_0})} \lesssim \|h\|_{L^p(X)},$$

Let us show the first estimate. We fix $t > 0$ and take $k_0 \in \mathbb{Z}$ such that $C_1\,\sigma^{k_0} \le t^{1/m} < C_1\,\sigma^{k_0+1}$. Then, we write $X = \cup_{Q\in\mathcal{D}_{k_0}} Q$ a.e.. Note that for each $Q \in \mathcal{D}_{k_0}$, there exists τ with $1 \le \tau < \sigma^m$ such that $t = \tau\, t_{\hat{B}_Q}$. As in Lemma 5.5, we fix $Q_0 \in \mathcal{D}_{k_0}$ and consider the family $\mathcal{J}_k = \{Q \in \mathcal{D}_{k_0} : \sigma^{k+1}\,\hat{B}_Q \cap \sigma^{k+1}\,\hat{B}_{Q_0} \ne \emptyset\}$. It is easy to see that each $Q \in \mathcal{J}_k$ satisfies $Q \subset \sigma^{k+2}\,\hat{B}_{Q_0} \subset \sigma^{k+3}\,\hat{B}_Q$. This and the fact that μ is doubling imply $\#\mathcal{J}_k \le c_\mu\,(C_1/c_1)^n\,\sigma^{n\,(k+3)}$. On the other hand, Example 3 gives (4.7) with $r = p$. Then, Minkowski's inequality and Lemma 5.4 imply

$$
\begin{aligned}
\|f - S_t f\|_{L^p(X)} &= \bigg(\sum_{Q\in\mathcal{D}_{k_0}} \int_Q |f - S_t f|^p\,d\mu\bigg)^{1/p} \\
&\le \bigg(\sum_{Q\in\mathcal{D}_{k_0}} \int_{\tau^{1/m}\,\hat{B}_Q} |f - S_{\tau\,t_{\hat{B}_Q}} f|^p\,d\mu\bigg)^{1/p} \\
&\lesssim \bigg(\sum_{Q\in\mathcal{D}_{k_0}} \mu(\tau^{1/m}\,\hat{B}_Q)\Big(\sum_{k\ge 0}\phi(k)\,r(\sigma^k\,\tau^{1/m}\,\hat{B}_Q)\,\Big(\fint_{\sigma^k\,\tau^{1/m}\,\hat{B}_Q} h^p\,d\mu\Big)^{1/p}\Big)^p\bigg)^{1/p} \\
&\le t^{1/m}\sum_{k\ge 0}\phi(k)\,\sigma^{k\,(1-\bar{n}/p)}\bigg(\sum_{Q\in\mathcal{D}_{k_0}}\int_{\sigma^{k+1}\,\hat{B}_Q} h^p\,d\mu\bigg)^{1/p} \\
&\lesssim t^{1/m}\sum_{k\ge 0}\phi(k)\,\sigma^{k\,(1+n/p-\bar{n}/p)}\bigg(\int_X h^p\,d\mu\bigg)^{1/p} \lesssim t^{1/m}\,\|h\|_{L^p(X)},
\end{aligned}
$$

where we have used that $\{\phi(k)\}_{k\ge 0}$ (given in Theorem 3.1) is a fast decay sequence by the decay of g.

In the weighted case with $w \in A_r(\mu)$, we use Example 4 for $r = 1$ and Example 5 for $r > 1$. For $r = 1$ we have $p^* > p$, and if $r > 1$ we observe that $\frac{n\,r\,p}{n-p} > r\,p$. Thus, in both cases we obtain

$$
\bigg(\fint_B |f - S_{t_B} f|^{r\,p}\,dw\bigg)^{1/(r\,p)} \le \sum_{k\ge 0}\phi(k)\,r(\sigma^k\,B)\bigg(\fint_{\sigma^k\,B} h^{r\,p}\,dw\bigg)^{1/(r\,p)}.
$$

Proceeding as before and using that the $w\,d\mu$ is doubling we obtain the desired inequality.

For the pseudo-Hardy inequalities one uses the same ideas with the weak-type norm in the left-hand side.

4.2. **Expanded Poincaré type inequalities.** We introduce some notation: given $1 \le p, q < \infty$ we say that $f \in \mathcal{M}$ satisfies an expanded $L^q - L^p$ Poincaré inequality if for all balls $B \subset X$

$$
\fint_B |f - S_{t_B} f|\,d\mu \le \sum_{k\ge 0}\alpha(k)\,r(\sigma^k\,B)\,\bigg(\fint_{\sigma^k\,B} h^p\,d\mu\bigg)^{1/p},
$$

where $\{\alpha(k)\}_{k\ge 0}$ is a sequence of non-negative numbers and h is some non-negative measurable function.

In this section we start with an expanded $L^1 - L^p$ Poincaré inequality and show that it self-improves to an expanded $L^q - L^p$ Poincaré inequality for q in the range

$(1,p^*)$. More precisely, our starting estimate is the following: let $p \geq 1$ and $f \in \mathcal{M}$ be such that

$$\fint_B |f - S_{t_B} f| \, d\mu \leq \sum_{k\geq 0} \alpha(k) \, r(\sigma^k B) \left(\fint_{\sigma^k B} h^p \, d\mu \right)^{1/p}, \tag{4.17}$$

for all balls $B \subset X$ and where $\{\alpha(k)\}_{k\geq 0}$ is a sequence of non-negative numbers and h is some non-negative measurable function.

In the classical situation, replacing $S_{t_B} f$ by f_B and taking $h = C\,|\nabla f|$ and $\alpha(k) = 0$ for $k \geq 1$, this inequality is nothing but the $L^1 - L^p$ Poincaré-Sobolev inequality. Let us also observe that if $\alpha(k) = 0$ for $k \geq 1$, we get back to (4.4) in the previous section. On the other hand, if h^p is doubling and $\{\alpha(k)\}_{k\geq 0}$ decays fast enough, then (4.17) leads us again to (4.5). As mentioned in [JM] and [Jim], we believe that the estimates (4.17) are more natural than (4.4) or (4.5) in the sense that they take into account the tail effects of the semigroup in place of looking only at a somehow local term.

Following the computations in [JM], assuming that $S_t 1 \equiv 1$ a.e. in X and for all $t > 0$, one can obtain that the following $L^1 - L^p$ Poincaré-Sobolev inequality

$$\fint_B |f - f_B| \, d\mu \leq C \, r(B) \left(\fint_B |Sf|^p \, d\mu \right)^{1/p},$$

for some (differential) operator S, implies (4.17) with $h = |Sf|$. As we show below, under some conditions on a Riemannian manifold we can obtain (4.17) without any kind of Poincaré-Sobolev inequality, thus our results are applicable in situations where such estimates do not hold or are unknown.

Starting with (4.17) we are going to apply our main results to obtain a self-improvement on the integrability of the left hand side. For the sake of simplicity, we are going to treat only the unweighted Poincaré-Sobolev inequality analogous to those in Example 3. We notice that the same ideas can be used to consider Example 4 and obtain (4.8) with $L^r(w)$, $1 < r < p^*$, in place of $L^{p^*,\infty}(w)$ (here one can show that $a \in D_{p^*-\epsilon}(w)$); Example 5 and obtain (4.9) for some $q > \frac{n\,r\,p}{n-p}$ (here one can show that $a \in D_{q_0-\epsilon}(w)$ and this allows us to pick such value of q); and Example 6 for which we can show (4.11) with $L^s(w)$, $1 < s < r$, in place of $L^{r,\infty}(w)$ if we further assume that $1 \leq p \leq q < r$ (here one can show that $a \in D_{r-\epsilon}(w)$). Further details are left to the interested reader.

We proceed as in [JM, Section 4.2.]. We fix $1 \leq p < n$ and define

$$a(B) = \sum_{k\geq 0} \alpha(k) \, a_0(\sigma^k B) \qquad \text{with} \quad a_0(B) = r(B) \left(\fint_B h^p \, d\mu \right)^{1/p}.$$

We are going to find another functional $\bar{a}$ with a similar expression so that $(a, \bar{a})$ satisfies a D_q condition as in Theorem 3.5.

Proposition 4.1. *Given a as above, let $1 \leq p < n$ and $1 < q < p^*$. There exists a sequence of non-negative numbers $\{\bar{\alpha}(k)\}_{k\geq 0}$, so that if we set*

$$\bar{a}(B) = \sum_{k\geq 0} \bar{\alpha}(k) \, a_0(\sigma^k B),$$

we have that $(a, \bar{a}) \in D_q$.

The proof of this result is postponed until Section 5.4. From the proof we obtain that $\bar{\alpha}(0) = C\,\alpha(0)$ and $\bar{\alpha}(l) = C\,\sigma^{l\,\bar{n}/\tilde{q}} \sum_{k\geq \max\{l-2,1\}} \sigma^{k\,(\frac{n}{p}-\frac{\bar{n}}{\tilde{q}})}\,\alpha(k)$ for $l \geq 1$ with $\tilde{q} = \max\{q,p\}$

This result, Theorem 3.5, and Kolmogorov's inequality (4.1) readily lead to the following corollary:

Corollary 4.2. *Given* $1 \leq p < n$, *let* $f \in \mathcal{M}$ *satisfy* (4.17). *Then, for all* $1 < q < p^*$ *there exists another sequence of non-negative numbers* $\{\tilde{\alpha}(k)\}_{k\geq 0}$ *so that*

$$\left(\fint_Q |f - S_{t_Q} f|^q\,d\mu\right)^{1/q} \leq \sum_{k\geq 0} \tilde{\alpha}(k)\,\ell(\sigma^k\,Q)\left(\fint_{\sigma^k\,Q} h^p\,d\mu\right)^{1/p}.$$

It is straightforward to show that $\tilde{\alpha}(k) = C\,\sum_{j=0}^{k} \sigma^{2\,n\,j}\,g(c\,\sigma^{m\,j})\,\bar{\alpha}(k-j)$.

Remark 4.3. We would like to call the reader's attention to the fact that in the case $p \geq n$, the functional a defined above is increasing since so it is a_0. Therefore the previous estimate holds for all $1 < q < \infty$ with a sequence $\tilde{\alpha}$ defined as before with $\bar{\alpha} = \alpha$.

As in [JM, Section 4.2] one can consider generalized Poincaré inequalities at the scale p^*. More precisely, one can push the exponent q to p^* and obtain an estimate in the Marcinkiewicz space associated with $\varphi(t) \approx t^{1/p^*}\,(1+\log^+ 1/t)^{-(1+\epsilon)/p^*}$, $\epsilon > 0$. Notice that φ is the fundamental function of the Orlicz space $L^{p^*}(\log L)^{-(1+\epsilon)}$, and the Marcinkiewicz space is the corresponding weak-type space (as $L^{q,\infty}$ is for L^q). Further details are left to the reader, see [JM].

Given $1 \leq p < \infty$, by Corollary 4.2 and Remark 4.3 both particularized to $q = p$, we immediately get that $f \in \mathcal{M}$ satisfies an expanded $L^1 - L^p$ Poincaré inequality (4.17) (with a fast decay sequence) if and only if it satisfies an expanded $L^p - L^p$ Poincaré inequality. Notice also that an expanded $L^1 - L^p$ Poincaré inequality implies trivially an expanded $L^1 - L^q$ (equivalently $L^q - L^q$) Poincaré inequality for every $q \geq p$. As a consequence of this, starting with (4.17) with a fast decay sequence $\{\alpha(k)\}_{k\geq 0}$ and repeating the argument in the previous section we obtain global pseudo-Poincaré inequalities: for all $q \geq p$ and all $t > 0$

$$\|f - S_t f\|_{L^q(X)} \lesssim t^{1/m}\,\|h\|_{L^q(X)}.$$

4.3. **Expanded Poincaré type inequalities on manifolds.** In this section we show that on Riemannian manifolds we can obtain expanded Poincaré type inequalities as (4.5) with different functions h on the right hand side. As observed before (see [JM]), assuming that $S_t 1 = 1$ μ-a.e., classical Poincaré-Sobolev inequalities imply (4.17). There are situations where such Poincaré inequalities do not hold or are unknown. However the arguments below lead us to obtain generalized expanded Poincaré type inequalities to whom the self-improving results are applicable.

We refer the reader to [ACDH] and the references therein for a complete account of this topic. Let M be a complete non-compact connected Riemannian manifold with d its geodesic distance. Assume that the volume form μ is doubling. Then M equipped with the geodesic distance and the volume form μ is a space of homogeneous type. Non-compactness of M implies infinite diameter, which together with the doubling volume property yields $\mu(M) = \infty$ (see for instance [Ma2]). Notice that connectedness implies that M has the non-empty annuli property, therefore we are in a setting where we can apply all the previous applications.

Let Δ be the positive Laplace-Beltrami operator on M given by

$$\langle \Delta f, g\rangle = \int_M \nabla f \cdot \nabla g \, d\mu$$

where ∇ is the Riemannian gradient on M and $\cdot$ is an inner product on TM. The Riesz transform is the tangent space valued operator $\nabla \Delta^{-1/2}$ and it is bounded from $L^2(M,\mu)$ into $L^2(M; TM, \mu)$ by construction.

One says that the heat kernel $p_t(x,y)$ of the semigroup $e^{-t\Delta}$ has Gaussian upper bounds if for some constants $c, C > 0$ and all $t > 0, x, y \in M$,

$$p_t(x,y) \le \frac{C}{\mu(B(x,\sqrt{t}))} \, e^{-c\frac{d^2(x,y)}{t}}. \tag{UE}$$

It is known that under doubling it is a consequence of the same inequality only at $y = x$ [Gr2, Theorem 1.1]. Notice that (UE) implies that $p_t(x,y)$ satisfies (2.2) with $m = 2$ (therefore $t_B = r(B)^2$) and $g(t) = c\, e^{-c\,t^2}$. Thus our results are applicable to the semigroup $S_t = e^{-t\,\Delta}$ and to the family of commuting operators $S_t = I - (I - e^{-t\,\Delta})^m$ with $m \ge 1$ —expanding the latter one trivially sees that its kernel satisfies (UE).

Under doubling and (UE), [CD] shows that

$$\big\| \, |\nabla \Delta^{-1/2} f| \, \big\|_{L^p} \le C_p \|f\|_{L^p} \tag{R_p}$$

holds for $1 < p < 2$ and all f bounded with compact support. Here, $|\cdot|$ is the norm on TM associated with the inner product. We define

$$q_+ = \sup \big\{ p \in (1,\infty) \, : \, (R_p) \text{ holds} \big\}$$

which satisfies $q_+ \ge 2$ under doubling and (UE). It can be equal to 2 ([CD]). It is bigger than 2 assuming further the stronger L^2-Poincaré inequalities ([AC]) and in some situations $q_+ = \infty$.

We also define $\tilde{q}_+$ as the supremum of those $p \in (1,\infty)$ such that for all $t > 0$,

$$\big\| \, |\nabla e^{-t\,\Delta} f| \, \big\|_{L^p} \le C\, t^{-1/2} \|f\|_{L^p}. \tag{G_p}$$

By analyticity of the heat semigroup, one always have $\tilde{q}_+ \ge q_+$; indeed (R_p) implies (G_p):

$$\big\| \, |\nabla e^{-t\,\Delta} f| \big\|_{L^p} \le C_p \, \|\Delta^{1/2} \, e^{-t\,\Delta} f\|_{L^p} \le C_p' \, t^{-1/2} \, \|f\|_{L^p}.$$

As we always have (R_2) then this estimate implies (G_2). Under the doubling volume property and L^2-Poincaré inequalities, $q_+ = \tilde{q}_+$, see [ACDH, Theorem 1.3]. It is not known if the equality holds or not under doubling and Gaussian upper bounds.

Proposition 4.4. *Let M be complete non-compact connected Riemannian manifold satisfying the doubling volume property and (UE).*

(a) *Given $m \ge 1$, let $S_t^m = I - (I - e^{-t\,\Delta})^m$. For any smooth function with compact support f we have*

$$\fint_B |f - S_{t_B}^m f| \, d\mu \le C \sum_{k \ge 1} \sigma^{-k\,(2\,m-n)} \, r(\sigma^k \, B) \fint_{\sigma^k \, B} |\Delta^{1/2} f| \, d\mu.$$

(*b*) *For any* $p \in ((\tilde{q}_+)', \infty) \cup [2, \infty)$ *and any smooth function with compact support* f *we have*

$$\left(\fint_B |f - e^{-t_B \Delta} f|^p \, d\mu\right)^{\frac{1}{p}} \leq C \sum_{k \geq 1} e^{-c\sigma^{2k}} r(\sigma^k B) \left(\fint_{\sigma^k B} |\nabla f|^p \, d\mu\right)^{1/p}$$

As a consequence of this result (whose proof is given below) and by Corollary 4.2 and Remark 4.3 we obtain Theorem 1.1 whose precise statement is given next:

Corollary 4.5. *Let* M *be complete non-compact connected Riemannian manifold satisfying the doubling volume property and* (UE). *Given* $1 \leq p < \infty$ *we set* $p^* = n\,p/(n-p)$ *if* $1 \leq p < n$ *and* $p^* = \infty$ *otherwise.*

(*a*) *Given* $m \geq 1$, *let* $S_t^m = I - (I - e^{-t\Delta})^m$ *and* $1 < q < p^*$. *Assume that* $m > (n + n/p - \bar{n}/\max\{q, p\})/2$ *if* $1 < p < n$. *Then, for any smooth function with compact support* f *we have*

$$\left(\fint_B |f - S_{t_B}^m f|^q \, d\mu\right)^{1/q} \leq C \sum_{k \geq 1} \phi(k)\, r(\sigma^k B) \left(\fint_{\sigma^k B} |\Delta^{1/2} f|^p \, d\mu\right)^{1/p},$$

where $\phi(k) = \sigma^{-k(2m - D - n/p)}$ *if* $1 < p < n$ *and* $\phi(k) = \sigma^{-k(2m-D)}$ *if* $p \geq n$.

(*b*) *For any* $p \in ((\tilde{q}_+)', \infty) \cup [2, \infty)$, *any* $1 < q < p^*$ *and any smooth function with compact support* f *we have*

$$\left(\fint_B |f - e^{-t_B \Delta} f|^q \, d\mu\right)^{1/q} \leq C \sum_{k \geq 1} e^{-c\sigma^k} r(\sigma^k B) \left(\fint_{\sigma^k B} |\nabla f|^p \, d\mu\right)^{1/p}.$$

Remark 4.6. As mentioned before we can also get similar estimates assuming further local Poincaré-Sobolev inequalities. Notice that our assumptions guarantee that $e^{-t\Delta} 1 \equiv 1$. We assume that M satisfies the $L^1 - L^p$ Poincaré, $1 \leq p < \infty$, that is, for every ball B and every $f \in L^1_{\mathrm{loc}}(M)$, $|\nabla f| \in L^p_{\mathrm{loc}}(M)$

$$\fint_B |f - f_B| d\mu \leq r(B) \left(\fint_B |\nabla f|^p \, d\mu\right)^{1/p}.$$

Then,

$$\fint_B |f - S_t f| \, d\mu \leq C \sum_{k \geq 1} e^{-c\sigma^k} r(\sigma^k B) \left(\fint_{\sigma^k B} |\nabla f|^p \, d\mu\right)^{1/p}.$$

with either $S_t = e^{-t\Delta}$ or $S_t = I - (I - e^{-t\Delta})^m$. Notice that Proposition 4.4 establishes this estimate for some values p, and for the first choice of S_t, without assuming any kind of Poincaré inequalities.

We would like to call the reader's attention to the fact that, as mentioned before, one could prove similar estimates in the spirit of Examples 4, 5 and 6. Besides, global pseudo-Poincaré inequalities can be derived in the same manner.

We finish this section exhibiting some examples of manifolds where the previous results can be applied. The most interesting example, where our results seem to be new is the following:

Consider two copies of $\mathbb{R}^n$ minus the unit ball glued smoothly along their unit circles with $n \geq 2$. It is shown in [CD] that this manifold has doubling volume form and

Gaussian upper bounds. $L^2 - L^2$ Poincaré does not hold: in fact, it satisfies $L^p - L^p$ Poincaré if and only if $p > n$ (see [HaK] in the case of a double-sided cone in $\mathbb{R}^n$, which is the same). If $n = 2$, (R_p) holds if and only if $p \leq 2$ ([CD]). If $n > 2$, (R_p) holds if and only if $p < n$ ([CCH]). In any case, we have $q_+ = n$, hence $\tilde{q}_+ \geq n$. We can apply Corollary 4.5 and obtain (a) and (b). Notice that although classical $L^p - L^p$ Poincaré holds if and only if $p > n$, (b) yields in particular expanded $L^p - L^p$ Poincaré estimates for all $n' < p < \infty$.

There are many examples of manifolds or submanifolds satisfying the doubling property and the classical $L^1 - L^1$ Poincaré. Since doubling and $L^1 - L^1$ Poincaré imply (UE), we can apply Proposition 4.4 and Corollary 4.5 on such manifolds. Note that in this case, (b) of Proposition 4.4 and Corollary 4.5 are not new since, as mentioned before, Poincaré inequalities are stronger than expanded Poincaré inequalities. However, (a) yields new expanded Poincaré inequalities involving the square root of the Laplace-Beltrami operator on the right hand side. From these manifolds, we would like mention the following:

- Complete Riemannian manifolds M that are quasi-isometric to a Riemannian manifold with non-negative Ricci curvature (in particular every Riemannian manifold with non-negative Ricci curvature) have doubling volume form and admit classical $L^1 - L^1$ Poincaré.
- Singular conical manifolds with closed basis admit classical $L^2 - L^2$ Poincaré inequalities for C^∞ functions (see [CL]). Using the methods of [Gr1] one can also see that classical $L^1 - L^1$ Poincaré holds. Such manifolds do not necessarily satisfy the doubling property, but they do, if for instance, one assumes that the basis is compact.
- Co-compact covering manifolds with polynomial growth deck transformation group satisfy the doubling property and the classical $L^1 - L^1$ Poincaré (see [Sa1]).
- Nilpotent Lie groups have polynomial growth, then they satisfy the doubling property and the classical $L^1 - L^1$ Poincaré inequality. Among the important nilpotent Lie groups we mention the *Carnot groups*.

5. Proofs of the main results

5.1. **Proof of Theorem 3.1.** We split the proof in two parts.

5.1.1. *Step I: Dyadic case.* We use some ideas from [JM]. First, we fix $\sigma \geq 4\, D_0^3$ large enough and take the dyadic structure given by Theorem 2.1. In this part of the proof, we show that for every $1 \leq \tau < \sigma^m$ and for every $Q \in \mathcal{D}$,

$$\|f - S_{\tau\, t_{\hat{B}_Q}} f\|_{L^{r,\infty},Q} \lesssim \sum_{k\geq 0} \sigma^{2\,n\,k}\, g\big(\sigma^{m\,(k-8)}\big)\, a(\sigma^k\, \hat{B}_Q).$$

In order to get it, we define a functional $\tilde{a} : \mathcal{B} \times \mathcal{F} \longrightarrow [0, +\infty)$ given by

$$\tilde{a}(B) = \sum_{k\geq 0} \sigma^{2\,n\,k}\, g\big(\sigma^{m\,(k-8)}\big)\, a(\sigma^k\, B).$$

Fix $Q \in \mathcal{D}$ and assume that $\tilde{a}(\hat{B}_Q) < \infty$, otherwise, there is nothing to prove. Let $G(x) = \big|f(x) - S_{\tau\, t_{\hat{B}_Q}} f(x)\big|\, \chi_{\sigma^2\, \hat{B}_Q}(x)$. The Lebesgue differentiation theorem implies that it is sufficient to estimate $\|MG\|_{L^{r,\infty},Q}$. Thus, we study the level sets $\Omega_t = \{x \in X : MG(x) > t\}$, $t > 0$. We split the proof in two cases. When t is large, we use the Whitney covering lemma (Theorem 5.2 below). When t is small, the estimate is straightforward.

The following auxiliary result will be very useful. Its proof is postponed until Section 5.1.3.

Lemma 5.1. *Assume that $a \in D_1$ and (3.1). For every $1 \le \tau < \sigma^m$, $k \ge 0$ and $R \in \mathcal{D}$, we have*

$$\fint_{\sigma^k \hat{B}_R} |f - S_{\tau\, t_{\hat{B}_R}} f| \, d\mu \le \|a\|_{D_1(\mu)} \, c_\mu^2 \, \sigma^{5\,n} \, (C_1/c_1)^n \, a(\sigma^{k+2} \, \hat{B}_R).$$

Take $c_0 = c_M \, \|a\|_{D_1(\mu)} \, c_\mu^3 \, (C_1/c_1)^{2\,n} \, g(1)^{-1}$, where c_M is the constant of the weak-type $(1,1)$ of M. Then, since $\tilde{a}(\hat{B}_Q) < \infty$ and as a consequence of the previous lemma, we get $G \in L^1(X)$ with

$$\|G\|_{L^1(X)} = \int_{\sigma^2 \, \hat{B}_Q} |f - S_{\tau\, t_{\hat{B}_Q}} f| \, d\mu \le \frac{c_0}{c_M} \, \tilde{a}(\hat{B}_Q) \, \mu(Q). \tag{5.1}$$

Thus, using that M is of weak type $(1,1)$ with constant c_M, we obtain

$$\mu(\Omega_t) \le \frac{c_M}{t} \, \|G\|_{L^1(X)} \le \frac{c_0}{t} \, \tilde{a}(\hat{B}_Q) \, \mu(Q). \tag{5.2}$$

Next, let $q > 1$ be large enough, to be chosen. Our goal is to show the following good-λ inequality: given $0 < \lambda < 1$, for all $t > 0$

$$\mu(\Omega_{q\,t} \cap Q) \lesssim \lambda \, \mu(\Omega_t \cap Q) + \left(\frac{\tilde{a}(\hat{B}_Q)}{\lambda\, t} \right)^r \mu(Q). \tag{5.3}$$

If $0 < t \le c_0 \, c_\mu \, (C_1/c_1)^n \sigma^{2\,n} \, \tilde{a}(\hat{B}_Q)$ and $0 < \lambda < 1$ then (5.3) is trivial:

$$\mu(\Omega_{q\,t} \cap Q) \le \mu(Q) \lesssim \left(\frac{\tilde{a}(\hat{B}_Q)}{\lambda\, t} \right)^r \mu(Q) \lesssim \lambda \, \mu(\Omega_t \cap Q) + \left(\frac{\tilde{a}(\hat{B}_Q)}{\lambda\, t} \right)^r \mu(Q).$$

In order to consider the other case, we need to state the following version of the Whitney covering lemma whose proof is given below.

Theorem 5.2. *Let $t > 0$ and $G \in L^1(X)$. Let $\Omega_t = \{x \in X : MG(x) > t\}$ be a proper open subset of X. Then, there is a family of Whitney cubes $\{Q_i^t\}_i$ such that*

(a) $\Omega_t = \bigcup_i Q_i^t$ *μ-almost everywhere.*

(b) $\{Q_i^t\}_i \subset \mathcal{D}$, *these cubes are maximal with respect to the inclusion and therefore they are pairwise disjoint.*

(c) $0 < (C_1/c_1) \, \sigma^6 \, r(\hat{B}_{Q_i^t}) < d(Q_i^t, \Omega_t^c) \le (1/2) \, (C_1/c_1) \, \sigma^8 \, r(\hat{B}_{Q_i^t})$ *and as a consequence* $\sigma^9 \, (C_1/c_1)^2 \, B_{Q_i^t} \cap \Omega_t^c \ne \emptyset$.

(d) $\fint_{\sigma^k \, \hat{B}_{Q_i^t}} G \, d\mu \lesssim t$, *for all* $k \ge 1$.

(e) $M(G \, \chi_{(\sigma \, \hat{B}_{Q_i^t})^c})(x) \lesssim t$, *for all* $x \in Q_i^t$.

Suppose that $t > c_0\, c_\mu\, (C_1/c_1)^n \sigma^{2\,n}\, \tilde{a}(\hat{B}_Q)$. Note that Ω_t is a level set of the lower semicontinuous function MG. Moreover, as we have already seen, $G \in L^1(X)$ and $\mu(\Omega_t) < \infty$. Thus, Ω_t is an open proper subset of X. Therefore, the set Ω_t can be covered by the family of Whitney cubes $\{Q_i^t\}_i$, by applying Theorem 5.2. From now on we restrict our attention to those cubes Q_i^t with $Q_i^t \cap Q \neq \emptyset$. Notice that as a consequence of (5.2) and $t > c_0\, c_\mu\, (C_1/c_1)^n \sigma^{2\,n}\, \tilde{a}(\hat{B}_Q)$, we have $\mu(\Omega_t) < \mu(Q)$ and therefore $Q_i^t \subsetneq Q$ for every $Q_i^t \cap Q \neq \emptyset$. Also for such cubes, by (5.2), we obtain

$$\begin{aligned}\mu(Q_i^t) \leq \mu(\Omega_t) \leq \frac{c_0}{t}\, \tilde{a}(\hat{B}_Q)\, \mu(Q) \leq \frac{c_0}{t}\, \tilde{a}(\hat{B}_Q)\, c_\mu \left(\frac{r(\hat{B}_Q)}{r(B_{Q_i^t})}\right)^n \mu(Q_i^t) \\ \leq \sigma^{-2\,n} \left(\frac{r(\hat{B}_Q)}{r(\hat{B}_{Q_i^t})}\right)^n \mu(Q_i^t)\end{aligned}$$

and therefore

$$r(\hat{B}_{Q_i^t}) \leq \sigma^{-2}\, r(\hat{B}_Q) \qquad \text{and} \qquad \sigma^2\, \hat{B}_{Q_i^t} \subset \sigma\, \hat{B}_Q. \tag{5.4}$$

We need the following estimate:

Proposition 5.3. *For every $x \in Q_i^t$,*

$$MG(x) \leq M(|f - S_{\tau\, t_{\hat{B}_{Q_i^t}}} f|\, \chi_{\sigma\, \hat{B}_{Q_i^t}})(x) + c_1\, t + c_2\, \tilde{a}(\hat{B}_Q).$$

This estimate gives

$$MG(x) \leq M(|f - S_{\tau\, t_{\hat{B}_{Q_i^t}}} f|\, \chi_{\sigma\, \hat{B}_{Q_i^t}})(x) + C_0\, t. \tag{5.5}$$

We choose q large enough so that $q > C_0$ and take $0 < \lambda < 1$. Using that the level sets are nested, we write

$$\begin{aligned}\mu(\Omega_{q\,t} \cap Q) &= \sum_{i:Q_i^t \subset Q} \mu\big(\{x \in Q_i^t : MG(x) > q\, t\}\big) \\ &\leq \sum_{i:Q_i^t \subset Q} \mu\big(\{x \in Q_i^t : M(|f - S_{\tau\, t_{\hat{B}_{Q_i^t}}} f|\, \chi_{\sigma\, \hat{B}_{Q_i^t}})(x) > (q - C_0)\, t\}\big) \\ &= \sum_{\Gamma_1} \cdots + \sum_{\Gamma_2} \cdots = I + II,\end{aligned} \tag{5.6}$$

where

$$\Gamma_1 = \Big\{Q_i^t \subset Q : \fint_{\sigma\, \hat{B}_{Q_i^t}} |f - S_{\tau\, t_{\hat{B}_{Q_i^t}}} f|\, d\mu \leq \lambda\, t\Big\},\ \Gamma_2 = \Big\{Q_i^t \subset Q : \fint_{\sigma\, \hat{B}_{Q_i^t}} |f - S_{\tau\, t_{\hat{B}_{Q_i^t}}} f|\, d\mu > \lambda\, t\Big\}.$$

Applying that M is of weak type $(1, 1)$, μ doubling and Theorem 5.2, we estimate I:

$$I \lesssim \frac{1}{t} \sum_{\Gamma_1} \int_{\sigma\, \hat{B}_{Q_i^t}} |f - S_{\tau\, t_{\hat{B}_{Q_i^t}}} f|\, d\mu \lesssim \lambda \sum_{i:Q_i^t \subset Q} \mu(Q_i^t) \lesssim \lambda\, \mu(\Omega_t \cap Q).$$

In order to estimate II, we first observe that if $Q_i^t \in \Gamma_2$ (by Lemma 5.1), we have

$$\lambda\, t < \fint_{\sigma\, \hat{B}_{Q_i^t}} |f - S_{\tau\, t_{\hat{B}_{Q_i^t}}} f|\, d\mu \lesssim a(\sigma^3\, \hat{B}_{Q_i^t}).$$

Thus,

$$II \le \sum_{\Gamma_2} \mu(Q_i^t) \lesssim \Big(\frac{1}{\lambda\, t}\Big)^r \sum_{i:Q_i^t \subset Q} a(\sigma^3\, \hat{B}_{Q_i^t})^r\, \mu(Q_i^t).$$

In principle, it is not possible to apply the condition $D_r(\mu)$ since the balls of the family $\{\sigma^3\, \hat{B}_{Q_i^t}\}_i$ may not be pairwise disjoint. Note that by (5.4) we have $\{\sigma^3\, \hat{B}_{Q_i^t}\}_i \subset \sigma^2\, \hat{B}_Q$. Next, we claim that $\{\sigma^3\, \hat{B}_{Q_i^t}\}_i$ splits in N families $\{\mathcal{E}_j\}_{j=1}^N$ of pairwise disjoint balls with $N \le c_\mu\, (C_1/c_1)^{3\,n}\, \sigma^{13\,n}$. Assuming this, we use that $a \in D_r(\mu)$ over each $\mathcal{E}_j$ and the fact that μ is doubling to obtain

$$\begin{aligned} II &\lesssim \Big(\frac{1}{\lambda\, t}\Big)^r \sum_{j=1}^N \sum_{i:Q_i^t \in \mathcal{E}_j} a(\sigma^3\, \hat{B}_{Q_i^t})^r\, \mu(\sigma^3\, \hat{B}_{Q_i^t}) \lesssim \Big(\frac{1}{\lambda\, t}\Big)^r a(\sigma^2\, \hat{B}_Q)^r\, \mu(\sigma^2\, \hat{B}_Q) \\ &\lesssim \Big(\frac{1}{\lambda\, t}\Big)^r \tilde{a}(\hat{B}_Q)^r\, \mu(Q). \end{aligned}$$

Plugging the estimates for I and II into (5.6), we conclude

$$\mu(\Omega_{q\,t} \cap Q) \lesssim \lambda\, \mu(\Omega_t \cap Q) + \Big(\frac{1}{\lambda\, t}\Big)^r \tilde{a}(\hat{B}_Q)^r\, \mu(Q),$$

for all $t > c_0\, c_\mu\, (C_1/c_1)^n \sigma^{2\,n}\, \tilde{a}(\hat{B}_Q)$ provided we check the previous claim. Note that by Lemma 5.4 below it suffices to fix Q_j^t and show that

$$\# E_j := \#\{Q_i^t : \sigma^3\, \hat{B}_{Q_i^t} \cap \sigma^3\, \hat{B}_{Q_j^t} \ne \emptyset\} \le c_\mu\, (C_1/c_1)^{3\,n}\, \sigma^{13\,n}.$$

As a consequence of Theorems 2.1 and 5.2, for any $Q_i^t \in E_j$ we have

$$0 < \sigma^5\, r(\hat{B}_{Q_i^t}) < d(\sigma^3\, \hat{B}_{Q_i^t}, \Omega_t^c) \le \sigma^8\, (C_1/c_1)\, r(\hat{B}_{Q_i^t}).$$

Then it is easy to see that

$$\sigma^{-4}\, (C_1/c_1)\, r(\hat{B}_{Q_i^t}) \le r(\hat{B}_{Q_j^t}) \le \sigma^4\, (C_1/c_1)\, r(\hat{B}_{Q_i^t})$$

and

$$Q_i^t \subset \sigma^8\, (C_1/c_1)\, \hat{B}_{Q_j^t} \subset \sigma^{13}\, (C_1/c_1)^2\, \hat{B}_{Q_i^t}.$$

Using this estimates we obtain

$$\begin{aligned} \mu(\sigma^8\, (C_1/c_1)\, \hat{B}_{Q_j^t})\, \# E_j &\le \sum_{Q_i^t \in E_j} \mu(\sigma^{13}\, (C_1/c_1)^2\, \hat{B}_{Q_i^t}) \le c_\mu\, \sigma^{13\,n} (C_1/c_1)^{3\,n} \sum_{Q_i^t \in E_j} \mu(Q_i^t) \\ &\le c_\mu\, \sigma^{13\,n} (C_1/c_1)^{3\,n}\, \mu(\cup_{Q_i^t \in E_j} Q_i^t) \le c_\mu\, \sigma^{13\,n} (C_1/c_1)^{3\,n}\, \mu(\sigma^8\, (C_1/c_1)\, \hat{B}_{Q_j^t}) \end{aligned}$$

and this readily leads to the desired bound for $\# E_j$.

Next, we fix $N > 0$. Note that the good-λ inequality (5.3) implies

$$\begin{aligned} \sup_{0<t\le N/q} t^r\, \frac{\mu(\Omega_{q\,t} \cap Q)}{\mu(Q)} &\le c\,\lambda \sup_{0<t\le N/q} t^r\, \frac{\mu(\Omega_t \cap Q)}{\mu(Q)} + c\left(\frac{\tilde{a}(\hat{B}_Q)}{\lambda}\right)^r \\ &\le c\,\lambda \sup_{0<t\le N} t^r\, \frac{\mu(\Omega_t \cap Q)}{\mu(Q)} + c\left(\frac{\tilde{a}(\hat{B}_Q)}{\lambda}\right)^r. \end{aligned}$$

Hence, we have

$$\sup_{0<t\le N} t^r\, \frac{\mu(\Omega_t \cap Q)}{\mu(Q)} \le c\,\lambda\, q^r \sup_{0<t\le N} t^r\, \frac{\mu(\Omega_t \cap Q)}{\mu(Q)} + c\, q^r \left(\frac{\tilde{a}(\hat{B}_Q)}{\lambda}\right)^r. \tag{5.7}$$

We observe that

$$\sup_{0<t\le N} t^r \frac{\mu(\Omega_t\cap Q)}{\mu(Q)} \le N^r < \infty.$$

Thus, if we take $\lambda > 0$ small enough, we can hide the first term in the right side of (5.7) and get

$$\sup_{0<t\le N} t^r \frac{\mu(\Omega_t\cap Q)}{\mu(Q)} \lesssim \tilde{a}(\hat{B}_Q)^r.$$

Taking limits as $N\to\infty$, we conclude

$$\|MG\|_{L^{r,\infty},Q} \lesssim \tilde{a}(\hat{B}_Q).$$

This estimate and the Lebesgue differentiation theorem yield the desired inequality, as observed at the beginning of the proof. □

5.1.2. *Step II: General case.* Fix a ball B. Let $k_0\in\mathbb{Z}$ be such that $C_1\,\sigma^{k_0}\le r(B) < C_1\,\sigma^{k_0+1}$ and $\mathcal{I}=\{Q\in\mathcal{D}_{k_0} : Q\cap B\neq\emptyset\}$. For every $Q\in\mathcal{I}$ it is easy to see that $\hat{B}_Q\subset\sigma\,B\subset\sigma^3\hat{B}_Q$. Then,

$$\mu(\sigma\,B)\,\#\mathcal{I} \le \sum_{Q\in\mathcal{I}} \mu(\sigma^3\hat{B}_Q) \le c_\mu\,\sigma^{3\,n}\,(C_1/c_1)^n\,\mu(\cup_{Q\in\mathcal{I}}Q) \le c_\mu\,\sigma^{3\,n}\,(C_1/c_1)^n\,\mu(\sigma\,B)$$

which leads to $\#\mathcal{I}\le c_\mu\,\sigma^{3\,n}\,(C_1/c_1)^n$. Note that $\mu(B)\approx\mu(Q)$ and also $t_B=\tau\,t_{\hat{B}_Q}$ with $1\le\tau<\sigma^m$. Then, the first part of the proof yields

$$\begin{aligned}\|f-S_{t_B}f\|_{L^{r,\infty},B} &\lesssim \sum_{Q\in\mathcal{I}} \|f-S_{t_B}f\|_{L^{r,\infty},Q}\\ &\lesssim \sum_{Q\in\mathcal{I}}\sum_{k\ge 0} \sigma^{2\,n\,k}\,g\big(\sigma^{m\,(k-8)}\big)\,a(\sigma^k\,\hat{B}_Q)\\ &\lesssim \sum_{k\ge 0} \sigma^{2\,n\,k}\,g\big(\sigma^{m\,(k-9)}\big)\,a(\sigma^k\,B).\end{aligned}$$

In the last estimate we have used that $\sigma^k\,\hat{B}_Q\subset\sigma^{k+1}\,B\subset\sigma^{k+3}\,\hat{B}_Q$ and that $a(\sigma^k\,\hat{B}_Q)\lesssim a(\sigma^{k+1}\,B)$: by $a\in D_1$,

$$a(\sigma^k\,\hat{B}_Q)\,\mu(\sigma^k\,\hat{B}_Q) \le \|a\|_{D_1}\,a(\sigma^{k+1}\,B)\,\mu(\sigma^{k+1}\,B) \le \|a\|_{D_1}\,c_\mu\,\sigma^{3\,n}\,a(\sigma^{k+1}\,B)\,\mu(\sigma^k\,\hat{B}_Q).$$

□

5.1.3. *Proofs of the auxiliary results.*

Lemma 5.4. *Let $N\ge 2$ and let $\mathcal{E}=\{E_j\}_j$ be a sequence of sets such that its overlapping is at most N, that is,*

$$\sup_j \#\{E_k : E_k\cap E_j\neq\emptyset\}\le N.$$

Then, there exist $\tilde{N}$ pairwise disjoint subfamilies $\mathcal{E}_k\subset\mathcal{E}$ comprised of disjoint sets so that $\mathcal{E}=\cup_{k=1}^{\tilde{N}}\mathcal{E}_k$ and $\tilde{N}\le N$.

Proof. By the axiom of choice we first take any set in $\mathcal{E}$. Then, we select another set among those that do not meet the one just chosen. We continue until there is not set to be chosen. All these selected sets define $\mathcal{E}_1$. We repeat this on $\mathcal{E}\setminus\mathcal{E}_1$ and obtain $\mathcal{E}_2$. Iterating this procedure we have a collection families $\{\mathcal{E}_k\}_{k=1}^{\tilde{N}}$, each of them being comprised of disjoint sets from $\mathcal{E}$. To estimate $\tilde{N}$ we suppose that we have already chosen $\mathcal{E}_1, \dots \mathcal{E}_N$ and that there is set $E_j \notin \mathcal{E}_k$, $1 \le k \le N$. Thus for every $1 \le k \le N$ there exists $E_k \in \mathcal{E}_k$ such that $E_j \cap E_k \neq \emptyset$ which violates our hypothesis since E_j meets $N+1$ sets. This shows that $\tilde{N} \le N$. $\square$

Lemma 5.5. *Let $R \in \mathcal{D}_{k_0}$ for some $k_0 \in \mathbb{Z}$, and set $\mathcal{J}_k = \{Q \in \mathcal{D}_{k_0} : Q \cap \sigma^k \hat{B}_R \neq \emptyset\}$ with $k \ge 0$. Then*

$$\sigma^k \hat{B}_R \subset \bigcup_{Q \in \mathcal{J}_k} Q \subset \bigcup_{Q \in \mathcal{J}_k} \hat{B}_Q \subset \sigma^{k+1} \hat{B}_R, \qquad \mu\text{-a.e.}, \tag{5.8}$$

and

$$\#\mathcal{J}_k \le c_\mu \, \sigma^{(k+2)\,n} \, (C_1/c_1)^n. \tag{5.9}$$

Also, given $1 \le \tau \le \sigma^m$, for each fixed $Q_0 \in \mathcal{J}_k$, we have

$$\#\mathcal{I}_k = \#\{Q \in \mathcal{J}_k : \tau^{1/m} \hat{B}_Q \cap \tau^{1/m} \hat{B}_{Q_0} \neq \emptyset\} \le c_\mu \, \sigma^{3\,n} \, (C_1/c_1)^n. \tag{5.10}$$

Proof. Note that (5.8) follows easily from Theorem 2.1. It is easy to see that for every $Q \in \mathcal{J}_k$ we have $\sigma^{k+1} \hat{B}_R \subset \sigma^{k+2} \hat{B}_Q$. Then, all these give

$$\begin{aligned}
\mu(\sigma^{k+1} \hat{B}_R) \, \#\mathcal{J}_k &\le \sum_{Q \in \mathcal{J}_k} \mu(\sigma^{k+2} \hat{B}_Q) \le c_\mu \, \sigma^{(k+2)\,n} \, (C_1/c_1)^n \sum_{Q \in \mathcal{J}_k} \mu(Q) \\
&\le c_\mu \, \sigma^{(k+2)\,n} \, (C_1/c_1)^n \, \mu(\cup_{Q \in \mathcal{J}_k} Q) \le c_\mu \, \sigma^{(k+2)\,n} \, (C_1/c_1)^n \, \mu(\sigma^{k+1} \hat{B}_R),
\end{aligned}$$

and this readily implies (5.9).

Next we observe that for every $Q \in \mathcal{I}_k$ we have $Q \subset \sigma^2 \hat{B}_{Q_0} \subset \sigma^3 \hat{B}_Q$. Then, proceeding as before we conclude (5.10):

$$\begin{aligned}
\mu(\sigma^2 \hat{B}_{Q_0}) \, \#\mathcal{I}_k &\le \sum_{Q \in \mathcal{I}_k} \mu(\sigma^3 \hat{B}_Q) \le c_\mu \, \sigma^{3\,n} \, (C_1/c_1)^n \sum_{Q \in \mathcal{I}_k} \mu(Q) \\
&\le c_\mu \, \sigma^{3\,n} \, (C_1/c_1)^n \, \mu(\cup_{Q \in \mathcal{I}_k} Q) \le c_\mu \, \sigma^{3\,n} \, (C_1/c_1)^n \, \mu(\sigma^2 \hat{B}_{Q_0}).
\end{aligned}$$

$\square$

Proof of Lemma 5.1. Fix $R \in \mathcal{D}_{k_0}$ for some $k_0 \in \mathbb{Z}$, $k \ge 0$ and $1 \le \tau < \sigma^m$. Then, (5.8) implies

$$\begin{aligned}
\int_{\sigma^k \hat{B}_R} |f - S_{\tau\, t_{\hat{B}_R}} f| \, d\mu &\le \sum_{Q \in \mathcal{J}_k} \int_{\tau^{1/m} \hat{B}_Q} |f - S_{t_{\tau^{1/m} \hat{B}_Q}} f| \, d\mu \\
&\le \sum_{Q \in \mathcal{J}_k} a(\tau^{1/m} \hat{B}_Q) \, \mu(\tau^{1/m} \hat{B}_Q) \\
&\le \|a\|_{D_1(\mu)} \, c_\mu \, \sigma^{3\,n} \, (C_1/c_1)^n \, a(\sigma^{k+2} \hat{B}_R) \, \mu(\sigma^{k+2} \hat{B}_R) \\
&\le \|a\|_{D_1(\mu)} \, c_\mu^2 \, \sigma^{5\,n} \, (C_1/c_1)^n \, a(\sigma^{k+2} \hat{B}_R) \, \mu(\sigma^k \hat{B}_R).
\end{aligned}$$

Note that we have used that $\{\tau^{1/m} \hat{B}_Q\}_{Q \in \mathcal{J}_k} \subset \sigma^{k+2} \hat{B}_R$, (5.10), Lemma 5.4 and that $a \in D_1(\mu)$. $\square$

Proof of Theorem 5.2. Items (a)–(d) follow as in the proof of Whitney covering lemma of [Jim, Theorem 5.3, Lemma 5.4] with the difference that now, for every $k \in \mathbb{Z}$, we take

$$\Omega_k = \Big\{x \in \Omega : C_1 \, \frac{C_1}{c_1} \, \sigma^{k+6} < d(x, \Omega^c) \le C_1 \, \frac{C_1}{c_1} \, \sigma^{k+7}\Big\}.$$

On the other hand, (e) follows from (c): Fix Q_i^t and $x \in Q_i^t$, by (c) we can take $z \in \sigma^9\,(C_1/c_1)^2\,B_{Q_i^t} \cap \Omega_t^c$. Let $B \ni x$ be such that $B \cap (\sigma\,\hat{B}_{Q_i^t})^c \neq \emptyset$. Thus, $z \in (C_1/c_1)^2\,\sigma^{10}\,B$ and using that μ is doubling, we have

$$\fint_B G\chi_{(\sigma\,\hat{B}_{Q_i^t})^c}\,d\mu \le c_\mu\,(C_1/c_1)^{2\,n}\,\sigma^{10\,n} \fint_{(C_1/c_1)^2\,\sigma^{10}\,B} G\,d\mu \lesssim MG(z) \lesssim t,$$

since $z \in \Omega_t^c$. Observe that this inequality holds for any ball B such that $B \ni x$ and $B \cap (\sigma\,\hat{B}_{Q_i^t})^c \neq \emptyset$. Taking the supremum over these balls, the desired estimate is proved. $\square$

Proof of Proposition 5.3. We claim that for every $x \in \sigma\,\hat{B}_{Q_i^t}$,

$$\big|S_{\tau\,\hat{B}_{t_{Q_i^t}}} f(x) - S_{\tau\,t_{\hat{B}_Q}} f(x)\big| \lesssim t + \tilde{a}(\hat{B}_Q),$$

Then, (e) in Theorem 5.2 leads us to the desired estimate: for every $x \in Q_i^t$,

$$MG(x) \le M(G\chi_{(\sigma\,\hat{B}_{Q_i^t})^c})(x) + M(G\chi_{\sigma\,\hat{B}_{Q_i^t}})(x) \lesssim t + \tilde{a}(\hat{B}_Q) + M\big(|f - S_{\tau\,t_{\hat{B}_{Q_i^t}}} f|\chi_{\sigma\,\hat{B}_{Q_i^t}}\big)(x).$$

We show our claim. Note that the commutation rule implies

$$\begin{aligned}\big|S_{\tau\,t_{\hat{B}_{Q_i^t}}} f(x) - S_{\tau\,t_{\hat{B}_Q}} f(x)\big| &\le \big|S_{\tau\,t_{\hat{B}_{Q_i^t}}}\big(f - S_{\tau\,t_{\hat{B}_Q}} f\big)(x)\big| + \big|S_{\tau\,t_{\hat{B}_Q}}\big(f - S_{\tau\,t_{\hat{B}_{Q_i^t}}} f\big)(x)\big| \\ &= I + II.\end{aligned}$$

We study each term in turn. Fix $x \in \sigma\,\hat{B}_{Q_i^t}$ and pick $k_i \in \mathbb{Z}$ such that

$$\sigma^{k_i}\,r(\hat{B}_{Q_i^t}) \le r(\hat{B}_Q) < \sigma^{k_i+1}\,r(\hat{B}_{Q_i^t}). \tag{5.11}$$

Thus, using (5.4), we have

$$k_i \ge 2 \qquad \text{and} \qquad \sigma^{k_i}\,\hat{B}_{Q_i^t} \subset \sigma\,\hat{B}_Q. \tag{5.12}$$

This implies that $\big|f(y) - S_{\tau\,t_{\hat{B}_Q}} f(y)\big| = G(y)$, when $y \in \sigma^{k_i}\,\hat{B}_{Q_i^t}$. Therefore, since $1 \le \tau < \sigma^m$, we can write

$$\begin{aligned}I &\le \frac{1}{\mu\big(B(x, r(\hat{B}_{Q_i^t}))\big)} \int_X g\bigg(\frac{d(x,y)^m}{\tau\,t_{\hat{B}_{Q_i^t}}}\bigg)\,|f(y) - S_{\tau\,t_{\hat{B}_Q}} f(y)|\,d\mu(y) \\ &\le \frac{1}{\mu\big(B(x, r(\hat{B}_{Q_i^t}))\big)} \int_X g\bigg(\frac{d(x,y)^m}{\tau\,t_{\hat{B}_{Q_i^t}}}\bigg)\,G(y)\,d\mu(y) \\ &\quad + \frac{1}{\mu\big(B(x, r(\hat{B}_{Q_i^t}))\big)} \int_{(\sigma^{k_i}\,\hat{B}_{Q_i^t})^c} \cdots\,d\mu(y) \\ &= I_1 + I_2.\end{aligned} \tag{5.13}$$

To take advantage of the decay of g we decompose X as the union of dyadic annuli $\{C_k(Q_i^t)\}_{k\ge 2}$. Thus, if $x \in \sigma\,\hat{B}_{Q_i^t}$ and $y \in C_k(Q_i^t)$, we have

$$\frac{d(x,y)^m}{\tau\,t_{\hat{B}_{Q_i^t}}} \ge \lambda_k \qquad \text{where} \qquad \lambda_k = \begin{cases} 0, & \text{if } k = 2, \\ \sigma^{m\,(k-3)}, & \text{if } k \ge 3. \end{cases}$$

Also for every $k \geq 2$, we have $\sigma^k \hat{B}_{Q_i^t} \subset \sigma^{k+1} B(x, r(\hat{B}_{Q_i^t}))$. Then, using that μ is doubling, the decay of g and applying (d) in Theorem 5.2, we obtain

$$I_1 \lesssim \sum_{k\geq 2} \sigma^{n\,k}\, g(\lambda_k) \fint_{\sigma^k \hat{B}_{Q_i^t}} G\, d\mu \lesssim t \sum_{k\geq 2} g(\lambda_k)\, \sigma^{n\,k} \lesssim t.$$

To estimate I_2 we note that $Q_i^t \subset Q$, (5.11) and (5.12) imply the following: for every $k \geq k_i + 1$

$$C_k(Q_i^t) \subset \sigma^k \hat{B}_{Q_i^t} \subset \sigma^{k-k_i+1} \hat{B}_Q \subset \sigma^{k-1} \hat{B}_Q \subset \sigma^{k+k_i+1} B\big(x, r(\hat{B}_{Q_i^t})\big), \tag{5.14}$$

with $x \in \sigma\, \hat{B}_{Q_i^t}$. Therefore, arguing as in Lemma 5.1 and using that $a \in D_1(\mu)$ and μ doubling, we get

$$\begin{aligned} I_2 &\leq \frac{1}{\mu\big(B(x, r(\hat{B}_{Q_i^t}))\big)} \sum_{k\geq k_i+1} g(\lambda_k) \int_{\sigma^{k-k_i+1} \hat{B}_Q} |f - S_{\tau\, t_{\hat{B}_Q}} f|\, d\mu \\ &\lesssim \sum_{k\geq k_i+1} \sigma^{n\,(k+k_i)}\, g\big(\sigma^{m\,(k-3)}\big)\, a(\sigma^{k+1} \hat{B}_Q) \lesssim \sum_{k\geq 3} \sigma^{2\,n\,k}\, g\big(\sigma^{m\,(k-3)}\big)\, a(\sigma^{k+1} \hat{B}_Q) \lesssim \tilde{a}(\hat{B}_Q). \end{aligned}$$

Collecting all the estimates, we obtain $I \lesssim t + \tilde{a}(\hat{B}_Q)$.

Next, let us show that $II \lesssim \tilde{a}(\hat{B}_Q)$. Notice that by (5.14), $\sigma^k \hat{B}_{Q_i^t} \subset \sigma^{k-k_i+1} \hat{B}_Q \subset \sigma^{k-k_i+2} B(x, r(\hat{B}_Q))$, $k \geq k_i + 1$, and then, proceeding as in Lemma 5.1 and using that μ is doubling, we obtain

$$\begin{aligned} II &\leq \frac{1}{\mu\big(B(x, r(\hat{B}_Q))\big)} \int_X g\bigg(\frac{d(x,y)^m}{\tau\, t_{\hat{B}_Q}}\bigg) \big|f(y) - S_{\tau\, t_{\hat{B}_{Q_i^t}}} f(y)\big|\, d\mu(y) \\ &\leq \frac{g(0)}{\mu\big(B(x, r(\hat{B}_Q))\big)} \int_{\sigma^{k_i+1} \hat{B}_{Q_i^t}} |f - S_{\tau\, t_{\hat{B}_{Q_i^t}}} f|\, d\mu \\ &\quad + \frac{1}{\mu\big(B(x, r(\hat{B}_Q))\big)} \sum_{k\geq k_i+2} g\bigg(\frac{\lambda_k\, t_{\hat{B}_{Q_i^t}}}{\tau\, t_{\hat{B}_Q}}\bigg) \int_{\sigma^k \hat{B}_{Q_i^t}} |f - S_{\tau\, t_{\hat{B}_{Q_i^t}}} f|\, d\mu \\ &\lesssim a(\sigma^4 \hat{B}_Q) + \sum_{k\geq k_i+2} g\big(\sigma^{m\,(k-k_i-5)}\big)\, \sigma^{n\,(k-k_i)}\, a(\sigma^{k-k_i+3} \hat{B}_Q) \\ &\lesssim \sum_{k\geq 2} \sigma^{n\,k}\, g\big(\sigma^{m\,(k-8)}\big)\, a(\sigma^k \hat{B}_Q) \lesssim \tilde{a}(\hat{B}_Q). \end{aligned}$$

□

5.2. **Proof of Theorem 3.2.** We follow the steps of the proof of Theorem 3.1. So, we only detail those points where both proofs are different. We recall that $w \in A_\infty(\mu)$ implies that there exist $1 < p, s < \infty$ such that $w \in A_p(\mu) \cap RH_s(\mu)$. In particular, for any ball B and any measurable set $S \subset B$,

$$\bigg(\frac{\mu(S)}{\mu(B)}\bigg)^p \lesssim \frac{w(S)}{w(B)} \lesssim \bigg(\frac{\mu(S)}{\mu(B)}\bigg)^{1/s'}. \tag{5.15}$$

The first inequality follows from $w \in A_p(\mu)$ and the second one from $w \in RH_s(\mu)$ (see [ST]). Note that in particular, this yields that w is doubling.

We fix $Q \in \mathcal{D}$ and suppose that $\tilde{a}(\hat{B}_Q) < \infty$ where

$$\tilde{a}(\hat{B}_Q) = \sum_{k\geq 0} \sigma^{2\,n\,k}\, g\big(\sigma^{m\,(k-9)}\big)\, a(\sigma^k\, \hat{B}_Q).$$

Set G and Ω_t as before, for all $t > 0$. Then, as we have assumed that $a \in D_1(\mu)$, we have (5.1) and (5.2). Taking $q > 1$ large enough, we show the following weighted version of (5.3): given $0 < \lambda < 1$, for all $t > 0$,

$$w(\Omega_{q\,t} \cap Q) \lesssim \lambda^{1/s'}\, w(\Omega_t \cap Q) + \left(\frac{\tilde{a}(\hat{B}_Q)}{\lambda\, t}\right)^r w(Q). \tag{5.16}$$

With this in hand, the proof follows the steps of Theorem 3.1. We explain how to obtain (5.16). If $0 < t \lesssim \tilde{a}(\hat{B}_Q)$ this estimate is trivial, since

$$w(\Omega_{q\,t} \cap Q) \leq w(Q) \lesssim \left(\frac{\tilde{a}(\hat{B}_Q)}{\lambda\, t}\right)^r w(Q).$$

Let us consider the case $t \gtrsim \tilde{a}(Q)$. Notice that $G \in L^1(X)$ and $\mu(\Omega_t) < \infty$, by (5.2). Then, by Theorem 5.2, we write Ω_t as the μ-a.e. union of Whitney cubes $\{Q_i^t\}_i$. Arguing as before, we obtain

$$\begin{aligned} w(\Omega_{q\,t} \cap Q) &\leq \sum_{i:Q_i^t \subset Q} w\big(\{x \in Q_i^t : M(|f - S_{\tau\, t_{\hat{B}_{Q_i^t}}} f|\, \chi_{\sigma\, \hat{B}_{Q_i^t}})(x) > (q - C_0)\, t\}\big) \\ &= \sum_{\Gamma_1} \cdots + \sum_{\Gamma_2} \cdots = I + II. \end{aligned}$$

To estimate I we use (5.15), that M is of weak type $(1,1)$, μ is doubling and Theorem 5.2:

$$\begin{aligned} I &\lesssim \sum_{\Gamma_1} \left(\frac{\mu\big(\{x \in Q_i^t : M(|f - S_{\tau\, t_{\hat{B}_{Q_i^t}}} f|\, \chi_{\sigma\, \hat{B}_{Q_i^t}})(x) > (q - C_0)\, t\}\big)}{\mu(Q_i^t)}\right)^{1/s'} w(Q_i^t) \\ &\lesssim \frac{1}{t^{1/s'}} \sum_{\Gamma_1} \left(\fint_{\sigma\, \hat{B}_{Q_i^t}} |f - S_{\tau\, t_{\hat{B}_{Q_i^t}}} f|\, d\mu\right)^{1/s'} w(Q_i^t) \\ &\lesssim \lambda^{1/s'} \sum_{i:Q_i^t \subset Q} w(Q_i^t) \lesssim \lambda^{1/s'}\, w(\Omega_t \cap Q). \end{aligned}$$

On the other hand, following the computations to estimate II in the proof of Theorem 3.1 (replacing the Lebesgue measure by w) and using Lemma 5.1, we conclude that

$$II \lesssim \left(\frac{a(\hat{B}_Q)}{\lambda\, t}\right)^r w(Q) \lesssim \left(\frac{\tilde{a}(\hat{B}_Q)}{\lambda\, t}\right)^r w(Q).$$

Note that we have used that w is doubling and that $a \in D_r(w) \cap D_1(\mu)$. Collecting the obtained estimates for I and II, we obtain (5.16) and therefore the proof is completed. $\square$

5.3. **Proof of Theorem 3.5.** We have to modify the previous argument: when passing from the dyadic case to the general case we used that $a \in D_1$ —indeed $a \in D_1$ implies $a(B_1) \lesssim a(B_2)$ if $B_1 \subset B_2 \subset \sigma^3 B_1$—. Here we do not have such property (unless we assume $\bar{a} \in D_1$) but we can use the following observation: if $(a, \bar{a}) \in D_r(\mu)$ then for all balls B, $\tilde{B}$ such that $B \subset \tilde{B}$, and for any family of pairwise disjoint balls $\{B_i\}_i \subset B$ we have

$$\sum_i a(B_i)^r \, \mu(B_i) \lesssim \bar{a}(\tilde{B})^r \, \mu(\tilde{B}). \tag{5.17}$$

We follow the lines in the proof of Theorem 3.1 pointing out the main changes. We start as in Step II and cover B with the dyadic cubes in $\mathcal{I}$. As the cardinal of $\mathcal{I}$ is controlled by a geometric constant, it suffices to get the desired estimate for a fixed cube $Q \in \mathcal{I}$. As mentioned before for every $k \geq 0$ we have $\sigma^k \hat{B}_Q \subset \sigma^{k+1} B$. We take $\tilde{a}$ given by

$$\tilde{a}(B) = \sum_{k \geq 0} \sigma^{2\,n\,k} \, g\big(\sigma^{m\,(k-9)}\big) \, \bar{a}(\sigma^k B).$$

Using that $(a, \bar{a})$ satisfies (3.3), we can see (as in the proof of Lemma 5.1) that for each $R \in \mathcal{D}$, $1 \leq \tau < \sigma^m$ and $k \geq 1$,

$$\fint_{\sigma^k \hat{B}_R} |f - S_{\tau\, t_{\hat{B}_R}} f| \, d\mu \lesssim \bar{a}(\sigma^{k+2} \hat{B}_R). \tag{5.18}$$

Furthermore, when $R = Q$ using that $\sigma^{k+2} \hat{B}_Q \subset \sigma^{k+3} B \subset \sigma^{k+5} \hat{B}_Q$, $\mu(\sigma^{k+3} B) \lesssim \mu(\sigma^k \hat{B}_Q)$ and (5.17), we can analogously obtain

$$\fint_{\sigma^k \hat{B}_Q} |f - S_{\tau\, t_{\hat{B}_Q}} f| \, d\mu \lesssim \bar{a}(\sigma^{k+3} B). \tag{5.19}$$

This implies that $G = |f - S_{\tau\, t_{\hat{B}_Q}} f| \, \chi_{\sigma^2 \hat{B}_Q} \in L^1(X)$ with $\|G\|_{L^1(X)} \lesssim \tilde{a}(B) \, \mu(Q)$. Also Ω_t, the t-level set of MG, satisfies $\mu(\Omega_t) \lesssim \tilde{a}(B) \, \mu(Q)/t$.

Our goal is to show the following good-λ type inequality: given $0 < \lambda < 1$, for all $t > 0$

$$\mu(\Omega_{q\,t} \cap Q) \lesssim \lambda \, \mu(\Omega_t \cap Q) + \left(\frac{\tilde{a}(B)}{\lambda\, t}\right)^r \mu(Q). \tag{5.20}$$

From here we obtain as before $\|MG\|_{L^{r,\infty},Q} \lesssim \tilde{a}(B)$ which in turn implies the desired estimate:

$$\|f - S_{t_B} f\|_{L^{r,\infty},B} \lesssim \sum_{Q \in \mathcal{I}} \|f - S_{t_B} f\|_{L^{r,\infty},Q} \leq \sum_{Q \in \mathcal{I}} \|MG\|_{L^{r,\infty},Q} \lesssim \tilde{a}(B) \, \#\mathcal{I} \lesssim \tilde{a}(B).$$

Notice that (5.20) is trivial if $0 < t \lesssim \tilde{a}(B)$. Otherwise, proceeding as before and using the ideas that led us to (5.18), (5.19) we can obtain an analog of Proposition 5.3 with $\tilde{a}(B)$ in the right hand side, which is written in terms of $\bar{a}$ in place of a. All these together yield (5.6). The estimate for I is done exactly as before. For II, we use the same ideas, but in this case, we do not want to use (5.18), because this would drive us to $\bar{a}$ before using (3.3). By applying Lemma 5.5, and proceeding as in Lemma 5.1, for every $Q_i^t \in \Gamma_2$ we take the family $\mathcal{J}(Q_i^t) = \mathcal{J}_1(Q_i^t)$ and obtain

(5.21)
$$\lambda\, t < \fint_{\sigma\, \hat{B}_{Q_i^t}} |f - S_{\tau\, t_{\hat{B}_{Q_i^t}}} f|\, d\mu \lesssim \sum_{R\in\mathcal{J}(Q_i^t)} \fint_{\tau^{1/m}\, \hat{B}_R} |f - S_{t_{\tau^{1/m}\, \hat{B}_R}} f|\, d\mu \le \sum_{R\in\mathcal{J}(Q_i^t)} a(\tau^{1/m}\, \hat{B}_R).$$

This and the fact that $\#\mathcal{J}(Q_i^t) \le C$ give

$$II \le \sum_{\Gamma_2} \mu(\hat{B}_{Q_i^t}) \lesssim \sum_{i:Q_i^t\subset Q} \sum_{R\in\mathcal{J}(Q_i^t)} \left(\frac{a(\tau^{1/m}\, \hat{B}_R)}{\lambda\, t}\right)^r \mu(\tau^{1/m}\, \hat{B}_R).$$

As before, we split the balls $\{\sigma^3\, \hat{B}_{Q_i^t}\}_i$ in K families $\{\mathcal{E}_k\}_{k=1}^K$ of pairwise disjoint balls. For every Q_i^t, by (5.10) and Lemma 5.4 we can split the family $I(Q_i^t) = \{\tau^{1/m}\, \hat{B}_R : R \in \mathcal{J}(Q_i^t)\}$ in $\{\mathcal{I}(Q_i^t)_j\}_{j=1}^{J_{Q_i^t}}$ disjoint families of disjoint subsets. Notice that $J_{Q_i^t} \le c_\mu\, \sigma^{3\,n}\, (C_1/c_1)^n$. Write $J = \max J_{Q_i^t}$ and set $\mathcal{I}(Q_i^t)_j = \emptyset$ for $J_{Q_i^t} < j \le J$. In this way, for every Q_i^t we have split $I(Q_i^t)$ in J pairwise disjoint families (some of them might be empty) so that for in each family the corresponding balls (if any) are pairwise disjoint. Notice that for each fixed $1 \le k \le K$, $1 \le j \le J$, we have that $\{\tau^{1/m}\, \hat{B}_R : R \in \mathcal{I}(Q_i^t)_j, Q_i^t \in \mathcal{E}_k\}$ is a disjoint family since so it is for a fixed Q_i^t, $\tau^{1/m}\, \hat{B}_R \subset \sigma^3\, \hat{B}_{Q_i^t}$, and $\{\sigma^3\, \hat{B}_{Q_i^t} : Q_i^t \in \mathcal{E}_k\}$ is also a disjoint family. Then, we use (5.17) and the fact $\tau^{1/m}\, \hat{B}_R \subset \sigma^3\, \hat{B}_{Q_i^t} \subset \sigma^2\, \hat{B}_Q \subset \sigma^3\, B$:

$$\begin{aligned} II &\lesssim \frac{1}{(\lambda\, t)^r} \sum_{k=1}^K \sum_{j=1}^J \sum_{R\in\mathcal{I}(Q_i^t), Q_i^t\in\mathcal{E}_k} \left(\frac{a(\tau^{1/m}\, \hat{B}_R)}{\lambda\, t}\right)^r \mu(\tau^{1/m}\, \hat{B}_R) \\ &\lesssim \frac{J\cdot K}{(\lambda\, t)^r}\, \bar{a}(\sigma^3\, B)^r\, \mu(\sigma^3\, B) \lesssim \Big(\frac{\tilde{a}(B)}{\lambda\, t}\Big)^r \mu(Q) \end{aligned}$$

From here one gets the good-lambda type inequality (5.20). Further details are left to the interested reader. ☐

5.4. **Proof of Proposition 4.1.** We adapt the argument in [JM] to the present situation. Fix $1 < q < p^*$. Let us recall that Hölder's inequality yields that the D_q conditions are decreasing, thus we can assume without loss of generality that $p \le q < p^*$. Fix a ball B and a family $\{B_i\}_i \subset B$ of pairwise disjoint balls. Minkowski's inequality and the fact that $q \ge p$ give

$$\begin{aligned} \Big(\sum_i a(B_i)^q\, \mu(B_i)\Big)^{1/q} &\le \sum_{k\ge 0} \alpha(k) \Big(\sum_i a_0(\sigma^k\, B_i)^q\, \mu(B_i)\Big)^{1/q} \\ &\le \sum_{k\ge 0} \alpha(k) \Big(\sum_i \frac{r(\sigma^k\, B_i)^p\, \mu(B_i)^{p/q}}{\mu(\sigma^k\, B_i)} \int_{2^k\, B_i} h^p\, d\mu\Big)^{\frac{1}{p}}. \end{aligned} \tag{5.22}$$

We estimate the inner sum as follows. First, if $k = 0$ we use $p \le q < p^*$, (4.2) and that the balls $B_i \subset B$ are pairwise disjoint:

$$\sum_i \frac{r(B_i)^p\, \mu(B_i)^{p/q}}{\mu(B_i)} \int_{B_i} h^p\, d\mu \lesssim \frac{r(B)^p}{\mu(B)^{1-p/q}} \int_B h^p\, d\mu = \mu(B)^{p/q} a_0(B)^p.$$

For $k \ge 1$ we arrange the balls according to their radii and give an estimate of the overlapping whose proof is given below:

Lemma 5.6. *Let B be a ball, $l \geq 0$ and $\mathcal{E}_l = \{B_i\}_i$ be a family of pairwise disjoint balls of B with $\sigma^{-l}\, r(B) < r(B_i) \leq \sigma^{-l+1}\, r(B)$. Given $B_i \in \mathcal{E}_l$ and $k \geq 1$, we have*

$$\#\mathcal{J}_k(B_i) = \#\{B_j \in \mathcal{E}_l : \sigma^k\, B_j \cap \sigma^k\, B_i \neq \emptyset\} \leq C_\mu\, \sigma^{n\,(k+2)}.$$

In addition, for every $B_i \in \mathcal{E}_l$, and $k \geq 1$, if $0 \leq l \leq k+1$, then $\sigma^k\, B_i \subset \sigma^{k-l+2}\, B$, and if $l \geq k+2$ then $\sigma^k B_i \subset \sigma\, B$.

For every $l \geq 0$, we write $\mathcal{E}_l = \{B_i : \sigma^{-l}\, r(B) < r(B_i) \leq \sigma^{-l+1}\, r(B)\}$. Then

$$\sum_i \frac{r(\sigma^k\, B_i)^p\, \mu(B_i)^{p/q}}{\mu(\sigma^k\, B_i)} \int_{\sigma^k\, B_i} h^p\, d\mu = \sum_{l=0}^{\infty} \sum_{B_i \in \mathcal{E}_l} \frac{r(\sigma^k\, B_i)^p\, \mu(B_i)^{p/q}}{\mu(\sigma^k\, B_i)} \int_{\sigma^k\, B_i} h^p\, d\mu$$
$$= \sum_{l=0}^{k+1} \cdots + \sum_{l=k+2}^{\infty} \cdots = \Sigma_1 + \Sigma_2.$$

We estimate Σ_1. Using the previous result, (4.2), (4.3) and Lemma 5.4 we have

$$\begin{aligned}
\Sigma_1 &= \sum_{l=0}^{k+1} \frac{r(\sigma^{k-l+2}\, B)^p\, \mu(B)^{p/q}}{\mu(\sigma^{k-l+2}\, B)} \sum_{B_i \in \mathcal{E}_l} \left(\frac{r(\sigma^k\, B_i)}{r(\sigma^{k-l+2}\, B)}\right)^p \left(\frac{\mu(B_i)}{\mu(B)}\right)^{p/q} \frac{\mu(\sigma^{k-l+2}\, B)}{\mu(\sigma^k\, B_i)} \\
&\qquad\qquad \times \int_{\sigma^k\, B_i} h^p\, d\mu \\
&\lesssim \sum_{l=0}^{k+1} \frac{r(\sigma^{k-l+2}\, B)^p\, \mu(B)^{p/q}}{\mu(\sigma^{k-l+2}\, B)}\, \sigma^{-l\, \bar{n}\, p/q} \sum_{B_i \in \mathcal{E}_l} \int_{\sigma^k\, B_i} h^p\, d\mu \\
&\lesssim \sum_{l=0}^{k+1} \frac{r(\sigma^{k-l+2}\, B)^p\, \mu(B)^{p/q}}{\mu(\sigma^{k-l+2}\, B)}\, \sigma^{-l\, \bar{n}\, p/q}\, \sigma^{n\, k} \int_{\sigma^{k-l+2}\, B} h^p\, d\mu \\
&= \mu(B)^{p/q}\, \sigma^{n\, k} \sum_{l=0}^{k+1} \sigma^{-l\, \bar{n}\, p/q}\, a_0(\sigma^{k-l+2}\, B)^p \\
&= \mu(B)^{p/q}\, \sigma^{k\, (n - \bar{n}\, p/q)} \sum_{l=1}^{k+2} \sigma^{l\, \bar{n}\, p/q}\, a_0(\sigma^l\, B)^p
\end{aligned}$$

On the other hand, the previous result, (4.2), (4.3), Lemma 5.4 and the fact that $p \leq q < p^*$ imply

$$\begin{aligned}
\Sigma_2 &= \frac{r(\sigma\, B)^p\, \mu(B)^{p/q}}{\mu(\sigma\, B)} \sum_{l=k+2}^{\infty} \sum_{B_i \in \mathcal{E}_l} \left(\frac{r(\sigma^k\, B_i)}{r(\sigma\, B)}\right)^p \left(\frac{\mu(B_i)}{\mu(\sigma^k\, B_i)}\, \frac{\mu(\sigma\, B)}{\mu(B)}\right)^{p/q} \\
&\qquad\qquad \times \left(\frac{\mu(\sigma\, B)}{\mu(\sigma^k\, B_i)}\right)^{1-p/q} \int_{\sigma^k\, B_i} h^p\, d\mu \\
&\lesssim \frac{r(\sigma\, B)^p\, \mu(B)^{p/q}}{\mu(\sigma\, B)}\, \sigma^{k\, p\, (1 + \frac{n - \bar{n}}{q})} \int_{\sigma\, B} h^p\, d\mu \sum_{l=k+2}^{\infty} \sigma^{-l\, (p + n\, p/q - n)} \\
&\lesssim \mu(B)^{p/q}\, \sigma^{k\, (n - \bar{n}\, p/q)} a_0(\sigma\, B)^p.
\end{aligned}$$

Plugging the obtained estimates in (5.22) we conclude that

$$\begin{aligned}
\Big(\sum_i a(B_i)^q\,\mu(B_i)\Big)^{1/q} &\lesssim \alpha(0)\mu(B)^{1/q}a_0(B)+\sum_{k\ge1}\alpha(k)\,(\Sigma_1+\Sigma_2)^{\frac1p}\\
&\lesssim \alpha(0)\mu(B)^{1/q}a_0(B)+\sum_{k\ge0}\alpha(k)\Big(\mu(B)^{p/q}\,\sigma^{k\,(n-\bar n\,p/q)}\sum_{l=1}^{k+2}\sigma^{l\,\bar n\,p/q}\,a_0(\sigma^l\,B)^p\Big)^{\frac1p}\\
&\lesssim \alpha(0)\mu(B)^{1/q}a_0(B)+\mu(B)^{1/q}\sum_{l=1}^{\infty}a_0(\sigma^l\,B)\Big(\sigma^{l\,\bar n/q}\sum_{k\ge\max\{l-2,1\}}\sigma^{k\,(\frac np-\frac{\bar n}{q})}\,\alpha(k)\Big)\\
&= \mu(B)^{1/q}\sum_{l=0}^{\infty}\bar\alpha(l)\,a_0(\sigma^l\,B)=\big(\bar a(B)^q\,\mu(B)\big)^{1/q}
\end{aligned}$$

where $\bar\alpha(0)=C\,\alpha(0)$ and $\bar\alpha(l)=\sigma^{l\,\bar n/q}\sum_{k\ge\max\{l-2,1\}}\sigma^{k\,(\frac np-\frac{\bar n}{q})}\,\alpha(k)$ for $l\ge1$. This shows as desired that $(a,\bar a)\in D_q$. □

Remark 5.7. We would like to call the reader's attention to the fact that, in the previous argument, it was crucial that $q<p^*$. Since otherwise, the geometric sum for the terms $l\ge k+2$ diverges.

Proof of Lemma 5.6. It is straightforward to show that for every $B_j\in\mathcal{J}_k(B_i)$,

$$\sigma^k\,B_j\subset\sigma^{k+1}\,B_i\subset\sigma^{k+2}\,B_j.$$

This and the fact that the balls $\{B_j\}_j$ are pairwise disjoint imply

$$\begin{aligned}
\mu(\sigma^{k+1}\,B_i)\,\#\mathcal{J}_k(B_i) &\le \sum_{B_j\in\mathcal{J}_k(B_i)}\mu(\sigma^{k+2}\,B_j)\le c_\mu\,\sigma^{(k+2)\,n}\sum_{B_j\in\mathcal{J}_k(B_i)}\mu(B_j)\\
&\le c_\mu\,\sigma^{(k+2)\,n}\,\mu(\cup_{B_j\in\mathcal{J}_k(B_i)}B_j)\le c_\mu\,\sigma^{(k+2)\,n}\,\mu(\sigma^{k+1}\,B_i).
\end{aligned}$$

From here the estimate for $\#\mathcal{J}_k(B_i)$ follows at once. The rest of the proof is trivial and left to the reader. □

5.5. **Proof of Proposition 4.4.** We first show (b). Fix $p\in((\tilde q_+)',\infty)\cup[2,\infty)$. We first observe that

$$\begin{aligned}
\Big(\fint_B|f-e^{-t_B\,\Delta}f|\,d\mu\Big)^{1/p} &= \Big(\fint_B\Big|-\int_0^{t_B}\frac{d}{ds}e^{-s\,\Delta}f(x)\,ds\Big|\,d\mu(x)\Big)^{1/p}\\
&\le\int_0^{t_B}\Big(\fint_B|e^{-s\,\Delta}\Delta f(x)|^p\,d\mu(x)\Big)^{1/p}ds.
\end{aligned}$$

Fix $0<s<t_B$, and take a smooth function φ supported in B with $\|\varphi\|_{L^{p'}(B,\frac{\mu}{\mu(B)})}=1$. Then,

$$\begin{aligned}
I &= \frac{1}{\mu(B)}\Big|\int_M e^{-s\,\Delta}\Delta f(x)\,\varphi(x)\,d\mu(x)\Big|=\frac{1}{\mu(B)}\Big|\int_M\nabla f(x)\cdot\nabla e^{-s\,\Delta}\varphi(x)\,d\mu(x)\Big|\\
&\le\sum_{k=1}^{\infty}\frac{\mu(\sigma^k\,B)^{1/p}}{\mu(B)}\Big(\fint_{\sigma^k\,B}|\nabla f|^p\,d\mu\Big)^{1/p}\Big(\int_{C_k(B)}|\nabla e^{-s\,\Delta}\varphi|^{p'}d\mu\Big)^{1/p'}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{k=1}^{\infty} \frac{\sigma^{k\,n/p}}{\mu(B)^{1/p'}} \Big(\fint_{\sigma^k B} |\nabla f|^p \, d\mu \Big)^{1/p} \Big(\int_{C_k(B)} |\nabla e^{-s\,\Delta} \varphi|^{p'} d\mu \Big)^{1/p'} \\
&= \sum_{k=1}^{\infty} \frac{\sigma^{k\,n/p}}{\mu(B)^{1/p'}} \Big(\fint_{\sigma^k B} |\nabla f|^p \, d\mu \Big)^{1/p} I_k.
\end{aligned}
$$

We estimate each I_k. For $k = 1$ we notice that $p' \in (1, 2] \cup (1, \tilde{q}_+)$ allows us to use $(G_{p'})$ —let us recall that $\tilde{q}_+ \geq q_+ \geq 2$, and that (G_2) always holds—:

$$
I_1 \leq \big\| |\nabla e^{-s\,\Delta} \varphi| \big\|_{L^{p'}} \leq C\, s^{-1/2} \, \|\varphi\|_{L^{p'}} = C\, s^{-1/2} \, \mu(B)^{1/p'}.
$$

Assume that $k \geq 2$. By definition of $\tilde{q}_+$ and the argument of [ACDH, p. 944] we have

$$
\Big(\int_M |\nabla_x \, p_s(x, y)|^{p'} \, e^{\gamma \frac{d^2(x,y)}{s}} \, d\mu(x) \Big)^{1/p'} \leq \frac{C}{\sqrt{s} \mu(B(y, \sqrt{s}))^{1/p}},
$$

for all $s > 0$ and $y \in M$, with $\gamma > 0$ depending on p'. Using this estimate and Minkowski's inequality we can control I_k:

$$
\begin{aligned}
I_k &= \Big(\int_{C_k(B)} \Big| \int_B \nabla_x p_s(x, y) \, \varphi(y) \, d\mu(y) \Big|^{p'} d\mu(x) \Big)^{1/p'} \\
&\leq e^{-c \frac{\sigma^{2\,k} \, r(B)^2}{s}} \int_B \Big(\int_{C_k(B)} |\nabla_x \, p_s(x, y)|^{p'} \, e^{\gamma \frac{d^2(x,y)}{s}} \, d\mu(x) \Big)^{1/p'} |\varphi(y)| \, d\mu(y) \\
&\lesssim s^{-1/2} e^{-c \frac{\sigma^{2\,k} \, r(B)^2}{s}} \int_B \frac{1}{\mu(B(y, \sqrt{s}))^{1/p}} \, \varphi(y) \, d\mu(y) \\
&\lesssim s^{-1/2} \left(\frac{r(B)}{\sqrt{s}} \right)^{n/p} e^{-c \frac{\sigma^{2\,k} \, r(B)^2}{s}} \frac{1}{\mu(B)^{1/p}} \int_B \varphi(y) \, d\mu(y) \\
&\lesssim s^{-1/2} \left(\frac{r(B)}{\sqrt{s}} \right)^{n/p} e^{-c \frac{\sigma^{2\,k} \, r(B)^2}{s}} \mu(B)^{1/p'},
\end{aligned}
$$

where we have used that $\mu(B) \approx \mu(B(y, r_B)) \leq c_\mu (r_B/\sqrt{s})^n \, \mu(B(y, \sqrt{s}))$ since $0 < s < t_B = r(B)^2$. Then,

$$
I \lesssim s^{-1/2} \Big(\fint_{\sigma\,B} |\nabla f|^p \, d\mu \Big)^{1/p} + s^{-1/2} \sum_{k=2}^{\infty} \left(\frac{\sigma^k \, r(B)}{\sqrt{s}} \right)^{n/p} e^{-c \frac{\sigma^{2\,k} \, r(B)^2}{s}} \Big(\fint_{\sigma^k B} |\nabla f|^p \, d\mu \Big)^{1/p}
$$

Taking the supremum over all such functions φ we obtain

$$
\begin{aligned}
\Big(\fint_B |f - e^{-t_B\,\Delta} f| \, d\mu \Big)^{1/p} &\lesssim \Big(\fint_{\sigma\,B} |\nabla f|^p \, d\mu \Big)^{1/p} \int_0^{t_B} s^{-1/2} \, ds \\
&\quad + \sum_{k=2}^{\infty} \Big(\fint_{\sigma^k B} |\nabla f|^p \, d\mu \Big)^{1/p} \int_0^{t_B} s^{-1/2} \left(\frac{\sigma^k \, r(B)}{\sqrt{s}} \right)^{n/p} e^{-c \frac{\sigma^{2\,k} \, r(B)^2}{s}} \, ds \\
&\lesssim \sum_{k=1}^{\infty} e^{-c\,\sigma^{2\,k}} \, r(\sigma^k \, B) \Big(\fint_{\sigma^k B} |\nabla f|^p \, d\mu \Big)^{1/p}.
\end{aligned}
$$

It remains to prove (a). We write $h = \Delta^{1/2} f$ and $h = \sum_{k=1}^{\infty} h_k$ with $h_k = h\,\chi_{C_k(B)}$. Since $\Delta^{1/2} = c\int_0^\infty \sqrt{t}\, e^{-t\,\Delta}\,\Delta\,\frac{dt}{t}$ we obtain

$$\begin{aligned}
\fint_B |f - S^m_{t_B} f|\,d\mu &= \fint_B |(I - e^{-t_B\,\Delta})^m f|\,d\mu \\
&= \fint_B \Big|(I - e^{-t_B\,\Delta})^{m-1}\Big(-\int_0^{t_B} \frac{d}{ds} e^{-s\,\Delta} f(x)\,ds\Big)\Big|\,d\mu \\
&\le \int_0^{t_B} \fint_B \big|(I - e^{-t_B\,\Delta})^{m-1} e^{-s\,\Delta}\Delta^{1/2} h\big| d\mu\,ds \\
&\lesssim \int_0^{t_B}\int_0^\infty \fint_B \big|(I - e^{-t_B\,\Delta})^{m-1} e^{-(s+t)\,\Delta}\Delta h\big| d\mu\,\sqrt{t}\,\frac{dt}{t}\,ds \\
&\le \sum_{k=1}^\infty \int_0^{t_B}\int_0^\infty \fint_B \big|(I - e^{-t_B\,\Delta})^{m-1} e^{-(s+t)\,\Delta}\Delta h_k\big| d\mu\,\sqrt{t}\,\frac{dt}{t}\,ds.
\end{aligned}$$

One has that $t\,\partial_t p_t(x,y)$ satisfies also (UE) (see [Dav, Theorem 4] or [Gr2, Corollary 3.3]) and this easily implies that $\{e^{-t\,\Delta}\,(t\,\Delta)\}_{t>0}$ satisfies $L^1 - L^1$ full off-diagonal estimates (see [AM1] for a discussion of off-diagonal estimates associated to semigroups): given E, F closed sets and $t > 0$

$$\|e^{-t\,\Delta}\,(t\,\Delta)(f\,\chi_E)\|_{L^1(F)} \le C\,e^{-c\,\frac{d(E,F)^2}{t}}\,\|f\|_{L^1(E)}. \tag{5.23}$$

This and (UE) imply that $e^{-t\,\Delta}\,(t\,\Delta)$ and $(I - e^{-t\,\Delta})^{m-1}$ are uniformly bounded on L^1. These facts allow us to estimate the term $k = 1$:

$$\begin{aligned}
&\int_0^{t_B}\int_0^\infty \fint_B \big|(I - e^{-t_B\,\Delta})^{m-1} e^{-(s+t)\,\Delta}\Delta h_1\big| d\mu\,\sqrt{t}\,\frac{dt}{t}\,ds \\
&\qquad\lesssim \fint_{\sigma B} |h|\,d\mu \int_0^{t_B}\int_0^\infty \frac{\sqrt{t}}{t+s}\,\frac{dt}{t}\,ds \lesssim r(\sigma\,B)\fint_{\sigma\,B} |h|\,d\mu.
\end{aligned}$$

For $k \ge 2$ we split the integral in the variable t in two pieces: $0 < t < m\,t_B$ and $t \ge m\,t_B$. We first fix $0 < t < m\,t_B$ and $0 < s < t_B$. Observe that

$$(I - e^{-t_B\,\Delta})^{m-1} e^{-(s+t)\,\Delta}\Delta = \sum_{j=0}^{m-1} C_{j,m} e^{-(j\,t_B + t + s)\,\Delta}$$

and that $t + s \le j\,t_B + t + s \le 2\,m\,t_B$. Then (5.23) implies

$$\begin{aligned}
\fint_B \big|(I - e^{-t_B\,\Delta})^{m-1} e^{-(s+t)\,\Delta}\Delta h_k\big| d\mu &\lesssim \frac{1}{\mu(B)}\sum_{j=0}^{m-1} (j\,t_B + t + s)^{-1}\, e^{-\frac{\sigma^{2\,k}\,r(B)^2}{j\,t_B + t + s}} \int_{\sigma^k\,B} |h|\,d\mu \\
&\lesssim \sigma^{k\,n}\, e^{-c\,\sigma^{2\,k}}\,(t+s)^{-1}\fint_{\sigma^k\,B} |h|\,d\mu.
\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}
&\int_0^{t_B}\int_0^{m\,t_B} \fint_B \big|(I - e^{-t_B\,\Delta})^{m-1} e^{-(s+t)\,\Delta}\Delta h_k\big| d\mu\,\sqrt{t}\,\frac{dt}{t}\,ds \\
&\qquad\lesssim e^{-c\,\sigma^{2\,k}} \fint_{\sigma^k\,B} |h|\,d\mu \int_0^{t_B}\int_0^{m\,t_B} \frac{\sqrt{t}}{t+s}\,\frac{dt}{t}\,ds
\end{aligned}$$

$$\lesssim e^{-c\,\sigma^{2\,k}}\,r(\sigma^k\,B)\fint_{\sigma^k\,B}|h|\,d\mu$$

Next for the case $t \ge m\,t_B$ we make the changes of variables $t' = t/(t_B\,m)$ and $s' = s/t_B$:

$$\begin{aligned} I &= \int_0^{t_B}\int_{m\,t_B}^{\infty}\fint_B \big|(I-e^{-t_B\,\Delta})^{m-1}e^{-(s+t)\,\Delta}\Delta h_k\big|d\mu\,\sqrt{t}\,\frac{dt}{t}\,ds \\ &\lesssim r(B)\int_0^1\int_1^{\infty}\fint_B \big|(I-e^{-t_B\,\Delta})^{m-1}\,e^{-t\,t_B\,(m-1)\,\Delta}\,e^{-(s+t)\,t_B\,\Delta}(t_B\,\Delta)h_k\big|d\mu\,\sqrt{t}\,\frac{dt}{t}\,ds \\ &\lesssim r(B)\int_0^1\int_1^{\infty}\fint_B \big|(e^{-t\,t_B\,\Delta}-e^{-(t\,t_B+t_B)\,\Delta})^{m-1}\,e^{-(s+t)\,t_B\,\Delta}((s+t)\,t_B\,\Delta)h_k\big|d\mu\,\frac{dt}{t^{\frac{3}{2}}}\,ds. \end{aligned}$$

We need the following lemma whose proof is below.

Lemma 5.8. *Given given E, F closed sets and $0 < t \le s$, we have*

$$\Big\|\frac{s}{t}\big(e^{-s\,\Delta}-e^{-(s+t)\,\Delta}\big)(f\,\chi_E)\Big\|_{L^1(F)} \le C\,e^{-c\,\frac{d(E,F)^2}{s}}\,\|f\|_{L^1(E)}. \tag{5.24}$$

Using this result, (5.23) and [HM, Lemma 2.3] we have for every $0 < s < 1 < t < \infty$

$$\begin{aligned} &\fint_B \big|(e^{-t\,t_B\,\Delta}-e^{-(t\,t_B+t_B)\,\Delta})^{m-1}\,e^{-(s+t)\,t_B\,\Delta}((s+t)\,t_B\,\Delta)h_k\big|d\mu\,\frac{dt}{t^{\frac{3}{2}}}\,ds \\ &\qquad\lesssim t^{-(m-1)}\,\frac{1}{\mu(B)}\,e^{-c\,\frac{\sigma^{2\,k}\,r(B)^2}{\max\{t\,t_B,(s+t)\,t_B\}}}\int_{\sigma^k\,B}|h|\,d\mu \lesssim t^{-(m-1)}\,\sigma^{k\,n}e^{-c\,\frac{\sigma^{2\,k}}{t}}\fint_{\sigma^k\,B}|h|\,d\mu. \end{aligned}$$

Thus,

$$\begin{aligned} I &\lesssim r(B)\,\sigma^{k\,n}\fint_{\sigma^k\,B}|h|\,d\mu\int_0^1\int_1^{\infty}t^{-(m-1)}e^{-c\,\frac{\sigma^{2\,k}}{t}}\,\frac{dt}{t^{\frac{3}{2}}}\,ds \\ &\lesssim \sigma^{-k\,(2\,m-n)}\,r(\sigma^k\,B)\fint_{\sigma^k\,B}|h|\,d\mu. \end{aligned}$$

Gathering the obtained estimates the proof is completed. □

Proof of Lemma 5.8. We proceed as in [HM, p. 504]:

$$\begin{aligned} \Big\|\frac{s}{t}\big(e^{-s\,\Delta}-e^{-(s+t)\,\Delta}\big)(f\,\chi_E)\Big\|_{L^1(F)} &= \Big\|-\frac{s}{t}\int_0^t\frac{d}{du}e^{-(s+u)\,\Delta}(f\,\chi_E)\,du\,\Big\|_{L^1(F)} \\ &\le \frac{s}{t}\int_0^t\big\|e^{-(s+u)\,\Delta}((s+u)\,\Delta)(f\,\chi_E)\big\|_{L^1(F)}\,\frac{du}{s+u} \\ &\le C\,\|f\|_{L^1(E)}\,\frac{s}{t}\int_0^t e^{-\frac{c\,d(E,F)^2}{s+u}}\,\frac{du}{s+u} \\ &\le C\,e^{-\frac{c\,d(E,F)^2}{s}}\,\|f\|_{L^1(E)}, \end{aligned}$$

where we have used (5.23) and that $s \le s+u \le s+t \le 2\,s$. □

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