# WEIGHTED NORM INEQUALITIES ON GRAPHS

NADINE BADR AND JOSÉ MARÍA MARTELL

ABSTRACT. Let $(\Gamma, \mu)$ be an infinite graph endowed with a reversible Markov kernel $p$ and let $P$ be the corresponding operator. We also consider the associated discrete gradient $\nabla$. We assume that $\mu$ is doubling, an uniform lower bound for $p(x, y)$ when $p(x, y) > 0$, and gaussian upper estimates for the iterates of $p$. Under these conditions (and in some cases assuming further some local Poincaré inequality) we study the comparability of $(I-P)^{1/2}f$ and $\nabla f$ in Lebesgue spaces with Muckenhoupt weights. Also, we establish weighted norm inequalities for a Littlewood-Paley-Stein square function, its formal adjoint, and commutators of the Riesz transform with bounded mean oscillation functions.

## 1. INTRODUCTION

It is well-known that the Riesz transforms are bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ and of weak type (1,1). By the weighted theory for classical Calderón-Zygmund operators, the Riesz transforms are also bounded on $L^p(\mathbb{R}^n, w(x)dx)$ for all $w \in A_p$, $1 < p < \infty$ and of weak type (1,1) with respect to $w$ when $w \in A_1$.

Besides, the Euclidean case, several works have considered the $L^p$ boundedness of the Riesz transforms on Riemannian manifolds. In general, the range of $p$ for which we have the $L^p$ boundedness is no longer $(1, \infty)$. Although there are numerous results on this subject, so far the picture is not complete, we refer the reader to [2] and [1] for more details and references. For weighted norm inequalities on manifolds see [5]. Another context of interest where one can study the $L^p$ boundedness of the Riesz transform is that given by graphs, see [16], [17], [6].

Our purpose in this paper is to develop the weighted theory on graphs for the associated Riesz transforms as was done in [5] for manifolds. We also consider the corresponding reverse weighted inequalities where one controls the discrete Laplacian by the gradient, in which case, taking into account the unweighted case thoroughly studied in [6], we further assume a local Poincaré inequality for $p < 2$. In doing that, we need to prove weighted estimates for a Littlewood-Paley-Stein square function and its formal adjoint which are interesting on their own right. Finally, weighted estimates for commutators of the Riesz transform with BMO functions are obtained.

The plan of the paper is as follows. We next give some preliminaries of graphs, the geometrical assumptions and recall the definitions of the Muckenhoupt weights. In Section 2 we state our main results on Riesz transforms, reverse inequalities and square functions. The proofs of these results are in Section 3. A short discussion on commutators of the Riesz transform with bounded mean oscillation functions is

*Date*: April 14, 2010.

2000 *Mathematics Subject Classification.* 60J10, 42B20, 42B25.

*Key words and phrases.* Graphs, discrete Laplacian, Riesz transforms, square functions, Muckenhoupt weights, spaces of homogeneous type.

The second author was supported by MICINN Grant MTM2007-60952, and by CSIC PIE 200850I015.

in Section 4. Finally, Appendix A contains some auxiliary Calderón-Zygmund type results from [3] used to proved our main results and also a weighted Calderón-Zygmund decomposition for the gradient that extends [6].

1.1. **Graphs.** The following presentation is partly borrowed from [10], [6]. Let $\Gamma$ be an infinite set and let $\mu_{xy} = \mu_{yx} \geq 0$ be a symmetric weight on $\Gamma \times \Gamma$. We call $(\Gamma, \mu)$ a weighted graph. In the sequel, we write $\Gamma$ instead of $(\Gamma, \mu)$. If $x, y \in \Gamma$, we say that $x \sim y$ if and only if $\mu_{xy} > 0$. Denote by $\mathcal{E}$ the set of edges in $\Gamma$:

$$\mathcal{E} = \{(x, y) \in \Gamma \times \Gamma;\ x \sim y\},$$

and notice that, due to the symmetry of $\mu$, $(x, y) \in \mathcal{E}$ if and only if $(y, x) \in \mathcal{E}$.

Given $x, y \in \Gamma$, a path joining $x$ to $y$ is a finite sequence of edges $x_0 = x, ..., x_n = y$ such that, for all $0 \leq i \leq n-1$, $x_i \sim x_{i+1}$. By definition, the length of such a path is $n$. Assume that $\Gamma$ is connected, which means that, for all $x, y \in \Gamma$, there exists a path joining $x$ to $y$. For all $x, y \in \Gamma$, the distance between $x$ and $y$, denoted by $d(x, y)$, is the shortest length of a path joining $x$ and $y$. For all $x \in \Gamma$ and all $r \geq 0$, let $B(x, r) = \{y \in \Gamma,\ d(y, x) \leq r\}$. In the sequel, we always assume that $\Gamma$ is locally uniformly finite, which means that there exists $N \geq 1$ such that, for all $x \in \Gamma$, $\#B(x, 1) \leq N$ ($\#E$ denotes the cardinal of any subset $E$ of $\Gamma$).

For all $x \in \Gamma$, set $m(x) = \sum_{y \sim x} \mu_{xy}$. Notice that as $\Gamma$ is connected we have that $m(x) > 0$ for all $x \in \Gamma$. If $E \subset \Gamma$, define $m(E) = \sum_{x \in E} m(x)$. For all $x \in \Gamma$ and $r > 0$, we write $V(x, r)$ in place of $m(B(x, r))$ and, if $B$ is a ball, $m(B)$ will be denoted by $V(B)$.

For all $1 \leq p < \infty$, we say that a function $f$ on $\Gamma$ belongs to $L^p(\Gamma, m)$ (or $L^p(\Gamma)$) if

$$\|f\|_{L^p(\Gamma)} = \left( \sum_{x \in \Gamma} |f(x)|^p\, m(x) \right)^{1/p} < \infty.$$

When $p = \infty$, we say that $f \in L^\infty(\Gamma, m)$ (or $L^\infty(\Gamma)$) if

$$\|f\|_{L^\infty(\Gamma)} = \sup_{x \in \Gamma} |f(x)| < \infty.$$

We define $p(x, y) = \mu_{xy}/m(x)$ for all $x, y \in \Gamma$. Observe that $p(x, y) = 0$ if $d(x, y) \geq 2$. For every $x, y \in \Gamma$ we set

$$p_0(x, y) = \delta(x, y), \qquad p_{k+1}(x, y) = \sum_{z \in \Gamma} p(x, z) p_k(z, y), \quad k \in \mathbb{N}$$

The $p_k$'s are called the iterates of $p$. Notice that $p_1 \equiv p$ and that for all $x \in \Gamma$, there are at most $N$ non-zero terms in this sum. Observe also that, for all $x \in \Gamma$,

$$\sum_{y \in \Gamma} p(x, y) = 1 \tag{1.1}$$

and, for all $x, y \in \Gamma$,

$$p(x, y) m(x) = p(y, x) m(y). \tag{1.2}$$

Given a function $f$ on $\Gamma$ and $x \in \Gamma$, we define

$$Pf(x) = \sum_{y \in \Gamma} p(x, y) f(y)$$

(again, this sum has at most $N$ non-zero terms). From $p(x,y)\geq 0$ for all $x,y\in\Gamma$ and (1.1), one has that for all $1\leq p\leq\infty$

$$\|Pf\|_{L^p(\Gamma)}\leq\|f\|_{L^p(\Gamma)}. \tag{1.3}$$

We observe that for every $k\geq 1$, $P^kf(x)=\sum_y p_k(x,y)\,f(y)$.

By means of the operator $P$ we define a Laplacian on $\Gamma$. Consider a function $f\in L^2(\Gamma)$, by (1.3), $(I-P)f\in L^2(\Gamma)$ and

$$\begin{aligned}\langle (I-P)f,f\rangle_{L^2(\Gamma)} &= \sum_{x,y}p(x,y)(f(x)-f(y))f(x)m(x)\\ &= \frac{1}{2}\sum_{x,y}p(x,y)\left|f(x)-f(y)\right|^2 m(x),\end{aligned} \tag{1.4}$$

where we have used (1.1) in the first equality and (1.2) in the second one. As in [9] we define the operator "length of the gradient" by

$$\nabla f(x)=\left(\frac{1}{2}\sum_{y\in\Gamma}p(x,y)\left|f(y)-f(x)\right|^2\right)^{\frac{1}{2}}$$

Then, (1.4) shows that

$$\langle (I-P)f,f\rangle_{L^2(\Gamma)}=\|\nabla f\|^2_{L^2(\Gamma)}. \tag{1.5}$$

Notice that (1.2) implies that $P$ is self-adjoint on $L^2(\Gamma)$. Thus, by (1.5), $I-P$ can be considered as a discrete "Laplace" operator which is non-negative and self-adjoint on $L^2(\Gamma)$. By means of spectral theory, one defines its square root $(I-P)^{1/2}$. The equality (1.5) exactly means that

$$\left\|(I-P)^{1/2}f\right\|_{L^2(\Gamma)}=\|\nabla f\|_{L^2(\Gamma)}. \tag{1.6}$$

1.2. **Assumptions.** We need some further assumptions on $\Gamma$. We say that $(\Gamma,\mu)$ satisfies the doubling property if there exists $C>0$ such that, for all $x\in\Gamma$ and all $r>0$,

$$V(x,2r)\leq CV(x,r). \tag{$D$}$$

Note that this assumption implies that there exist $C,D\geq 1$ such that, for any ball $B$ and $\lambda>1$,

$$V(\lambda B)\leq C\,\lambda^D\,V(B). \tag{1.7}$$

Under doubling $(\Gamma,d,\mu)$ becomes a space of homogeneous type (see [8]). Notice that since $\Gamma$ is infinite set, it is also unbounded (as it is locally uniformly finite) and therefore $m(\Gamma)=\infty$ (see [15]).

The second assumption on $(\Gamma,\mu)$ is an uniform lower bound for $p(x,y)$ when $x\sim y$, *i.e.* when $p(x,y)>0$. Given $\alpha>0$, we say that $(\Gamma,\mu)$ satisfies the condition $\Delta(\alpha)$ if, for all $x,y\in\Gamma$,

$$x\sim y\Longleftrightarrow \mu_{xy}\geq\alpha m(x),\qquad\text{and}\qquad x\sim x. \tag{$\Delta(\alpha)$}$$

The next assumption on $(\Gamma,\mu)$ is a pointwise upper bound for the iterates of $p$. We say that $(\Gamma,\mu)$ satisfies $(UE)$ (an upper estimate for the iterates of $p$) if there exist $C,c>0$ such that, for all $x,y\in\Gamma$ and all $k\geq 1$,

$$p_k(x,y)\leq\frac{Cm(y)}{V(x,\sqrt{k})}e^{-c\frac{d^2(x,y)}{k}}. \tag{$UE$}$$

It is known that under doubling this is equivalent to the same inequality only at $y = x$, (also ($D$) and ($UE$) are equivalent to a Faber-Krahn inequality, [9, Theorem 1.1]). Notice that, when ($D$) holds, ($UE$) is also equivalent to

$$p_k(x, y) \le \frac{C m(y)}{V(y, \sqrt{k})} e^{-c\frac{d^2(x,y)}{k}}, \tag{1.8}$$

which will be used in the sequel.

1.3. **Muckenhoupt weights.** As $\Gamma$ is a space of homogeneous type one can consider weights (positive and finite functions) in the Muckenhoupt class $A_\infty(\Gamma)$ (see [18]). We say that $w \in A_p(\Gamma)$, $1 < p < \infty$, if there exists a constant $C$ such that for every ball $B \subset \Gamma$,

$$\Big(\frac{1}{V(B)} \sum_{x\in B} w(x)\, m(x)\Big) \Big(\frac{1}{V(B)} \sum_{x\in B} w(x)^{1-p'}\, m(x)\Big)^{p-1} \le C.$$

For $p = 1$, we say that $w \in A_1(\Gamma)$ if there is a constant $C$ such that for every ball $B \subset \Gamma$

$$\frac{1}{V(B)} \sum_{x\in B} w(x)\, m(x) \le C\, w(y), \qquad \text{for } y \in B.$$

Finally, $A_\infty(\Gamma)$ is the union of all the $A_p(\Gamma)$ classes. The reverse Hölder classes are defined in the following way: $w \in RH_q(\Gamma)$, $1 < q < \infty$, if

$$\Big(\frac{1}{V(B)} \sum_{x\in B} w(x)^q\, m(x)\Big)^{\frac{1}{q}} \le C\, \frac{1}{V(B)} \sum_{x\in B} w(x)\, m(x)$$

for every ball $B$. The endpoint $q = \infty$ is given by the condition $w \in RH_\infty(\Gamma)$: for any ball $B$,

$$w(y) \le C\, \frac{1}{V(B)} \sum_{x\in B} w(x)\, m(x), \qquad \text{for } y \in B.$$

Notice that we have excluded the case $q = 1$ since the class $RH_1(\Gamma)$ consists of all the weights and that is the way that $RH_1(\Gamma)$ is understood.

For every $1 \le p < \infty$ and $w \in A_\infty(\Gamma)$ we write $L^p(\Gamma, w)$ for the corresponding Lebesgue space with measure $w(x)\, m(x)$. Finally, for every $E \subset \Gamma$ and $w \in A_\infty$, we write $w(E) := \sum_{x\in E} w(x)\, m(x)$.

Given $w \in A_\infty(\Gamma)$ we define $r_w = \inf\{p > 1; w \in A_p(\Gamma)\}$ and $s_w = \sup\{s > 1; w \in RH_s(\Gamma)\}$ for which we have $1 \le r_w < \infty$ and $1 < s_w \le \infty$. Given $1 \le p_0 < q_0 \le \infty$ we introduce the (possible empty) set

$$\mathcal{W}_w(p_0, q_0) = \Big(p_0\, r_w, \frac{q_0}{(s_w)'}\Big) = \{p;\ p_0 < p < q_0,\ w \in A_{p/p_0}(\Gamma) \cap RH_{(\frac{q_0}{p})'}(\Gamma)\}.$$

The reader is referred to [18] for more details on Muckenhoupt weights in spaces of homogeneous type. Some of the needed properties can be also found in [3, Section 2].

## 2. Main results

2.1. **Riesz Transform.** We recall a result from [16].

**Theorem 2.1.** [16] *Under the assumptions* ($D$), ($\Delta(\alpha)$) *and* ($UE$), *we have that*

$$\big\|\nabla(I - P)^{-1/2} f\big\|_p \le C_p\, \|f\|_p \tag{$R_p$}$$

*holds for* $1 < p < 2$ *and all* $f \in L_0^\infty(\Gamma)$*. Moreover, the Riesz transform is of weak-type* $(1,1)$*.*

We notice that for $p = 2$ this estimate is indeed an equality (see (1.5)). We define

$$q_+ = \sup\{p \in (1,\infty) : (R_p) \text{ holds}\},$$

which satisfies $q_+ \geq 2$ under the assumptions of Theorem 2.1. It can be equal to 2 ([16, Section 4] with $n = 2$). It is bigger than 2 assuming further the stronger $L^2$-Poincaré inequalities [6].

The main result of this paper gives the following weighted estimates for the Riesz transform:

**Theorem 2.2.** *Let* $(\Gamma,\mu)$ *be a weighted graph and assume that it satisfies* $(D)$*,* $(\Delta(\alpha))$ *and* $(UE)$*. Let* $w \in A_\infty(\Gamma)$*.*

(i) *If* $p \in \mathcal{W}_w(1,q_+)$*, then the Riesz transform is of strong-type* $(p,p)$ *with respect to* $w(x)\,m(x)$*, that is,*

$$\|\nabla(I-P)^{-1/2}f\|_{L^p(\Gamma,w)} \leq C_{p,w}\|f\|_{L^p(\Gamma,w)}$$

*for all* $f \in L_0^\infty(\Gamma)$*.*

(ii) *If* $w \in A_1(\Gamma) \cap RH_{(q_+)'}(\Gamma)$*, then the Riesz transform is of weak-type (1,1) with respect to* $w(x)\,m(x)$*, that is,*

$$\|\nabla(I-P)^{-1/2}f\|_{L^{1,\infty}(\Gamma,w)} \leq C_{p,w}\|f\|_{L^1(\Gamma,w)}$$

*for all* $f \in L_0^\infty(\Gamma)$*.*

Notice that if $q_+ = \infty$ then the Riesz transform is bounded on $L^p(\Gamma,w)$ for $r_w < p < \infty$, that is, for $w \in A_p(\Gamma)$, and we obtain the same weighted theory as for the Riesz transforms on $\mathbb{R}^n$:

**Corollary 2.3.** *Let* $(\Gamma,\mu)$ *be a weighted graph satisfying* $(D)$*,* $(\Delta(\alpha))$ *and* $(UE)$*. Assume that the Riesz transform has strong type* $(p,p)$ *with respect to* $m$ *for all* $1 < p < \infty$*. Then the Riesz transform has strong type* $(p,p)$ *with respect to* $w(x)\,m(x)$ *for all* $w \in A_p(\Gamma)$ *and* $1 < p < \infty$ *and it is of weak-type* $(1,1)$ *with respect to* $w(x)\,m(x)$ *for all* $w \in A_1(\Gamma)$*.*

## 2.2. **Reverse Inequalities.**

For the reverse inequalities, we also obtain a weighted norm inequalities assuming further Poincaré inequalities:

**Definition 2.4** (Poincaré inequality)**.** We say that $(\Gamma,\mu)$ satisfies a scaled $L^p$-Poincaré inequality on balls if there exists $C > 0$ such that, for any $x \in \Gamma$, any $r > 0$ and any function $f \in L^p_{\text{loc}}(\Gamma)$ such that $\nabla f \in L^p_{\text{loc}}(\Gamma)$,

$$\sum_{y \in B(x,r)} |f(y) - f_B|^p\, m(y) \leq C r^p \sum_{y \in B(x,r)} |\nabla f(y)|^p\, m(y), \tag{$P_p$}$$

where

$$f_B = \frac{1}{V(B)} \sum_{x \in B} f(x) m(x)$$

is the mean value of $f$ on $B$.

If $1 \le p < q < +\infty$, then $(P_p)$ implies $(P_q)$ (this is a very general statement on spaces of homogeneous type, see [13]). Thus the set of $p$'s such that $(P_p)$ holds is, if not empty, an interval unbounded on the right. A deep result from [14] implies that this set is open in $[1, \infty)$. We define

$$r_- = \inf \big\{p \in [1, \infty) \,:\, (P_p) \text{ holds}\big\}.$$

We recall the already known unweighted estimates for the reverse inequalities:

**Theorem 2.5.** [6] *Assume $(D)$, $(\Delta(\alpha))$ and that $1 \le r_- < 2$. Then, for all $r_- < p < \infty$*

$$\|(I-P)^{\frac{1}{2}} f\|_p \le C \|\nabla f\|_p. \tag{$RR_p$}$$

*Moreover, if $(P_1)$ holds, there exists $C > 0$ such that, for all $\lambda > 0$,*

$$m\big\{x \in \Gamma : \,\big|(I-P)^{1/2} f(x)\big| > \lambda\big\} \le \frac{C}{\lambda} \|\nabla f\|_1. \tag{2.1}$$

For $p < 2$, this result is proved in [6, Theorem 1.11]. For $p > 2$, it is well known that $(RR_p)$ follows from $(R_{p'})$ which holds by Theorem 2.1 (see [16]). Let us observe that in that case $(UE)$ follows from $(D)$, $(\Delta(\alpha))$ and $(P_2)$ (see [10]).

The weighted version is the following:

**Theorem 2.6.** *Let $(\Gamma, \mu)$ be a weighted graph and assume that it satisfies $(D)$, $(\Delta(\alpha))$ and that $1 \le r_- < 2$.*

*$(i)$ For every $r_- < p < \infty$ and $w \in A_{\frac{p}{r_-}}(\Gamma)$ we have*

$$\|(I-P)^{\frac{1}{2}} f\|_{L^p(\Gamma,w)} \le C \|\nabla f\|_{L^p(\Gamma,w)}. \tag{2.2}$$

*$(ii)$ If $(P_1)$ holds, for every $w \in A_1(\Gamma)$ there exists $C > 0$ such that, for all $\lambda > 0$,*

$$\|(I-P)^{\frac{1}{2}} f\|_{L^{1,\infty}(\Gamma,w)} \le C \|\nabla f\|_{L^1(\Gamma,w)}. \tag{2.3}$$

2.3. **Square function.** To prove Theorem 2.6 we need to obtain a weak-type estimate for an operator that turns out to be the adjoint of a discrete version of the Littlewood-Paley-Stein square function. The boundedness of this square function is interesting on its own right so we give here the weighted estimates that we obtain using our techniques.

For every function $f$ on $\Gamma$ we define Littlewood-Paley-Stein square function

$$g_P f(x) = \Big(\sum_{k=1}^{\infty} \big|\sqrt{k}\,(I-P)P^k f(x)\big|^2\Big)^{1/2}$$

In [6] it is shown that $g_P$ is bounded on $L^p(\Gamma)$ for every $1 < p < \infty$. The adjoint of $g_P$ (as an $\ell^2$-valued operator) is defined as follows: given $\vec{f} = \{f_k\}_{k\ge 1}$

$$S_P \vec{f}(x) = \sum_{k=1}^{\infty} \sqrt{k}\,(I-P)P^k f_k(x)$$

It is straightforward to show that for every $\vec{f} = \{f_k\}$ and $h$ we have

$$\sum_{x\in\Gamma} S_P \vec{f}(x)\, h(x)\, m(x) = \sum_{x\in\Gamma} \langle \vec{f}(x), \vec{g}_P h(x)\rangle_{\ell^2}\, m(x), \tag{2.4}$$

where $\vec{g}_P h(x) = \{\sqrt{k}\,(I-P)P^k h(x)\}_{k\geq 1}$. Therefore, by duality it follows that $S_P$ is bounded from $L^p_{\ell^2}(\Gamma)$ to $L^p(\Gamma)$ for every $1<p<\infty$:

$$\|S_P\vec{f}\|_{L^p(\Gamma)} \leq C\,\|\vec{f}\|_{L^p_{\ell^2}(\Gamma)} = C\left\|\Big(\sum_{k=1}^{\infty}|f_k|^2\Big)^{1/2}\right\|_{L^p(\Gamma)}$$

for every $\vec{f}\in L^\infty_{\ell^2,0}(\Gamma)$, that is, $\|\vec{f}\|_{\ell^2}\in L^\infty_0(\Gamma)$.

**Theorem 2.7.** *Let $(\Gamma,\mu)$ be a weighted graph and assume that it satisfies $(D)$, $(\Delta(\alpha))$ and $(UE)$.*

*(i) If $1<p<\infty$ and $w\in A_p$, then*

$$\|g_P f\|_{L^p(\Gamma,w)} \leq C_{p,w}\|f\|_{L^p(\Gamma,w)}, \qquad f\in L^\infty_0(\Gamma).$$

*(ii) If $w\in A_1$ then*

$$\|g_P f\|_{L^{1,\infty}(\Gamma,w)} \leq C_{p,w}\|f\|_{L^1(\Gamma,w)}, \qquad f\in L^\infty_0(\Gamma)$$

*(iii) If $1<p<\infty$ and $w\in A_p$, then*

$$\|S_P\vec{f}\|_{L^p(\Gamma,w)} \leq C_{p,w}\|\vec{f}\|_{L^p_{\ell^2}(\Gamma,w)}, \qquad \vec{f}\in L^\infty_{\ell^2,0}(\Gamma).$$

*(iv) If $w\in A_1$ then*

$$\|S_P\vec{f}\|_{L^{1,\infty}(\Gamma,w)} \leq C_{p,w}\|\vec{f}\|_{L^1_{\ell^2}(\Gamma,w)}, \qquad \vec{f}\in L^\infty_{\ell^2,0}(\Gamma).$$

Let us notice that in the unweighted case for $p=1$ the weak-type estimates are new.

## 3. Proofs of the main results

We will use the following notation. If $B=B(x,r)$ is a ball with $r>0$, set $\lambda B = B(x,\lambda(r+1))$ for all $\lambda\geq 1$, and write $C_1(B)=4B$ and $C_j(B)=2^{j+1}B\setminus 2^jB$ for all $j\geq 2$. Also, for any subset $A\subset\Gamma$, set

$$\partial A = \{x\in A:\ \exists\, y\sim x,\ y\notin A\}.$$

### 3.1. Proof of Theorem 2.2.

The proof follows the ideas in [5] which in turn uses some arguments from [7].

We need to introduce some notation. Given $1\leq p<\infty$ and $w\in A_\infty(\Gamma)$ we write $L^p_{\ell^2}(\Gamma,w)$ for the $\ell^2$-valued $L^p(\Gamma,w)$ space, that is, $F\in L^p_{\ell^2}(\Gamma,w)$ if and only if

$$\|F\|_{L^p_{\ell^2}(\Gamma,w)} = \big\|\|F(x,y)\|_{\ell^2_y}\big\|_{L^p(\Gamma,w)_x} = \left\|\Big(\sum_{y\in\Gamma}|F(\cdot,y)|^2\Big)^{\frac12}\right\|_{L^p(\Gamma,w)}$$
$$= \left(\sum_{x\in\Gamma}\Big(\sum_{y\in\Gamma}|F(x,y)|^2\Big)^{\frac{p}{2}} w(x)\,m(x)\right)^{\frac1p}.$$

Analogously, one defines the unweighted $L^\infty_{\ell^2}(\Gamma)$. We say that $F$ has compact support if there exists a ball $B$ so that $\operatorname{supp} F(\cdot,y)\subset B$ for all $y\in\Gamma$.

We consider the $\ell^2(\Gamma)$-valued gradient

$$\vec{\nabla} f(x) = \left\{\frac{1}{\sqrt{2}}\,p(x,y)^{\frac12}\,\big(f(y)-f(x)\big)\right\}_{y\in\Gamma},$$

so that

$$\|\vec{\nabla} f(x)\|_{\ell^2(\Gamma)} = \Big(\frac{1}{2}\sum_{y\in\Gamma} p(x,y)\,|f(y)-f(x)|^2\Big)^{\frac{1}{2}} = \nabla f(x).$$

We notice that in the previous expressions, as $\Gamma$ is locally uniformly finite, there are at most $N$ non-zero terms.

*Proof of Theorem 2.2, (i).* We fix $w \in A_\infty(\Gamma)$ and $p \in \mathcal{W}_w(1, q_+)$. Using [3, Proposition 2.1] there exist $p_0, q_0$ such that

$$1 < p_0 < p < q_0 < q_+ \qquad \text{and} \qquad w \in A_{\frac{p}{p_0}}(\Gamma) \cap RH_{(\frac{q_0}{p})'}(\Gamma).$$

By [3, Lemma 4.4] we have that $v = w^{1-p'} \in A_{p'/q_0'}(\Gamma) \cap RH_{(p_0'/p')'}(\Gamma)$.

Let us write $\mathcal{T} f = \vec{\nabla}(I-P)^{-1/2} f$ for the corresponding $\ell^2$-valued linear operator and observe that $Tf(x) = \nabla(I-P)^{-1/2}f(x) = \|\mathcal{T}f(x)\|_{\ell^2(\Gamma)}$. Thus, the boundedness of $T$ on $L^p(\Gamma,w)$ is equivalent to that of $\mathcal{T}$ from $L^p(\Gamma,w)$ to $L^p_{\ell^2}(\Gamma,w)$. Notice that $\mathcal{T}$ is a linear operator and so we can consider its adjoint which is written as $\mathcal{T}^*$. Thus, the boundedness of $\mathcal{T}$ from $L^p(\Gamma,w)$ to $L^p_{\ell^2}(\Gamma,w)$ is equivalent to that of $\mathcal{T}^*$ from $L^{p'}_{\ell^2}(\Gamma,v)$ to $L^{p'}(\Gamma,v)$.

We are going to use Theorem A.1 (see below). Given $f \in L^\infty_{\ell^2,0}(\Gamma)$, we write $h = \mathcal{T}^* f$ and $F = |h|^{q_0'}$. Notice that $F \in L^1(\Gamma)$: $\mathcal{T}^*$ is bounded from $L^{q_0'}_{\ell^2}(\Gamma)$ to $L^{q_0'}(\Gamma)$ since $1 < q_0 < q_+$ implies that $\mathcal{T}$ is bounded from $L^{q_0}(\Gamma)$ to $L^{q_0}_{\ell^2}(\Gamma)$. We pick $\mathcal{A}_k = I - (I - P^{2\,(1+k^2)})^m$ with $m$ large enough. Let $B$ be a ball of radius $k$ and center $x_B$. We write

$$F \le G_B + H_B \equiv 2^{q_0'-1}\,|(I-\mathcal{A}_k)^* h|^{q_0'} + 2^{q_0'-1}\,|\mathcal{A}_k^* h|^{q_0'}.$$

We first estimate $H_B$. Set $q = p_0'/q_0'$ and observe that by duality there exists $g \in L^{p_0}(B, m/V(B))$ with norm 1 such that for all $x \in B$

$$\begin{aligned}
\Big(\frac{1}{V(B)}\sum_{y\in B} H_B(y)^q\, m(y)\Big)^{\frac{1}{q\,q_0'}} &\lesssim V(B)^{-1}\sum_{y\in\Gamma} |h(y)|\,|\mathcal{A}_k g(y)|\, m(y)\\
&\lesssim \sum_{j=1}^{\infty} 2^{j\,D}\Big(\frac{1}{V(2^{j+1}B)}\sum_{C_j(B)} |h(y)|^{q_0'}\, m(y)\Big)^{\frac{1}{q_0'}}\Big(\frac{1}{V(2^{j+1}B)}\sum_{C_j(B)} |\mathcal{A}_k g(y)|^{q_0}\, m(y)\Big)^{\frac{1}{q_0}}\\
&\le MF(x)^{\frac{1}{q_0'}}\sum_{j=1}^{\infty} 2^{j\,D}\Big(\frac{1}{V(2^{j+1}B)}\sum_{C_j(B)} |\mathcal{A}_k g(y)|^{q_0}\, m(y)\Big)^{\frac{1}{q_0}}.
\end{aligned} \tag{3.1}$$

Next, we use (1.8) to obtain that for all $j \ge 1$ and $1 \le i \le m$,

$$\sup_{y\in C_j(B)} |P^{2\,i\,(1+k^2)} g(y)| \le C\, e^{-c\,4^j}\,\frac{1}{V(B)}\sum_{y\in B} |g(y)|\, m(y) \tag{3.2}$$

Then expanding $\mathcal{A}_k$ we conclude that

$$\begin{aligned}
\Big(\frac{1}{V(B)}\sum_{y\in B} H_B(y)^q\, m(y)\Big)^{\frac{1}{q\,q_0'}} &\lesssim MF(x)^{\frac{1}{q_0'}}\sum_{j=1}^{\infty} 2^{j\,D} e^{-c\,4^j}\Big(\frac{1}{V(B)}\sum_{y\in B} |g(y)|^{p_0}\, m(y)\Big)^{\frac{1}{p_0}}\\
&\lesssim MF(x)^{\frac{1}{q_0'}}.
\end{aligned} \tag{3.3}$$

We next estimate $G_B$. Using duality there exists $g \in L^{q_0}(B, m/V(B))$ with norm 1 such that for all $x \in B$

$$\begin{aligned}\Big(\frac{1}{V(B)}\sum_{y\in B} G_B(y)\, m(y)\Big)^{\frac{1}{q_0'}} &\lesssim V(B)^{-1}\sum_{y\in\Gamma}\|f(y)\|_{\ell^2}\,\|\mathcal{T}(I-\mathcal{A}_k)g(y)\|_{\ell^2}\, m(y)\\ &\lesssim \sum_{j=1}^{\infty} 2^{j\,D}\Big(\frac{1}{V(2^{j+1}\,B)}\sum_{C_j(B)}\|f(y)\|_{\ell^2}^{q_0'}\, m(y)\Big)^{\frac{1}{q_0'}}\\ &\qquad\times\Big(\frac{1}{V(2^{j+1}\,B)}\sum_{C_j(B)}|T(I-\mathcal{A}_k)g(y)|^{q_0}\, m(y)\Big)^{\frac{1}{q_0}}\\ &\le M(\|f\|_{\ell^2}^{q_0'})(x)^{\frac{1}{q_0'}}\sum_{j=1}^{\infty} 2^{j\,D}\Big(\frac{1}{V(2^{j+1}\,B)}\sum_{C_j(B)}|T(I-\mathcal{A}_k)g(y)|^{q_0}\, m(y)\Big)^{\frac{1}{q_0}}.\end{aligned}$$

In order to control the terms in the sum we use the following auxiliary lemma whose proof is given below.

**Lemma 3.1.** *Let $\beta \in [1, \tilde{q}_+) \cup [1, 2]$. Then, for all $m \ge 1$, there exists $C > 0$ such that for all $j \ge 2$, all ball $B$ with radius $k$, and all $g \in L^1(\Gamma)$ with support in $B$,*

$$\Big(\frac{1}{V(2^{j+1}\,B)}\sum_{x\in C_j(B)}|\nabla(I-P)^{-1/2}(I-P^{2\,(1+k^2)})^m g(x)|^{\beta}\, m(x)\Big)^{\frac{1}{\beta}} \le C 4^{-jm}\,\frac{1}{V(B)}\sum_{x\in B}|g(x)|\, m(x). \tag{3.4}$$

Here, $\tilde{q}_+$ is defined as the supremum of those $p \in (1, \infty)$ such that for all $k \ge 1$,

$$\|\nabla P^k f\|_{L^p(\Gamma)} \le C\, k^{-1/2}\|f\|_{L^p(\Gamma)}. \tag{3.5}$$

By the analyticity of $P$, one always have $\tilde{q}_+ \ge q_+$. Under the doubling volume property and $L^2$-Poincaré, it is shown in [6, Theorem 1.4] that $q_+ = \tilde{q}_+$.

Using this lemma one easily estimates the terms where $j \ge 2$. For $j = 1$, we use that $T$ is bounded on $L^{q_0}(\Gamma)$ as $1 < q_0 < q_+$ and (3.2):

$$\begin{aligned}\sum_{y\in 4\,B}|T(I-\mathcal{A}_k)g(y)|^{q_0}\, m(y) &\lesssim \Big(\sum_{y\in B}|g(y)|^{q_0}\, m(y) + \sum_{l=1}^{\infty}\sum_{y\in C_l(B)}|\mathcal{A}_k g(y)|^{q_0}\, m(y)\Big)\\ &\lesssim \sum_{y\in B}|g(y)|^{q_0}\, m(y)\sum_{l=1}^{\infty} 2^{l\,D}\, e^{-c\,4^l} \lesssim \sum_{x\in B}|g(y)|^{q_0}\, m(y).\end{aligned} \tag{3.6}$$

Using this and (3.4) (with $\beta = q_0 < q_+ \le \tilde{q}_+$) we conclude the estimate for $G_B$:

$$\begin{aligned}\Big(\frac{1}{V(B)}\sum_{x\in B} G_B(y) m(y)\Big)^{\frac{1}{q_0'}} &\lesssim M(\|f\|_{\ell^2}^{q_0'})(x)^{\frac{1}{q_0'}}\sum_{j=1}^{\infty} 2^{j\,D} 4^{-j\,m}\Big(\frac{1}{V(B)}\sum_{y\in B}|g(y)|^{q_0}\, m(y)\Big)^{\frac{1}{q_0}}\\ &\le C\, M(\|f\|_{\ell^2}^{q_0'})(x)^{\frac{1}{q_0'}} = G(x)^{\frac{1}{q_0'}},\end{aligned} \tag{3.7}$$

provided $m > D/2$. With (3.3) and (3.7) in hand we can use Theorem A.1 with $r = p'/q_0'$, $q = p_0'/q_0'$ and $H_1 \equiv 0$. Notice that $v \in RH_{s'}(\mu)$ with $s = p_0'/p'$, $1 < s < q < \infty$

and $r = q/s$. Hence, using $v \in A_r(\mu)$ we obtain the desired estimate

$$\|\mathcal{T}^* f\|_{L^{p'}(\Gamma,v)}^{q_0'} \le \|MF\|_{L^r(\Gamma,v)} \lesssim \|M(\|f\|_{\ell^2}^{q_0'})\|_{L^r(\Gamma,v)} \lesssim \big\|\|f\|_{\ell^2}\big\|_{L^{p'}(\Gamma,v)}^{q_0'} = \|f\|_{L^{p'}_{\ell^2}(\Gamma,v)}^{q_0'}.$$

□

*Proof of Lemma 3.1.* We first observe that the desired estimate for a fixed $\beta_0$ implies the same one for all $\beta$ with $1 \le \beta \le \beta_0$. Since $\tilde{q}_+ \ge q_+ \ge 2$ it follows that if $\tilde{q}_+ = 2$ it suffices to obtain the case $\beta = 2$, and if $\tilde{q}_+ > 2$ it suffices to treat the case $2 \le \beta < \tilde{q}_+$. Thus, we fix $\beta \ge 2$.

As in [6] we write $(I-P)^{-1/2} = \sum_{l=0}^{\infty} a_l\, P^l$ where the coefficients $a_l$ are those from the Taylor expansion centered at 0 of the function $(1-x)^{-1/2}$. Then, expanding $(I - P^{2\,(1+k^2)})^m$ we have

$$\begin{aligned}
(I-P)^{-1/2}(I-P^{2\,(1+k^2)})^m &= \sum_{l=0}^{\infty} a_l\, P^l \sum_{j=0}^{m} C_m^j\,(-1)^j\, P^{2\,j\,(1+k^2)} \\
&= \sum_{j=0}^{m} C_m^j\,(-1)^j \sum_{l=0}^{\infty} a_l\, P^{l+2\,j\,(1+k^2)} \\
&= \sum_{j=0}^{m} C_m^j\,(-1)^j \sum_{l=2\,j\,(1+k^2)}^{\infty} a_{l-2\,j\,(1+k^2)}\, P^l \\
&= \sum_{l=0}^{\infty} \Big( \sum_{0\le j\le m:2j(1+k^2)\le l} C_m^j\,(-1)^j\, a_{l-2\,j\,(1+k^2)} \Big) P^l \\
&= \sum_{l=0}^{\infty} d_l\, P^l
\end{aligned}$$

and therefore

$$\nabla (I-P)^{-1/2}(I-P^{2\,(1+k^2)})^m g(x) = \sum_{l=0}^{\infty} d_l\, \nabla P^l g(x).$$

Notice that for $0 \le l \le 3$ we have $\nabla P^l g(x) = 0$ for every $x \in C_j(B)$ with $j \ge 2$. Indeed in that case we have that $\nabla P^l g$ is supported in $B(x_B, k+4)$ (as $g$ is supported in $B(x_B,k)$) and $B(x_B, k+4) \cap C_j(B) = \emptyset$ for every $j \ge 2$. This implies that the previous series runs from $l = 4$ to $\infty$. We claim that there exists $\gamma = \gamma(\beta)$ such that for every $l \ge 2$ and $z \in B$,

$$\Big( \sum_{x\in\Gamma} |\nabla_x p_l(x,z)|^\beta\, e^{\gamma \frac{d^2(x,z)}{l}}\, m(x) \Big)^{\frac{1}{\beta}} \le C\, \frac{m(z)}{\sqrt{l}\, V(z,\sqrt{l})^{1-1/\beta}} \tag{3.8}$$

This and Minkowski's inequality imply

$$\begin{aligned}
&\Big( \frac{1}{V(2^{j+1}B)} \sum_{x\in C_j(B)} |\nabla (I-P)^{-1/2}(I-P^{2\,(1+k^2)})^m g(x)|^\beta\, m(x) \Big)^{\frac{1}{\beta}} \\
&\quad \le \sum_{l=4}^{\infty} |d_l| \sum_{z\in B} |g(z)| \Big( \frac{1}{V(2^{j+1}B)} \sum_{x\in C_j(B)} |\nabla_x p_l(x,z)|^\beta\, m(x) \Big)^{\frac{1}{\beta}}
\end{aligned}$$

$$\lesssim \sum_{l=4}^{\infty} |d_l|\, e^{-\gamma' \frac{4^j\,(1+k^2)}{l}} \sum_{z\in B} |g(z)| \frac{m(z)}{\sqrt{l}\, V(z,\sqrt{l})^{1-1/\beta}\, V(2^{j+1}\,B)^{1/\beta}}$$

$$\lesssim \sum_{l=4}^{\infty} |d_l|\, l^{-1/2}\, e^{-\gamma' \frac{4^j\,(1+k^2)}{l}} \max\Big\{1, \frac{2^{j+1}\,(k+1)}{\sqrt{l}}\Big\}^{D\,(1-\frac{1}{\beta})} \Big(\frac{1}{V(B)} \sum_{z\in B} |g(z)| m(z)\Big)$$

$$\lesssim \sum_{l=4}^{\infty} |d_l|\, l^{-1/2}\, e^{-c \frac{4^j\,(1+k^2)}{l}} \Big(\frac{1}{V(B)} \sum_{z\in B} |g(z)| m(z)\Big).$$

Thus, it suffices to show that

$$\sum_{l=4}^{\infty} |d_l|\, l^{-1/2}\, e^{-c \frac{4^j\,(1+k^2)}{l}} \lesssim 4^{-j\,m}. \tag{3.9}$$

We follow the ideas in [6, Section 4]. For every $0 \le i \le m$ we write

$$\Sigma_i = \{l \ge 4 : 2\,i\,(1+k^2) < l \le 2\,(i+1)\,(1+k^2)\},$$

and $\Sigma_{m+1} = \{l : l > 2\,(m+1)\,(1+k^2)\}$. Then we have,

$$|d_l| \le C_m \begin{cases} (l - 2\,i\,(1+k^2))^{-1/2}, & l \in \Sigma_i,\ 0 \le i \le m \\ (1+k^2)^m\, l^{-m-1/2}, & l \in \Sigma_{m+1}. \end{cases}$$

Following *mutatis mutandis* [6, Lemma 4.1] we obtain

$$\sum_{l=4}^{\infty} |d_l|\, l^{-1/2}\, e^{-c \frac{4^j\,(1+k^2)}{l}} \le \sum_{i=0}^{m} \sum_{l\in\Sigma_i} \cdots + \sum_{l\in\Sigma_{m+1}} \cdots = I + II$$

and we estimate each term in turn. For $I$, considering the cases $i = 0$ and $i \ge 1$ separately we obtain

$$I \lesssim \sum_{i=0}^{m} \int_{2\,i\,(1+k^2)}^{2\,(i+1)\,(1+k^2)} \frac{e^{-c \frac{4^j\,(1+k^2)}{t}}}{\sqrt{t}\,\sqrt{t - 2\,i\,(1+k^2)}}\, dt \lesssim \sum_{i=0}^{m} \int_0^1 \frac{e^{-c \frac{4^j}{i+t}}}{\sqrt{t}\,\sqrt{i+t}}\, dt$$

$$\lesssim \int_0^1 e^{-c \frac{4^j}{t}} \frac{dt}{t} + e^{-c\,4^j} \int_0^1 \frac{dt}{\sqrt{t}} \lesssim e^{-c\,4^j}.$$

To control $II$ we use the estimate for $|d_l|$ to conclude that

$$II \lesssim \int_{2\,(m+1)\,(1+k^2)}^{\infty} e^{-c \frac{4^j\,(1+k^2)}{t}}\, t^{-1/2}\, (1+k^2)^m\, t^{-m-1/2}\, dt$$

$$\lesssim 4^{-j\,m} \int_0^{\frac{4^j}{2\,(m+1)}} e^{-c\,t}\, t^m \frac{dt}{t} \lesssim 4^{-j\,m}.$$

Gathering the obtained estimates we conclude (3.9).

To finish we need to show (3.8). We recall [16, Lemma 7]: there exists $\gamma_0 > 0$ such that for every $l \ge 1$ we have

$$\Big(\frac{1}{V(z,\sqrt{l})} \sum_{x\in\Gamma} |\nabla_x p_l(x,z)|^2\, e^{\gamma_0 \frac{d^2(x,z)}{l}}\, m(x)\Big)^{\frac{1}{2}} \le C\, \frac{m(z)}{\sqrt{l}\, V(z,\sqrt{l})} \tag{3.10}$$

Notice that this gives the case $\beta = 2$, and this completes the proof when $\tilde{q}_+ = 2$. We consider the case $\tilde{q}_+ > 2$ and then it suffices to take $2 < \beta < \tilde{q}_+$. We pick $\beta < q < \tilde{q}_+$.

For every $z \in \Gamma$ and $l \geq 1$ we have

$$\begin{aligned}\sum_{x\in\Gamma} e^{-c\frac{d^2(x,z)}{l}} m(x) &\leq \sum_{x:d(z,x)\leq\sqrt{l}} m(x) + \sum_{i=0}^{\infty} e^{-c\,4^i} \sum_{x:2^i\sqrt{l}<d(z,x)<2^{i+1}\sqrt{l}} m(x)\\ &\leq V(z,\sqrt{l}) + \sum_{i=0}^{\infty} e^{-c\,4^i} V(z,2^{i+1}\sqrt{l})\\ &\lesssim V(z,\sqrt{l}) \sum_{i=0}^{\infty} e^{-c\,4^i} 2^{i\,D}\\ &\lesssim V(z,\sqrt{l}).\end{aligned} \tag{3.11}$$

Given $l \geq 2$ we write $l = n + n'$ where $n' = n = l/2$ if $l$ is even and $n' = n + 1 = (l+1)/2$ if $l$ is odd. For every $l \geq 1$ we have $p_{l+1}(x,z) = P\big(p_l(\cdot,z)\big)(x)$, then it is immediate to see that

$$p_l(x,z) = p_{n+n'}(x,z) = P^n(p_{n'}(\cdot,z))(x).$$

Then, as $2 \leq q < \tilde{q}_+$ we can use (3.5). This, (1.8) and (3.11) yield

$$\begin{aligned}\Big(\frac{1}{V(z,\sqrt{l})}\sum_{x\in\Gamma}|\nabla_x p_l(x,z)|^q\, m(x)\Big)^{\frac{1}{q}} &= \Big(\frac{1}{V(z,\sqrt{l})}\sum_{x\in\Gamma}|\nabla_x P^n(p_{n'}(\cdot,z))(x)|^q\, m(x)\Big)^{\frac{1}{q}}\\ &\lesssim n^{-1/2}\Big(\frac{1}{V(z,\sqrt{l})}\sum_{x\in\Gamma}|p_{n'}(x,z)|^q\, m(x)\Big)^{\frac{1}{q}}\\ &\lesssim l^{-1/2}\frac{m(z)}{V(z,\sqrt{n'})}\Big(\frac{1}{V(z,\sqrt{l})}\sum_{x\in\Gamma} e^{-c\frac{d^2(x,z)}{n'}} m(x)\Big)^{\frac{1}{q}}\\ &\lesssim \frac{m(z)}{\sqrt{l}\,V(z,\sqrt{l})}.\end{aligned} \tag{3.12}$$

To complete the proof we take $r = (q-2)/(q-\beta)$ so that $r' = (q-2)/(\beta-2)$ and $2/r + q/r' = \beta$. We set $\gamma = \gamma_0/r$, then Hölder's inequality, (3.10), and (3.12) imply

$$\begin{aligned}&\Big(\frac{1}{V(z,\sqrt{l})}\sum_{x\in\Gamma}|\nabla_x p_l(x,z)|^\beta\, e^{\gamma\frac{d^2(x,z)}{l}}\, m(x)\Big)^{\frac{1}{\beta}}\\ &\leq \Big(\frac{1}{V(z,\sqrt{l})}\sum_{x\in\Gamma}|\nabla_x p_l(x,z)|^2\, e^{\gamma_0\frac{d^2(x,z)}{l}}\, m(x)\Big)^{\frac{1}{\beta\,r}}\Big(\frac{1}{V(z,\sqrt{l})}\sum_{x\in\Gamma}|\nabla_x p_l(x,z)|^q\, m(x)\Big)^{\frac{1}{\beta\,r'}}\\ &\lesssim \Big(\frac{m(z)}{\sqrt{l}\,V(z,\sqrt{l})}\Big)^{\frac{2}{\beta\,r}+\frac{q}{\beta\,r'}} = \frac{m(z)}{\sqrt{l}\,V(z,\sqrt{l})},\end{aligned}$$

and this readily leads to the desired estimate. □

*Proof of Theorem 2.2, (ii).* The proof is similar to that in [5, Theorem 1.2, $(ii)$], applying Theorem A.2 with $p_0 = 1$, $q_0 = q_+$. We see that the four items hold. Fix $w \in A_1(\Gamma) \cap RH_{(q_+)'}(\Gamma)$. By [3, Proposition 2.1], there exists $1 < q < q_+$ such that $w \in A_q(\Gamma) \cap RH_{(q_+/q)'}(\Gamma)$. This means that $q \in \mathcal{W}_w(1,q_+)$, therefore by Theorem 2.2 part $(i)$, $T = \nabla(I-P)^{-1/2}$ is bounded on $L^q(\Gamma,w)$ and thus $(a)$ holds.

We pick $\mathcal{A}_k = I - (I - P^{2\,(1+k^2)})^m$ with $m$ large enough to be chosen. Notice that expanding $\mathcal{A}_k$, (3.2) yields $(b)$ with $\alpha_j = C\,e^{-c\,4^j}$. To see $(c)$ we apply Lemma 3.1 with $(s_w)' < \beta$ —notice that such $\beta$ exists: we have $q_+ \leq \tilde{q}_+$ and $w \in RH_{(q_+)'}(\Gamma)$ implies $(s_w)' < q_+$—. Then, we obtain $(c)$ with $\alpha_j = C\,4^{-j\,m}$. Finally, we pick $m > D_w/2$ so that $(d)$ holds and therefore Theorem A.2 gives the weak-type $(1,1)$ with respect to $w(x)\,m(x)$. □

3.2. **Proof of Theorem 2.6.** We need the following auxiliary result whose proof is given below.

**Lemma 3.2.** *Under the hypotheses of Theorem 2.6, take $r_- < r < p < q < \infty$. Let $B$ be a ball with radius $k$ and $m \geq 1$ be a large enough integer. Then,*

$$\left(\frac{1}{V(B)}\sum_{x\in B}\left|(I-P)^{\frac12}\left(I-P^{2(1+k^2)}\right)^m f\right|^r m(x)\right)^{\frac1r} \leq \sum_j g_1(j)\left(\frac{1}{V(2^{j+1}B)}\sum_{x\in 2^{j+1}B}|\nabla f|^r m(x)\right)^{\frac1r}, \quad (3.13)$$

*for $m$ large enough depending on $q$ and $r$ ,and*

$$\left(\frac{1}{V(B)}\sum_{x\in B}\left|(I-P)^{\frac12}\left(I-\left(I-P^{2(1+k^2)}\right)^m\right)f\right|^q m(x)\right)^{\frac1q} \leq \sum_{j\geq1} g_2(j)\left(\frac{1}{V(2^{j+1}B)}\sum_{x\in 2^{j+1}B}|(I-P)^{\frac12}f|^r m(x)\right)^{\frac1r}, \quad (3.14)$$

*where $g_1(j) = C_m 2^j 4^{-mj}$ and $g_2(j) = C_m e^{-c\,4^j}$ for some $\theta > 0$.*

*Proof of Theorem 2.6, $(i)$.* Since $w \in A_{\frac{p}{r_-}}(\Gamma)$, then [3, Proposition 2.1] implies that there exist $r$, $q$, with $r_- < r < p < q < \infty$, such that

$$w \in A_{\frac{p}{r}}(\Gamma) \cap RH_{(\frac{q}{p})'}(\Gamma).$$

Note that (3.13) and (3.14) are respectively the conditions (A.5) and (A.6) of Theorem A.3 with $p_0 = r$, $q_0 = q$, $T = (I-P)^{\frac12}$, $\mathcal{A}_k = I - (I-P^{2(1+k^2)})^m$, with $k$ the radius of the ball $B$, $m$ large enough and $Sf = \nabla f$. Therefore, we obtain the desired inequality $(RR_p)$. □

*Proof of Lemma 3.2.* We write $h = (I-P)^{\frac12}f$ and $h = \sum_{j\geq1} h\,\chi_{C_j(B)} = \sum_{j\geq1} h_j$. Then we obtain that for every $1 \leq l \leq m$

$$\left(\frac{1}{V(B)}\sum_{x\in B}\left|P^{2l\,(1+k^2)}h_j(x)\right|^q m(x)\right)^{\frac1q} \leq \sup_{x\in B}\left|P^{2l\,(1+k^2)}h_j(x)\right|$$

$$\lesssim e^{-c\,4^j}\,\frac{1}{V(2^{j+1}B)}\sum_{x\in C_j(B)}|h(x)|\,m(x) \lesssim e^{-c\,4^j}\left(\frac{1}{V(2^{j+1}B)}\sum_{x\in C_j(B)}|h(x)|^r\,m(x)\right)^{\frac1r},$$

where we have used (1.8). This and the commutation rule readily lead to (3.14) after expanding $(I-P^{2(k^2+1)})^m$.

To estimate (3.13) we recall the following estimate that follows from $(D)$ and $(UE)$, see [6]: if $B = B(x_0, k)$ and $f$ is supported in $C_j(B)$, $j \geq 2$, one has that for $l \geq 1$,

$$\sup_{x\in B}\big(|P^l f(x)| + l|(I-P)P^l f(x)|\big) \leq C\, e^{-c\frac{4^j k^2}{l}} \frac{1}{V(2^{j+1}B)} \sum_{x\in C_j(B)} |f(x)|\, m(x). \tag{3.15}$$

We first observe that

$$(I-P)^{\frac12}(I-\mathcal{A}_k)f = (I-P)^{\frac12}(I-\mathcal{A}_k)(f-f_{4B}) = \sum_{j\geq 1}(I-P)^{\frac12}(I-\mathcal{A}_k)h_j$$

where $h = f - f_{4B}$, $h_j = h\,\phi_j$, $\phi_j = \chi_{C_j(B)}$ for $j \geq 3$, $\phi_2 = \chi_{8B} - \phi_1$ and

$$\phi_1(x) = \begin{cases} 1 & d(x,x_0) \leq 2\,k \\ \frac{4\,k-j}{2\,k} & d(x,x_0) = j,\ 2\,k \leq j \leq 4\,k \\ 0 & d(x,x_0) \geq 4\,k \end{cases}.$$

Notice that $\sum_{j\geq 1}\phi_j \equiv 1$. We estimate the term $j = 1$:

$$\|(I-P)^{\frac12}(I-\mathcal{A}_k)h_1\|_r = \|(I-\mathcal{A}_k)(I-P)^{\frac12}h_1\|_r \leq C\|(I-P)^{\frac12}h_1\|_r \leq C\|\nabla h_1\|_r$$

where we have used $(RR_r)$ in Theorem 2.5 (notice that $r_- < r < \infty$). It is easy to show that $\nabla h_1$ is supported in $4\,B$ (since $\phi_1$ is supported in $B(x_0, 4\,k-1)$) and also that $|\phi_1(x) - \phi_1(y)| \leq (2\,k)^{-1}$ whenever $d(x,y) \leq 1$. This allows one to obtain that for every $x \in 4\,B$

$$\nabla h_1(x) = \nabla(h\,\phi_1)(x) \leq \nabla h(x) + (2^{3/2}\,k)^{-1}\,|h(x)| = \nabla f(x) + (2^{3/2}\,k)^{-1}\,|f(x) - f_{4\,B}|.$$

Then, $(P_r)$ implies

$$\|(I-P)^{\frac12}(I-\mathcal{A}_k)h_1\|_r \leq C\,\|\nabla h_1\|_{L^r(4\,B)} \leq C\,\|\nabla f\|_{L^r(4\,B)}$$

and therefore

$$\bigg(\frac{1}{V(B)}\sum_{x\in B}|(I-P)^{\frac12}(I-\mathcal{A}_k)h_1(x)|^r m(x)\bigg)^{\frac1r} \leq C\bigg(\frac{1}{V(4B)}\sum_{x\in 4B}\nabla f(x)^r m(x)\bigg)^{\frac1r}.$$

Next, we consider the case $j \geq 3$ and write

$$\begin{aligned}(I-P)^{\frac12}(I-\mathcal{A}_k)h_j &= \sum_{l=0}^{\infty} a_l (I-P)P^l(I-P^{2(1+k^2)})^m h_j \\ &= \sum_{l=0}^{\infty} d_l (I-P)P^l h_j\end{aligned}$$

where $a_l$ are the coefficients considered in Lemma 3.1 and

$$d_l = \sum_{0\leq i\leq m;\, 2i(1+k^2)\leq l} (-1)^i C_m^i a_{l-2\,i\,(1+k^2)}.$$

We observe that under the notation of Lemma 3.1 if $l \in \Sigma_0$ we can use the estimate $|d_l| \leq C_m\, l^{-1/2}$ and then it is easy to see that

$$\sum_{l=4}^{2\,(1+k^2)} \frac{|d_l|}{l} e^{-c\frac{4^j k^2}{l}} = \sum_{l\in\Sigma_0}\frac{|d_l|}{l}e^{-c\frac{4^j k^2}{l}} \leq C\int_3^{2\,(1+k^2)} t^{-\frac32} e^{-c\frac{4^j k^2}{t}}\,dt \leq C\,k^{-1}e^{-c'\,4^j}. \tag{3.16}$$

This and (3.9) imply

$$\sum_{l=4}^{\infty}\frac{|d_l|}{l}e^{-c\frac{4^jk^2}{l}} \le \sum_{l=4}^{2\,(1+k^2)}\frac{|d_l|}{l}e^{-c\frac{4^jk^2}{l}} + k^{-1}\sum_{l>2\,(1+k^2)}|d_l|\,l^{-1/2}e^{-c\frac{4^jk^2}{l}} \le k^{-1}\,4^{-j\,m}.$$

Therefore, as in Lemma 3.1 it follows that for $x\in B$

$$\begin{aligned}
|(I-P)^{\frac{1}{2}}(I-\mathcal{A}_k)h_j(x)| &\le \sum_{l=0}^{\infty}|d_l||(I-P)P^lh_j(x)| \\
&\le C\left(\frac{1}{V(2^{j+1}B)}\sum_{x\in C_j(B)}|h(x)|\,m(x)\right)\sum_{l=4}^{\infty}\frac{|d_l|}{l}e^{-c\frac{4^jk^2}{l}} \\
&\le C\,4^{-j\,m}\,k^{-1}\left(\frac{1}{V(2^{j+1}B)}\sum_{x\in 2^{j+1}\,B}|f(x)-f_{4\,B}|^r\,m(x)\right)^{\frac{1}{r}} \\
&\le C\,4^{-j\,m}\,k^{-1}\sum_{i=3}^{j+1}\left(\frac{1}{V(2^iB)}\sum_{x\in 2^i\,B}|f(x)-f_{2^i\,B}|^r\,m(x)\right)^{\frac{1}{r}} \\
&\le C\,4^{-j\,m}\sum_{i=3}^{j+1}2^i\left(\frac{1}{V(2^iB)}\sum_{x\in 2^i\,B}|\nabla f(x)|^r\,m(x)\right)^{\frac{1}{r}},
\end{aligned}$$

where we have used (3.15), ($D$) and ($P_r$). Consequently, for $j\ge 3$ we have shown

$$\begin{aligned}
&\left(\frac{1}{V(B)}\sum_{x\in B}|(I-P)^{\frac{1}{2}}(I-\mathcal{A}_k)h_j(x)|^rm(x)\right)^{\frac{1}{r}} \\
&\qquad\le C\,4^{-j\,m}\sum_{i=3}^{j+1}2^i\left(\frac{1}{V(2^iB)}\sum_{x\in 2^i\,B}|\nabla f(x)|^r\,m(x)\right)^{\frac{1}{r}}. \qquad (3.17)
\end{aligned}$$

For the remaining term $j=2$, notice that in (3.15) we can also take $j=2$ and $f=h_2$ whose support is contained in $8\,B\setminus 2\,B$. Using this and the fact that $|h_2|\le|f-f_{4B}|\,\chi_{8B\setminus 2B}$ we can argue similarly and obtain that the last estimate (3.17) can be extended to the case $j=2$.

Gathering the obtained estimates we conclude as desired (3.13). □

It remains to prove part ($ii$) of Theorem 2.6. The proof follows the method in the unweighted case using Proposition A.4.

*Proof of Theorem 2.6, (ii).* We follow the proof of in [6, Theorem 1.11]. Consider $f$ such that $\nabla f\in L^\infty_c$, fix $w\in A_1(\Gamma)$ and let $\lambda>0$. We perform the Calderón-Zygmund decomposition of $f$ given by Proposition A.4.

We use that the measure $w(x)\,m(x)$ is doubling and (A.11):

$$w(\cup_i 4\,B_i)\le C\sum_i w(B_i)\le\frac{C}{\lambda}\sum_{x\in\Gamma}\nabla f(x)\,w(x)\,m(x). \qquad (3.18)$$

Next, we take $r > 1$, then $w \in A_r(\Gamma)$. Applying Theorem 2.6, part $(i)$ (note that $r_- = 1$), (A.8) and (3.18) one gets

$$\begin{aligned} I = w\{x \in \Gamma : |(I-P)^{1/2} g(x)| > \lambda/3\} &\le \frac{C}{\lambda^r} \sum_{x\in\Gamma} \nabla g(x)^r \, w(x)\, m(x) \\ &\le C\, w(\cup_i 4\, B_i) + \frac{C}{\lambda} \sum_{x \notin \cup_i 4\, B_i} \nabla g(x)\, w(x)\, m(x) \\ &\le \frac{C}{\lambda} \sum_{x\in\Gamma} \nabla f(x)\, w(x)\, m(x) + I_1. \end{aligned}$$

To estimate $I_1$ we write $F = \Gamma \setminus (\cup_i B_i)$. Following the computations in [6, pp. 305–306], if $x \notin \cup_i 4\, B_i$ and $y \sim x$ then $x, y \in F$ and therefore $g(x) = f(x)$ and $g(y) = f(y)$. Thus, we conclude that $\nabla g(x) = \nabla f(x)$ for every $x \notin \cup_i 4\, B_i$ and consequently

$$I_1 = \frac{C}{\lambda} \sum_{x \notin \cup_i 4\, B_i} \nabla f(x)\, w(x)\, m(x) \le \frac{C}{\lambda} \sum_{x\in\Gamma} \nabla f(x)\, w(x)\, m(x).$$

Gathering the obtained estimates we conclude that

$$I \le \frac{C}{\lambda} \sum_{x\in\Gamma} \nabla f(x)\, w(x)\, m(x). \tag{3.19}$$

Next, we use the following expansion of $(I-P)^{1/2}$:

$$(I-P)^{1/2} = \sum_{k=0}^{+\infty} a_k (I-P) P^k \tag{3.20}$$

where $\{a_k\}_k$ is the sequence in Lemma 3.1. For each $i \in I$, pick the integer $j \in \mathbb{Z}$ such that $2^{j-1} \le r(B_i) < 2^j$ and define $r_i = 2^j$ (we notice that $r(B_i) \ge 1/2$ and then $j \ge 0$). We split the expansion (3.20) into two parts:

$$(I-P)^{1/2} = \sum_{k=0}^{r_i^2} a_k (I-P) P^k + \sum_{k=r_i^2+1}^{+\infty} a_k (I-P) P^k = T_i + U_i.$$

We claim that

$$II = w\Big\{x \notin \cup_i 4\, B_i : \Big| \sum_{i\in I} T_i b_i(x) \Big| > \lambda/3 \Big\} \le \frac{C}{\lambda} \sum_{x\in\Gamma} \nabla f(x)\, w(x)\, m(x). \tag{3.21}$$

and

$$III = w\Big\{x \in \Gamma : \Big| \sum_{i\in I} U_i b_i(x) \Big| > \lambda/3 \Big\} \le \frac{C}{\lambda} \sum_{x\in\Gamma} \nabla f(x)\, w(x)\, m(x). \tag{3.22}$$

Deferring the proof of these estimates for the moment, gathering them with (3.18) and (3.19) we easily conclude as desired

$$\begin{aligned} w\{x \in \Gamma : |(I-P)^{1/2} f(x)| > \lambda\} &\le w(\cup_i 4\, B_i) + I + II + III \\ &\le \frac{C}{\lambda} \sum_{x\in\Gamma} \nabla f(x)\, w(x)\, m(x). \end{aligned}$$

We show (3.21). We need the following estimate, see [6]: if $f$ is supported in $B$, $j \geq 2$ and $l \geq 1$,

$$\sup_{x \in C_j(B)} \big(|P^l f(x)| + l|(I-P)P^l f(x)|\big) \leq C\, e^{-c\frac{4^j r(B)^2}{l}} \frac{1}{V(B)} \sum_{x \in B} |f(x)|\, m(x). \tag{3.23}$$

Using some ideas from [4], the fact that $\operatorname{supp} b_i \subset B_i$, (3.23) and that $0 \leq a_k \lesssim k^{-1/2}$ we obtain

$$\begin{aligned}
II &\leq \frac{3}{\lambda} \sum_{i \in I} \sum_{x \notin 4\,B_i} |T_i b_i(x)| w(x)\, m(x) \leq \frac{3}{\lambda} \sum_{i \in I} \sum_{j=2}^{\infty} \sum_{x \in C_j(B_i)} |T_i b_i(x)| w(x)\, m(x) \\
&\leq \frac{3}{\lambda} \sum_{i \in I} \sum_{j=2}^{\infty} \sum_{x \in C_j(B_i)} \sum_{k=1}^{r_i^2} |a_k|\, |(I-P)P^k b_i(x)|\, w(x)\, m(x) \\
&\leq \frac{C}{\lambda} \sum_{i \in I} \sum_{j=2}^{\infty} \frac{w(2^{j+1}\,B_i)}{V(B_i)} \sum_{k=1}^{r_i^2} k^{-3/2} e^{-c\frac{4^j r_i^2}{k}} \sum_{x \in B_i} |b_i(x)|\, m(x) \\
&\leq \frac{C}{\lambda} \sum_{i \in I} \sum_{j=2}^{\infty} 2^{j\,D} \sum_{k=1}^{r_i^2} k^{-3/2} e^{-c\frac{4^j r_i^2}{k}} \sum_{x \in B_i} |b_i(x)|\, \frac{w(2^{j+1}\,B_i)}{V(2^{j+1}\,B_i)} m(x) \\
&\leq \frac{C}{\lambda} \sum_{i \in I} \sum_{j=2}^{\infty} 2^{j\,D} \sum_{k=1}^{r_i^2} k^{-3/2} e^{-c\frac{4^j r_i^2}{k}} \sum_{x \in B_i} |b_i(x)|\, w(x)\, m(x) \\
&\leq C \sum_{i \in I} w(B_i) \sum_{j=2}^{\infty} 2^{j\,D} \sum_{k=1}^{r_i^2} k^{-3/2} r_i\, e^{-c\frac{4^j r_i^2}{k}}.
\end{aligned}$$

where we have used that $w \in A_1(\Gamma)$ and (A.10). Easy computations lead to

$$\sum_{k=1}^{r_i^2} k^{-3/2} r_i\, e^{-c\frac{4^j r_i^2}{k}} \leq C\, r_i \int_1^{r_i^2+1} t^{-3/2}\, e^{-c\frac{4^j r_i^2}{t}}\, dt \leq C \int_{1/2}^{r_i^2} s^{1/2}\, e^{-c\,4^j\,s}\, \frac{ds}{s} \leq C\, e^{-\alpha\, 4^j}$$

which, by (A.11), yields

$$II \leq C \sum_{i \in I} w(B_i) \leq \frac{C}{\lambda} \sum_{x \in \Gamma} \nabla f(x)\, w(x)\, m(x).$$

This shows (3.21).

Let us obtain (3.22). For every $j \geq 0$ and $k \geq 2$ we define

$$\beta_j = \sum_{i \in I,\ r_i = 2^j} \frac{b_i}{r_i}, \qquad\qquad f_k = \frac{a_k}{\sqrt{k}} \sum_{j:\ 4^j < k} 2^j\, \beta_j.$$

We also set $f_1 = 0$. Then,

$$\begin{aligned}
\sum_{i \in I} U_i b_i &= \sum_{i \in I} \sum_{k=r_i^2+1}^{\infty} a_k (I-P)P^k b_i = \sum_{j=0}^{\infty} \sum_{k=4^j+1}^{\infty} a_k (I-P)P^k (2^j\, \beta_j) \\
&= \sum_{k=2}^{\infty} \sqrt{k}\, (I-P)P^k f_k
\end{aligned}$$

$$= \sum_{k=1}^{\infty} \sqrt{k}\,(I-P)P^k f_k = S_P \vec{f},$$

where $\vec{f} = \{f_k\}_{k\geq 1}$. By Theorem 2.7 it follows that $S_P$ is bounded from $L^1_{\ell^2}(\Gamma, w)$ into $L^{1,\infty}(\Gamma, w)$. Then,

$$III = w\{x\in\Gamma : |S_P\vec{f}(x)| > \lambda/3\} \leq \frac{C}{\lambda}\sum_{x\in\Gamma}\Big(\sum_{k=1}^{\infty}|f_k(x)|^2\Big)^{\frac{1}{2}} w(x)\,m(x).$$

By Cauchy-Schwarz and using that $0 \leq a_k \lesssim k^{-1/2}$ we have

$$\sum_{k=1}^{\infty}|f_k|^2 \lesssim \sum_{k=2}^{\infty} k^{-2}\Big|\sum_{j:\ 4^j<k} 2^j\,\beta_j\Big|^2 \lesssim \sum_{k=1}^{\infty} k^{-3/2}\sum_{j:\ 4^j<k} 2^j\,|\beta_j|^2$$
$$\lesssim \sum_{j=0}^{\infty}|\beta_j|^2 \leq \Big(\sum_{j=0}^{\infty}|\beta_j|\Big)^2 \leq \Big(\sum_{i\in I}\frac{|b_i|}{r_i}\Big)^2$$

Then, (A.10) and (A.11) yield

$$III \lesssim \frac{1}{\lambda}\sum_{i\in I} r_i^{-1}\sum_{x\in B_i}|b_i(x)|\,w(x)\,m(x)$$
$$\lesssim \sum_{i\in I} r_i^{-1}\,w(B_i)\,r(B_i) \lesssim \sum_{i\in I} w(B_i) \lesssim \frac{1}{\lambda}\sum_{x\in\Gamma}\nabla f(x)\,w(x)\,m(x).$$

This shows (3.22) and therefore the proof is completed. □

3.3. **Proof of Theorem 2.7.** We first observe that $(i)$ and $(iii)$ are equivalent and therefore it suffices to obtain $(ii)$, $(iii)$ and $(iv)$. Fix $1 < p < \infty$ and $w \in A_p$. We see that $(i)$ implies $(iii)$: by (2.4) for $\vec{f} \in L^\infty_{\ell^2,0}(\Gamma)$ and $h \in L^\infty_0(\Gamma)$ we have

$$\Big|\sum_{x\in\Gamma} S_P\vec{f}(x)\,h(x)\,w(x)\,m(x)\Big| \leq \sum_{x\in\Gamma}\|\vec{f}(x)\|_{\ell^2}\,g_P(h\,w)(x)\,m(x)$$
$$\leq \|\vec{f}\|_{L^p_{\ell^2}(\Gamma,w)}\,\|g_P(h\,w)\|_{L^{p'}(\Gamma,w^{1-p'})} \leq C\,\|\vec{f}\|_{L^p_{\ell^2}(\Gamma,w)}\,\|h\|_{L^{p'}(\Gamma,w)}$$

where we have used $(i)$ and the fact that $w^{1-p'} \in A_{p'}(\Gamma)$. To conclude $(iii)$ it suffices to take the supremum over all those $h$ with $\|h\|_{L^{p'}(\Gamma,w)} \leq 1$. Conversely, to obtain that $(iii)$ implies $(i)$ we proceed as follows: by (2.4) for $\vec{f} \in L^\infty_{\ell^2,0}(\Gamma)$ and $h \in L^\infty_0(\Gamma)$ we have

$$\Big|\sum_{x\in\Gamma}\langle\vec{f}(x), \vec{g}_P h(x)\rangle_{\ell^2}\,w(x)\,m(x)\Big| \leq \sum_{x\in\Gamma}|S_P(\vec{f}\,w)(x)|\,|h(x)|\,m(x)$$
$$\leq \|S_P(\vec{f}\,w)\|_{L^{p'}(\Gamma,w^{1-p'})}\,\|h\|_{L^p(\Gamma,w)} \leq C\,\|\vec{f}\|_{L^{p'}_{\ell^2}(\Gamma,w)}\,\|h\|_{L^p(\Gamma,w)}$$

where we have used $(iii)$ and the fact that $w^{1-p'} \in A_{p'}(\Gamma)$. To conclude $(i)$ it suffices to take the supremum over all those $\vec{f}$ with $\|\vec{f}\|_{L^{p'}_{\ell^2}(\Gamma,w)} \leq 1$.

*Proof of Theorem 2.7, $(iii)$.* Fix $1 < p < \infty$ and $w \in A_p(\Gamma)$. There exists $1 < p_0 < \min\{p, 2\}$ such that $w \in A_{p/p_0}$. We use Theorem A.3 (indeed we use its vector-valued extension) with the $p_0$ just chosen and $q_0 = \infty$. We pick $T = S_P$, $\mathcal{A}_k = I - (I -$

$P^{2(1+k^2)})^m$, with $k$ the radius of the ball $B$, $m$ large enough and $S\vec{f}=\vec{f}$. We notice that $T\mathcal{A}_k=\mathcal{A}_kT$. Then, expanding $\mathcal{A}_k$ and writing $T\vec{f}(x)=\sum_{j=1}^{\infty}T\vec{f}(x)\,\chi_{C_j(B)}(x)$, (3.15) implies (A.6) with $\alpha_j=C_m\,e^{-c\,4^j}$.

To complete the proof we need to show (A.5). Given $\vec{f}\in L^{\infty}_{\ell^2,0}(\Gamma)$ we write $\vec{f}=\sum_{j=1}^{\infty}\vec{f}_j$ with $\vec{f}_j=\vec{f}\,\chi_{C_j}$. Set $f_{l,j}=f_l\,\chi_{C_j}$. For $j=1$ we use that $S_P$ is bounded from $L^{p_0}(\Gamma)$ into $L^{p_0}_{\ell^2}(\Gamma)$:

$$\begin{aligned}\Big(\frac{1}{V(B)}\sum_{x\in B}|S_P(I-\mathcal{A}_k)\vec{f}_1(x)|^{p_0}m(x)\Big)^{\frac{1}{p_0}}&\lesssim\frac{1}{V(B)^{\frac{1}{p_0}}}\|(I-P^{2(1+k^2)})^m\vec{f}_1\|_{L^{p_0}_{\ell^2}(\Gamma)}\\&\lesssim\frac{1}{V(B)^{\frac{1}{p_0}}}\|\vec{f}_1\|_{L^{p_0}_{\ell^2}(\Gamma)}+\sum_{n=1}^{m}\frac{1}{V(B)^{\frac{1}{p_0}}}\|P^{2\,n\,(1+k^2)}\vec{f}_1\|_{L^{p_0}_{\ell^2}(\Gamma)}\\&=\Big(\frac{1}{V(4\,B)}\sum_{x\in 4\,B}\|\vec{f}(x)\|^{p_0}_{\ell^2}m(x)\Big)^{\frac{1}{p_0}}+\sum_{n=1}^{m}I_n\end{aligned}$$

Fixed $1\le n\le m$ we estimate $I_n$ using (1.8) for $i=1$ , (3.23) for $i\ge 2$, and Minkowski's inequality:

$$\begin{aligned}I_n&\le\sum_{i=1}^{\infty}\Big(\frac{V(2^{i+3}\,B)}{V(B)}\Big)^{\frac{1}{p_0}}\Big(\sum_{l=1}^{\infty}\sup_{x\in C_i(4\,B)}|P^{2\,n\,(1+k^2)}(f_l\,\chi_{4\,B})(x)|^2\Big)^{\frac12}\\&\lesssim\sum_{i=1}^{\infty}2^{i\,D/p_0}\,e^{-c4^i}\Big(\sum_{l=1}^{\infty}\Big(\frac{1}{V(4\,B)}\sum_{x\in 4\,B}|f_l(x)|m(x)\Big)^2\Big)^{\frac12}\\&\lesssim\frac{1}{V(4\,B)}\sum_{x\in 4\,B}\|\vec{f}(x)\|_{\ell^2}m(x)\\&\le\Big(\frac{1}{V(4\,B)}\sum_{x\in 4\,B}\|\vec{f}(x)\|^{p_0}_{\ell^2}m(x)\Big)^{\frac{1}{p_0}}.\end{aligned}$$

Thus we conclude that

$$\Big(\frac{1}{V(B)}\sum_{x\in B}|S_P(I-\mathcal{A}_k)\vec{f}_1(x)|^{p_0}m(x)\Big)^{\frac{1}{p_0}}\lesssim\Big(\frac{1}{V(4\,B)}\sum_{x\in 4\,B}\|\vec{f}(x)\|^{p_0}_{\ell^2}m(x)\Big)^{\frac{1}{p_0}}$$

Next, we consider $j\ge 2$:

$$\begin{aligned}&\Big(\frac{1}{V(B)}\sum_{x\in B}|S_P(I-\mathcal{A}_k)\vec{f}_j(x)|^{p_0}m(x)\Big)^{\frac{1}{p_0}}\\&\qquad\le\sum_{l=1}^{\infty}\sqrt{l}\,\Big(\frac{1}{V(B)}\sum_{x\in B}|(I-P)\,P^l(I-\mathcal{A}_k)f_{l,j}(x)|^{p_0}m(x)\Big)^{\frac{1}{p_0}}\\&\qquad\le\sum_{l=1}^{2^{j+1}\,(1+k^2)\,m}\cdots+\sum_{l>2^{j+1}\,(1+k^2)\,m}\cdots=I+II.\end{aligned}$$

To estimate $I$ we expand $\mathcal{A}_k$ and use (3.15) to conclude that

$$I\lesssim\sum_{n=0}^{m}\sum_{l=1}^{2^{j+1}\,(1+k^2)\,m}\frac{\sqrt{l}\,e^{-c\frac{4^j\,k^2}{l+2\,(1+k^2)\,n}}}{l+2\,(1+k^2)\,n}\,\frac{1}{V(2^{j+1}\,B)}\sum_{x\in 2^{j+1}\,B}|f_l(x)|\,m(x)$$

$$\leq \frac{1}{V(2^{j+1}B)} \sum_{x\in 2^{j+1}B} \|\vec{f}(x)\|_{\ell^2}\, m(x) \sum_{n=0}^{m} \Big( \sum_{l=1}^{2^{j+1}(1+k^2)\,m} \frac{l\, e^{-c\frac{4^j k^2}{l+2(1+k^2)\,n}}}{(l+2(1+k^2)\,n)^2} \Big)^{\frac{1}{2}}$$

and we estimate the inner sum. It is straightforward to show that

$$\sum_{l=1}^{2^{j+1}(1+k^2)\,m} \frac{l\, e^{-c\frac{4^j k^2}{l+2(1+k^2)\,n}}}{(l+2(1+k^2)\,n)^2} \leq \int_1^{2^{j+1}(1+k^2)\,m+1} \frac{e^{-c\frac{4^j (1+k^2)}{t+2(1+k^2)\,n}}}{t+2(1+k^2)\,n}\, dt$$
$$\lesssim \int_{c_m 2^j}^{\infty} e^{-c\,s} \frac{ds}{s} \lesssim e^{-c\,2^j}. \tag{3.24}$$

Therefore we obtain

$$I \leq C_m\, e^{-c\,2^j} \Big( \frac{1}{V(2^{j+1}B)} \sum_{x\in 2^{j+1}B} \|\vec{f}(x)\|_{\ell^2}^{p_0}\, m(x) \Big)^{\frac{1}{p_0}}.$$

We next consider $II$. We claim that the following estimates hold (the proof is given at the end of this section): for every $n \geq 1$

$$|(I-P)\,P^n g(x)| \leq \frac{C}{n\, V(x,\sqrt{n})} \|g\|_{L^1(\Gamma)} \tag{3.25}$$

and

$$\|P^{n\,2(1+k^2)} - P^{(n+1)\,2(1+k^2)} g\|_{L^1(\Gamma)} \leq C\, n^{-1} \|g\|_{L^1(\Gamma)} \tag{3.26}$$

with $C$ independent of $k$.

Fix $l > 2^{j+1}(1+k^2)\,m$. Then, there exists an integer $q \geq 2^{j-1}$ such that $q\,4\,m\,(1+k^2) < l \leq (q+1)\,4\,m\,(1+k^2)$. We set $n = l - q\,2\,m\,(1+k^2)$. By (3.25) and (3.26), for every $x \in B = B(x_0,k)$ we have

$$|(I-P)\,P^l\,(I-P^{2(1+k^2)})^m g(x)| = |(I-P)\,P^n\,(P^{q\,2(1+k^2)} - P^{(q+1)\,2(1+k^2)})^m g(x)|$$
$$\leq \frac{C_m}{n\, q^m\, V(x,\sqrt{n})} \|g\|_{L^1(\Gamma)} \leq \frac{C_m}{l\, V(x_0,\sqrt{l})} \Big(\frac{k^2}{l}\Big)^m \|g\|_{L^1(\Gamma)}, \tag{3.27}$$

where we have used that $B(x_0,\sqrt{l}) \subset B(x, 2\sqrt{l})$ and as a consequence $V(x_0,\sqrt{l}) \leq C\,V(x,\sqrt{l})$. We define $\gamma_l = l^{-1/2}\, V(x_0,\sqrt{l})^{-1}\,(k^2/l)^m$ for $l > 2^{j+1}(1+k^2)\,m$ and $\gamma_l = 0$ for $1 \leq l \leq 2^{j+1}(1+k^2)\,m$. Using the previous estimate and Hölder's inequality we obtain

$$II \leq \sum_{x\in 2^{j+1}B} \sum_{l=1}^{\infty} \gamma_l\, |f_l(x)|\, m(x)$$
$$\leq \|\vec{\gamma}\|_{\ell^2} \sum_{x\in 2^{j+1}B} \|\vec{f}(x)\|_{\ell^2} m(x)$$
$$\leq \|\vec{\gamma}\|_{\ell^2}\, V(2^{j+1}B) \Big( \frac{1}{V(2^{j+1}B)} \sum_{x\in 2^{j+1}B} \|\vec{f}(x)\|_{\ell^2}^{p_0}\, m(x) \Big)^{\frac{1}{p_0}}.$$

On the other hand, it is straightforward to show that

$$\|\vec{\gamma}\|_{\ell^2}^2\, V(2^{j+1}B)^2 = \sum_{l>2^{j+1}(1+k^2)\,m} \Big( \frac{V(x_0, 2^{j+1}k)}{V(x_0,\sqrt{l})} \Big)^2 \Big(\frac{k^2}{l}\Big)^{2m} \frac{1}{l}$$

$$\lesssim \sum_{l>2^{j+1}(1+k^2)\,m} \Big(1+\frac{2^{j+1}\,k}{\sqrt{l}}\Big)^{2D}\Big(\frac{k^2}{l}\Big)^{2m}\frac{1}{l}$$
$$\lesssim \int_{2^{j+1}k^2 m}^{\infty} \Big(1+\frac{2^{j+1}\,k}{\sqrt{t}}\Big)^{2D}\Big(\frac{k^2}{t}\Big)^{2m}\frac{dt}{t}$$
$$\lesssim 2^{-j\,(2m-D)}.$$

Gathering this with the estimates obtained before we conclude that for every $j\ge 2$:

$$\Big(\frac{1}{V(B)}\sum_{x\in B}|S_P(I-\mathcal{A}_k)\vec{f}_j(x)|^{p_0}m(x)\Big)^{\frac{1}{p_0}}$$
$$\lesssim 2^{-j\,(m-\frac{D}{2})}\Big(\frac{1}{V(2^{j+1}B)}\sum_{x\in 2^{j+1}B}\|\vec{f}(x)\|_{\ell^2}^{p_0}\,m(x)\Big)^{\frac{1}{p_0}}.$$

This and the corresponding estimate for $j=1$ lead to (A.5) with $\alpha_j = 2^{-j\,(m-\frac{D}{2})}$ in which case we take $m>D/2$ to obtain $\sum_j \alpha_j<\infty$. □

*Proof of Theorem 2.7, (iv).* We use Theorem A.2 (indeed its vector-valued extension) with $p_0=1$, $q_0=\infty$, $T=S_P$, $\mathcal{A}_k = I-(I-P^{2(1+k^2)})^m$, with $k$ the radius of the ball $B$ and $m$ large enough. Fix $w\in A_1$. Note that $(a)$ follows from $(iii)$ as $S_P$ is bounded on $L^q(w)$ for every $1<q<\infty$. After expanding $\mathcal{A}_k$, it suffices to show $(b)$ for $P^{2n(1+k^2)}$ with $1\le n\le m$: fixed such an $n$, (3.2) yields as desired

$$\sup_{x\in C_j(B)}\|P^{2n(1+k^2)}\vec{f}(x)\|_{\ell^2}\lesssim e^{-c\,4^j}\left\|\Big\{\frac{1}{V(B)}\sum_{x\in B}|f_l(x)|\,m(x)\Big\}_l\right\|_{\ell^2}$$
$$\le e^{-c\,4^j}\frac{1}{V(B)}\sum_{x\in B}\|\vec{f}(x)\|_{\ell^2}\,m(x).$$

To see $(c)$ we take $\beta=\infty$ and fix $j\ge 2$. We proceed as in the Proof of Theorem 2.7, $(iii)$, to obtain

$$\sup_{x\in C_j(B)}|S_P(I-\mathcal{A}_k)\vec{f}(x)|\le \sum_{l=1}^{\infty}\sqrt{l}\sup_{x\in C_j(B)}|(I-P)\,P^l(I-\mathcal{A}_k)f_l(x)|$$
$$\le \sum_{l=1}^{2^{j+1}(1+k^2)\,m}\cdots+\sum_{l>2^{j+1}(1+k^2)\,m}\cdots = I+II.$$

To estimate $I$ we expand $\mathcal{A}_k$ and use (3.23) and (3.24) to conclude that

$$I\lesssim \sum_{n=0}^{m}\sum_{l=1}^{2^{j+1}(1+k^2)\,m}\frac{\sqrt{l}\,e^{-c\frac{4^j\,k^2}{l+2\,(1+k^2)\,n}}}{l+2\,(1+k^2)\,n}\,\frac{1}{V(B)}\sum_{x\in B}|f_l(x)|\,m(x)$$
$$\le \frac{1}{V(B)}\sum_{x\in B}\|\vec{f}(x)\|_{\ell^2}\,m(x)\sum_{n=0}^{m}\Bigg(\sum_{l=1}^{2^{j+1}(1+k^2)\,m}\frac{l\,e^{-c\frac{4^j\,k^2}{l+2\,(1+k^2)\,n}}}{(l+2\,(1+k^2)\,n)^2}\Bigg)^{\frac{1}{2}}$$
$$\le C_m\,e^{-c\,2^j}\frac{1}{V(B)}\sum_{x\in B}\|\vec{f}(x)\|_{\ell^2}\,m(x).$$

To estimate $II$ we proceed as in (3.27) to obtain that for every $x \in C_j(B)$

$$\begin{aligned} |(I-P)\,P^l\,(I-P^{2\,(1+k^2)})^m g(x)| &\le \frac{C_m}{l\,V(x,\sqrt{l})}\left(\frac{k^2}{l}\right)^m \|g\|_{L^1(\Gamma)} \\ &\le \frac{C_m\,2^{j\,\frac{D}{2}}}{l\,V(x_0,\sqrt{l})}\left(\frac{k^2}{l}\right)^m \|g\|_{L^1(\Gamma)} \end{aligned}$$

since $B(x_0,\sqrt{l}) \subset B(x, 2^{(j+3)/2}\sqrt{l})$. We take the same sequence as before $\vec{\gamma}$ and obtain

$$\begin{aligned} II &\le 2^{j\,\frac{D}{2}} \sum_{x\in B}\sum_{l=1}^{\infty} \gamma_l\,|f_l(x)|\,m(x) \\ &\le 2^{j\,\frac{D}{2}}\,\|\vec{\gamma}\|_{\ell^2} \sum_{x\in B} \|\vec{f}(x)\|\,m(x) \\ &\le 2^{-j\,(m-\frac{D}{2})}\,\frac{1}{V(B)} \sum_{x\in B} \|\vec{f}(x)\|\,m(x), \end{aligned}$$

where we have repeated the computations to estimate $\|\vec{\gamma}\|_{\ell^2}$ and we have used that $B \subset B(x_0,\sqrt{l})$. Collecting $I$ and $II$ we conclude $(c)$ with $\alpha_j = 2^{-j\,(m-\frac{D}{2})}$. To obtain $(d)$ we just need to take $m > D_w + D/2$. □

*Proof of Theorem 2.7, (ii).* We use Theorem A.2 (indeed its vector-valued extension) with $p_0 = 1$, $q_0 = \infty$, $T = g_P$, $\mathcal{A}_k = I - (I - P^{2(1+k^2)})^m$, with $k$ the radius of the ball $B$ and $m$ large enough. Fix $w \in A_1$. Note that $(a)$ follows from $(i)$ as $g_P$ is bounded on $L^q(w)$ for every $1 < q < \infty$. Notice that expanding $\mathcal{A}_k$, (3.2) yields $(b)$ with $\alpha_j = C\,e^{-c\,4^j}$. To see $(c)$ we take any $\beta > (s_w)'$, fix $j \ge 2$ and $f$ supported in $B$. We proceed by duality. Let $\vec{h} \in L^\infty_{\ell^2}(\Gamma)$ with $\operatorname{supp}\vec{h} \subset C_j(B)$ and $\|h\|_{L^{\beta'}_{\ell^2}(\Gamma)} \le 1$. Then, using (2.4) and that $S_P$ and $\mathcal{A}_k$ commute we have

$$\begin{aligned} \Big|\sum_{x\in\Gamma} \langle \vec{g}_P(I-\mathcal{A}_k)f(x), \vec{h}(x)\rangle_{\ell^2}\,m(x)\Big| &\le \sum_{x\in\Gamma} |f(x)|\,|S_P(I-\mathcal{A}_k)\vec{h}(x)|\,m(x) \\ &\le \sum_{x\in B} \|\vec{f}(x)\|\,m(x)\,\sup_{x\in B} |S_P(I-\mathcal{A}_k)\vec{h}(x)|. \end{aligned}$$

Fixed $x \in B$ and proceeding as in $(iii)$ we obtain

$$\begin{aligned} |S_P(I-\mathcal{A}_k)\vec{h}(x)| &\le \sum_{l=1}^{\infty} \sqrt{l}\,\sup_{x\in B} |(I-P)\,P^l(I-\mathcal{A}_k)h_l(x)| \\ &\le \sum_{l=1}^{2^{j+1}\,(1+k^2)\,m} \cdots + \sum_{l>2^{j+1}\,(1+k^2)\,m} \cdots = I + II. \end{aligned}$$

To estimate $I$ we expand $\mathcal{A}_k$, use (3.15) and (3.24) to conclude that

$$\begin{aligned} I &\lesssim \sum_{n=0}^{m} \sum_{l=1}^{2^{j+1}\,(1+k^2)\,m} \frac{\sqrt{l}\,e^{-c\frac{4^j\,k^2}{l+2\,(1+k^2)\,n}}}{l+2\,(1+k^2)\,n}\,\frac{1}{V(2^{j+1}\,B)} \sum_{x\in 2^{j+1}\,B} |h_l(x)|\,m(x) \\ &\le \frac{1}{V(2^{j+1}\,B)} \sum_{x\in 2^{j+1}\,B} \|\vec{h}(x)\|_{\ell^2}\,m(x) \sum_{n=0}^{m} \left( \sum_{l=1}^{2^{j+1}\,(1+k^2)\,m} \frac{l\,e^{-c\frac{4^j\,k^2}{l+2\,(1+k^2)\,n}}}{(l+2\,(1+k^2)\,n)^2} \right)^{\frac{1}{2}} \end{aligned}$$

$$\leq C_m\, e^{-c\,2^j}\Big(\frac{1}{V(2^{j+1}\,B)}\sum_{x\in 2^{j+1}\,B}\|\vec{h}(x)\|_{\ell^2}^{\beta'}\,m(x)\Big)^{\frac{1}{\beta'}}$$
$$\leq C_m\, e^{-c\,2^j}\,V(2^{j+1}\,B)^{-\frac{1}{\beta'}}.$$

We next estimate $II$ as we did in the proof of $(iii)$:

$$II \leq \|\vec{\gamma}\|_{\ell^2}\sum_{x\in 2^{j+1}\,B}\|\vec{h}(x)\|\,m(x) \leq 2^{-j\,(m-\frac{D}{2})}\,\frac{1}{V(2^{j+1}\,B)}\sum_{x\in 2^{j+1}\,B}\|\vec{h}(x)\|\,m(x)$$
$$\leq 2^{-j\,(m-\frac{D}{2})}\,V(2^{j+1}\,B)^{-\frac{1}{\beta'}}\Big(\sum_{x\in 2^{j+1}\,B}\|\vec{h}(x)\|^{\beta'}\,m(x)\Big)^{\frac{1}{\beta'}} \leq 2^{-j\,(m-\frac{D}{2})}\,V(2^{j+1}\,B)^{-\frac{1}{\beta'}}.$$

Gathering the obtained estimates and taking the sup over all such $\vec{h}$ we conclude that

$$\Big(\frac{1}{V(2^{j+1}\,B)}\sum_{x\in C_j(B)}(g_P(I-\mathcal{A}_k)f(x))^{\beta}\,m(x)\Big)^{\frac{1}{\beta}}$$
$$\leq C_m\,2^{-j\,(m-\frac{D}{2})}\frac{1}{V(B)}\sum_{x\in B}\|\vec{f}(x)\|\,m(x).$$

Thus, we have proved $(c)$ with $\alpha_j = 2^{-j\,(m-\frac{D}{2})}$. Finally, if $m$ is taken so that $m > D_w + D/2$ then we obtain $(d)$ and the proof is completed. □

*Proof of* (3.25). We need the following estimate (see [6, p. 288] and the references therein): for every $x, y\in\Gamma$ and $n\geq 1$

$$|p_n(x,y)-p_{n+1}(x,y)| \leq \frac{C\,m(y)}{n\,V(x,\sqrt{n})}\,e^{-c\,\frac{d(x,y)^2}{n}}. \tag{3.28}$$

Then we clearly have

$$|(I-P)\,P^n g(x)| \leq \sum_{y\in\Gamma}\frac{C\,m(y)}{n\,V(x,\sqrt{n})}\,e^{-c\,\frac{d(x,y)^2}{n}}\,|g(y)| \leq \frac{C}{n\,V(x,\sqrt{n})}\sum_{y\in\Gamma}|g(y)|\,m(y).$$

□

*Proof of* (3.26). Given $l\geq 1$ using (3.28) we have

$$\|(I-P)\,P^l g\|_{L^1(\Gamma)} \leq \sum_{x,y\in\Gamma}\frac{C\,m(y)}{l\,V(x,\sqrt{l})}\,e^{-c\,\frac{d(x,y)^2}{l}}\,|g(y)|\,m(x)$$
$$\leq \frac{C}{l}\sum_{y\in\Gamma}|g(y)|\,m(y)\sum_{x\in\Gamma}\frac{e^{-c\,\frac{d(x,y)^2}{l}}}{V(x,\sqrt{l})}\,m(x)$$
$$\leq \frac{C}{l}\sum_{y\in\Gamma}|g(y)|\,m(y) \tag{3.29}$$

where the last estimate is as follows:

$$\sum_{x\in\Gamma}\frac{e^{-c\,\frac{d(x,y)^2}{l}}}{V(x,\sqrt{l})}\,m(x) \leq \sum_{x:d(x,y)\leq 4\,\sqrt{l}}\frac{1}{V(x,\sqrt{l})}\,m(x)+\sum_{j=2}^{\infty}\sum_{x:2^j<d(x,y)/\sqrt{l}\leq 2^{j+1}}\frac{e^{-c\,4^j}}{V(x,\sqrt{l})}\,m(x)$$

$$\leq \sum_{j=1}^{\infty} \sum_{x\in B(y,2^{j+1}\sqrt{l})} \frac{e^{-c\,4^j}}{V(x,\sqrt{l})}\, m(x) \leq C \sum_{j=1}^{\infty} e^{-c\,4^j}\, 2^{j\,D} \leq C$$

where we have used that $B(y,2^{j+1}\sqrt{l}) \subset B(x,2^{j+2}\sqrt{l})$ and therefore $V(y,2^{j+1}\sqrt{l}) \leq C\,2^{j\,D}\,V(x,\sqrt{l})$.

To obtain (3.26) we use (3.29):

$$\|P^{n\,2\,(1+k^2)} - P^{(n+1)\,2\,(1+k^2)} g\|_{L^1(\Gamma)} \leq \sum_{l=n\,2\,(1+k^2)}^{(n+1)\,2\,(1+k^2)-1} \|(P^l - P^{l+1})g\|_{L^1(\Gamma)}$$

$$\leq \sum_{y\in\Gamma} |g(y)|\, m(y) \sum_{l=n\,2\,(1+k^2)}^{(n+1)\,2\,(1+k^2)-1} \frac{C}{l} \leq \frac{C}{n} \sum_{y\in\Gamma} |g(y)|\, m(y).$$

□

## 4. Commutators

Let $b \in \mathrm{BMO}(\Gamma)$, that is,

$$\|b\|_{\mathrm{BMO}(\Gamma)} = \sup_B \frac{1}{V(B)} \sum_{x\in B} |b(x) - b_B|\, m(x)$$

where the sup is taken over all balls and $b_B$ is the average of $b$ over $B$. Write $T = \nabla(I-P)^{-1/2}$ which is a sublinear operator. Given $k \geq 0$ we define the $k$th order commutator of the Riesz transform as

$$T_b^k f(x) = T((b(x)-b)^k\, f)(x), \qquad f \in L_0^\infty(\Gamma), \quad x\in\Gamma.$$

Note that $T_b^0 = T$. One can alternatively define the commutators using the associated linearization of $T$. Let us write again $\mathcal{T} = \vec{\nabla}(I-P)^{-1/2}$ which is a linear operator. We define the first order commutator $\mathcal{T}_b^1 g = [b,\mathcal{T}]g = b\,\mathcal{T} g - \mathcal{T}(b\,g)$, and for $k\geq 2$ the $k$-th order commutator is $\mathcal{T}_b^k = [b,\mathcal{T}_b^{k-1}]$. Here $g, b$ are scalar valued and $\mathcal{T}_b^k g$ is valued in $\ell^2$. It is straightforward to see that $T_b^k f = \|\mathcal{T}_b^k f\|_{\ell^2}$.

**Theorem 4.1.** *Under the assumptions of Theorem 2.2, for every $k\geq 1$, and $w \in A_\infty(\Gamma)$ we have*

$$\|T_b^k f\|_{L^p(\Gamma,w)} \leq C_{p,w} \|b\|_{\mathrm{BMO}(\Gamma)}^k\, \|f\|_{L^p(\Gamma,w)}, \qquad f\in L_0^\infty(\Gamma)$$

*for all $p \in \mathcal{W}_w(1,q_+)$.*

Note that even the unweighted $L^p$ estimates for the commutators are new.

*Proof.* The proof is similar to that of Theorem 2.2 using again the ideas from [5] and [7], we point out the main changes. We only consider the case $k=1$: the general case follows by induction and the details are left to the reader (see [3, Section 6.2] for similar arguments). As in [3, Lemma 6.1] it suffices to assume qualitatively $b\in L^\infty(\Gamma)$ and quantitatively $\|b\|_{\mathrm{BMO}(\Gamma)} = 1$ and get uniform bounds.

We proceed as before working with $\mathcal{T}_b^1$ in place of $\mathcal{T}$. Write $F = |(\mathcal{T}_b^1)^* f|^{q_0'}$ with $f \in L_{\ell^2}^\infty(\Gamma)$ with compact support. Observe that $F\in L^1(\Gamma)$ as $b\in L^\infty(\Gamma)$ and $\mathcal{T}^*$ is bounded from $L_{\ell^2}^{q_0'}(\Gamma)$ into $L^{q_0'}(\Gamma)$ (as $1<q_0<q_+$) —we note that this is the only place where we use that $b\in L^\infty(\Gamma)$—. Fixing $B$, we write $\hat{b} = b - b_B$ and decompose

$\mathcal{T}_b^1$ as $\mathcal{T}_b^1 g = -\mathcal{T}(\hat{b} g) + \hat{b}\,\mathcal{T} g$. Using this equality one sees that $(\mathcal{T}_b^1)^* = -(\mathcal{T}^*)_b^1$. Then we have

$$\begin{aligned} F &= |(\mathcal{T}_b^1)^* f|^{q_0'} = |(\mathcal{T}^*)_b^1 f|^{q_0'} \le 2^{q_0'-1}\,|\hat{b}\,\mathcal{T}^* f|^{q_0'} + 2^{q_0'-1}\,|\mathcal{T}^*(\hat{b}\,f)|^{q_0'} \\ &\le \big(2^{q_0'-1}\,|\hat{b}\,\mathcal{T}^* f|^{q_0'} + 4^{q_0'-1}\,|(I-\mathcal{A}_k^*)\mathcal{T}^*(\hat{b}\,f)|^{q_0'}\big) + 4^{q_0'-1}\,|\mathcal{A}_k^*\mathcal{T}^*(\hat{b}\,f)|^{q_0'} \\ &= G_B + H_B. \end{aligned}$$

We estimate $H_B$. By duality we take $g \in L^{p_0}(B, m/V(B))$ with norm 1 and obtain

$$\begin{aligned} \Big(\frac{1}{V(B)} \sum_{y\in B} H_B(y)^q\, m(y)\Big)^{\frac{1}{q\,q_0'}} &= C\,\Big|\frac{1}{V(B)} \sum_{y\in\Gamma} \mathcal{T}^*(\hat{b}\,f)(x)\,\mathcal{A}_k g(y)\, m(y)\Big| \\ &= C\,\Big|\frac{1}{V(B)} \sum_{y\in\Gamma} \big(-(\mathcal{T}^*)_b^1 f + \hat{b}\,\mathcal{T}^* f\big)(y)\,\mathcal{A}_k g(y)\, m(y)\Big| \\ &\lesssim \frac{1}{V(B)} \sum_{y\in\Gamma} |(\mathcal{T}^*)_b^1 f(y)|\,|\mathcal{A}_k g(y)|\, m(y) + \frac{1}{V(B)} \sum_{y\in\Gamma} |\hat{b}(y)|\,|\mathcal{T}^* f(y)|\,|\mathcal{A}_k g(y)|\, m(y) \\ &= I + II. \end{aligned}$$

The estimate for $I$ follows as in (3.1), (3.3) by using (3.2): for all $x \in B$,

$$\begin{aligned} I &\lesssim \sum_{j=1}^{\infty} 2^{j\,D}\Big(\frac{1}{V(2^{j+1}B)} \sum_{y\in C_j(B)} F(y)\, m(y)\Big)^{\frac{1}{q_0'}}\Big(\frac{1}{V(2^{j+1}B)} \sum_{y\in C_j(B)} |\mathcal{A}_k g(y)|^{q_0}\, m(y)\Big)^{\frac{1}{q_0}} \\ &\lesssim MF(x)^{\frac{1}{q_0'}}. \end{aligned}$$

Regarding $II$ we use Hölder's inequality to obtain that for all $\bar{x} \in B$

$$\begin{aligned} II &\lesssim \sum_{j=1}^{\infty} 2^{j\,D}\Big(\frac{1}{V(2^{j+1}B)} \sum_{y\in C_j(B)} |\mathcal{T}^* f(x)|^{q_0'}\, m(y)\Big)^{\frac{1}{q_0'}} \sup_{y\in C_j(B)} |\mathcal{A}_k g(y)| \\ &\qquad\qquad \times \Big(\frac{1}{V(2^{j+1}B)} \sum_{y\in 2^{j+1}B} |\hat{b}(y)|^{q_0}\, m(y)\Big)^{\frac{1}{q_0}} \\ &\lesssim \|b\|_{\mathrm{BMO}(\Gamma)} M(|\mathcal{T}^* f|^{q_0'})(\bar{x})^{\frac{1}{q_0'}} \sum_{j=1}^{\infty} 2^{j\,D} e^{-c\,4^j}\,(1+j) \lesssim M(|\mathcal{T}^* f|^{q_0'})(\bar{x})^{\frac{1}{q_0'}}, \end{aligned}$$

where we have used (3.2) and John-Nirenberg's inequality. Collecting $I$ and $II$, we conclude the first estimate in (A.1) with $H_1 = M(|\mathcal{T}^* f|^{q_0'})$. Let us write $G_{B,1}$ and $G_{B,2}$ for each of the terms that define $G_B$ and we estimate them in turn. Let $\delta > 1$ to be chosen and use John-Nirenberg's inequality: for any $x \in B$, we have

$$\begin{aligned} \frac{1}{V(B)} \sum_{y\in B} G_{B,1}(y)\, m(y) &= C\,\frac{1}{V(B)} \sum_{y\in B} |(\hat{b}\,\mathcal{T}^* f)(y)|^{q_0'}\, m(y) \\ &\lesssim \Big(\frac{1}{V(B)} \sum_{y\in B} |\mathcal{T}^* f|^{q_0'\,\delta}(x)\, m(y)\Big)^{\frac{1}{\delta}} \\ &\qquad \Big(\frac{1}{V(B)} \sum_{y\in B} |b(y) - b_B|^{q_0'\,\delta'}\, m(y)\Big)^{\frac{1}{\delta'}} \\ &\lesssim \|b\|_{\mathrm{BMO}(\Gamma)}^{q_0'} M(|\mathcal{T}^* f|^{q_0'\,\delta})(x)^{\frac{1}{\delta}}. \end{aligned}$$

To estimate $G_{B,2}$ we proceed as with $G_B$ in the proof of Theorem 2.2. Let $g$ be the corresponding dual function and use again John-Nirenberg's inequality: for any $x \in B$,

$$\begin{aligned}
\Big(\frac{1}{V(B)}\sum_{y\in B} G_{B,2}(y)\,m(y)\Big)^{\frac{1}{q_0'}} &= C\,\Big(\frac{1}{V(B)}\sum_{y\in B} |(I-\mathcal{A}_k)^*\,\mathcal{T}^*(\hat{b}f)(y)|^{q_0'}\,m(y)\Big)^{\frac{1}{q_0'}} \\
&\lesssim \frac{1}{V(B)}\sum_{y\in\Gamma} |\hat{b}(y)|\,\|f(y)\|_{\ell^2}\,\|\mathcal{T}(I-\mathcal{A}_k)g(y)\|_{\ell^2}\,m(y) \\
&\lesssim \sum_{j=1}^{\infty} 2^{j\,D}\Big(\frac{1}{V(2^{j+1}B)}\sum_{y\in 2^{j+1}B} |b(y)-b_B|^{q_0'\,\delta'}\,m(x)\Big)^{\frac{1}{q_0'\delta'}} \\
&\qquad\times\Big(\frac{1}{V(2^{j+1}B)}\sum_{y\in C_j(B)} \|f(y)\|_{\ell^2}^{q_0'\,\delta}\,m(y)\Big)^{\frac{1}{q_0'\,\delta}} \\
&\qquad\qquad\times\Big(\frac{1}{V(2^{j+1}B)}\sum_{y\in C_j(B)} |T(I-\mathcal{A}_k)g|^{q_0}\,m(y)\Big)^{\frac{1}{q_0}} \\
&\lesssim M(\|f\|_{\ell^2}^{\delta\,q_0'})(x)^{\frac{1}{\delta\,q_0'}}\sum_{j=1}^{\infty} 2^{j\,D}\,(1+j)\Big(\frac{1}{V(2^{j+1}B)}\sum_{y\in C_j(B)} |T(I-\mathcal{A}_k)g(y)|^{q_0}\,m(x)\Big)^{\frac{1}{q_0}} \\
&\lesssim M(\|f\|_{\ell^2}^{\delta\,q_0'})(x)^{\frac{1}{\delta\,q_0'}}.
\end{aligned}$$

where in the last estimate we have proceeded as in (3.7) using (3.4) for $j \ge 2$ and (3.6) for $j = 1$. Gathering the estimates for $G_{B,1}$ and $G_{B,2}$ we conclude the second estimate in (A.1) with $G = M(|\mathcal{T}^* f|^{q_0'\,\delta})^{\frac{1}{\delta}} + M(\|f\|_{\ell^2}^{\delta\,q_0'})^{\frac{1}{\delta}}$.

We apply Theorem A.1 as in the proof of Theorem 2.2. In this case, we observe that as $v \in A_r(\Gamma)$, we can take $1 < \delta < r$ so that $v \in A_{r/\delta}(\Gamma)$. Then, the desired estimate follows

$$\begin{aligned}
\|(\mathcal{T}_b^1)^* f\|_{L^{p'}(\Gamma,v)}^{q_0'} &\le \|MF\|_{L^r(\Gamma,v)} \lesssim \|G\|_{L^r(\Gamma,v)} + \|H_1\|_{L^r(\Gamma,v)} \\
&\le \|M(\|f\|_{\ell^2}^{\delta\,q_0'})^{\frac{1}{\delta}}\|_{L^r(\Gamma,v)} + \|M(|\mathcal{T}^* f|^{\delta\,q_0'})^{\frac{1}{\delta}}\|_{L^r(\Gamma,v)} \\
&\lesssim \|f\|_{L^{p'}_{\ell^2}(\Gamma,v)}^{q_0'} + \|\mathcal{T}^* f\|_{L^{p'}(\Gamma,v)}^{q_0'} \lesssim \|f\|_{L^{p'}_{\ell^2}(\Gamma,v)}^{q_0'}
\end{aligned}$$

where we have used that $\mathcal{T}^*$ is bounded from $L^{p'}_{\ell^2}(\Gamma, v)$ into $L^{p'}(\Gamma, v)$. □

## Appendix A. Auxiliary results

We use the following version of [3, Theorem 3.1] in the setting of spaces of homogeneous type (see [3, Section 5]).

**Theorem A.1.** *Fix $1 < q < \infty$, $a \ge 1$ and $v \in RH_{s'}(\Gamma)$, $1 < s < q$. Then, there exist $C$ and $K_0 \ge 1$ with the following property: Assume that $F$, $G$ and $H_1$ are non-negative functions on $\Gamma$ such that for any ball $B$ there exist non-negative functions $G_B$ and $H_B$ with $F(x) \le G_B(x) + H_B(x)$ for a.e. $x \in B$ and, for all $x, \bar{x} \in B$,*

$$\Big(\frac{1}{V(B)}\sum_{x\in B} H_B(x)^q\,m(x)\Big)^{\frac{1}{q}} \le a\,MF(x) + H_1(\bar{x}), \qquad \frac{1}{V(B)}\sum_{x\in B} G_B(x)\,m(x) \le G(x). \tag{A.1}$$

*If* $1 < r \le q/s$ *and* $F \in L^1(\Gamma)$ *(this assumption being only qualitative) we have*

$$\|MF\|_{L^r(\Gamma,v)} \le C\,\|G\|_{L^r(\Gamma,v)} + C\,\|H_1\|_{L^r(\Gamma,v)}. \tag{A.2}$$

The following result is taken from [5, Theorem 3.3], see also [3, Theorem 8.8].

**Theorem A.2.** *Let* $1 \le p_0 < q_0 \le \infty$ *and* $w \in A_\infty(\Gamma)$. *Let* $T$ *be a sublinear operator defined on* $L^2(\Gamma)$ *and* $\{\mathcal{A}_k\}_{k\ge1}$ *be a family of operator acting from* $L^\infty_c(\Gamma)$ *into on* $L^2(\Gamma)$. *Assume the following conditions:*

(a) *There exists* $q \in \mathcal{W}_w(p_0,q_0)$ *such that* $T$ *is bounded from* $L^q(\Gamma,w)$ *to* $L^{q,\infty}(\Gamma,w)$.

(b) *For all* $j \ge 1$, *there exist a constant* $\alpha_j$ *such that for any ball* $B$ *with* $k$ *its radius and for any* $f \in L^\infty_c(\Gamma)$ *supported in* $B$,

$$\Big(\frac{1}{V(2^{j+1}B)}\sum_{x\in C_j(B)}|\mathcal{A}_k f(x)|^{q_0}\,m(x)\Big)^{\frac{1}{q_0}} \le \alpha_j\,\Big(\frac{1}{V(B)}\sum_{x\in B}|f(x)|^{p_0}\,m(x)\Big)^{\frac{1}{p_0}}. \tag{A.3}$$

(c) *There exists* $\beta > (s_w)'$, *i.e.* $w \in RH_{\beta'}(\Gamma)$, *with the following property: for all* $j \ge 2$, *there exist a constant* $\alpha_j$ *such that for any ball* $B$ *with* $k$ *its radius and for any* $f \in L^\infty_c(\Gamma)$ *supported in* $B$ *and for* $j \ge 2$,

$$\Big(\frac{1}{V(2^{j+1}B)}\sum_{x\in C_j(B)}|T(I-\mathcal{A}_{r(B)})f(x)|^{\beta}\,m(x)\Big)^{\frac{1}{\beta}} \le \alpha_j\,\Big(\frac{1}{V(B)}\sum_{x\in B}|f(x)|^{p_0}\,m(x)\Big)^{\frac{1}{p_0}}. \tag{A.4}$$

(d) $\sum_j \alpha_j\,2^{D_w j} < \infty$ *for* $\alpha_j$ *in* (b) *and* (c), *where* $D_w$ *is the doubling constant of the measure* $w(x)\,m(x)$.

*If* $w \in A_1(\Gamma) \cap RH_{(q_0/p_0)'}(\Gamma)$ *then, for all* $f \in L^\infty_c(\Gamma)$,

$$\|Tf\|_{L^{p_0,\infty}(\Gamma,w)} \le C\,\|f\|_{L^{p_0}(\Gamma,w)}.$$

The following result is borrowed from [3], see Theorem 3.7 for the Euclidean version and Section 5 there for the extension to spaces of homogeneous type.

**Theorem A.3.** *Let* $1 \le p_0 < q_0 \le \infty$. *Suppose that* $T$ *is a sublinear operator acting on* $L^{p_0}(\Gamma)$, $(\mathcal{A}_k)_{k\ge1}$ *is a family of operators acting from a subspace* $\mathcal{D}$ *of* $L^{p_0}(\Gamma)$ *into* $L^{p_0}(\Gamma)$ *and* $S$ *is an operator from* $\mathcal{D}$ *into the space of measurable functions on* $\Gamma$. *Assume that*

$$\Big(\frac{1}{V(B)}\sum_{x\in B}|T(I-\mathcal{A}_k)f|^{p_0}m(x)\Big)^{\frac{1}{p_0}} \le C\sum_{j\ge1}\alpha_j\,\Big(\frac{1}{V(2^{j+1}B)}\sum_{x\in 2^{j+1}B}|Sf(x)|^{p_0}\,m(x)\Big)^{\frac{1}{p_0}} \tag{A.5}$$

*and*

$$\Big(\frac{1}{V(B)}\sum_{x\in B}|T\mathcal{A}_k f|^{q_0}m(x)\Big)^{\frac{1}{q_0}} \le C\sum_{j\ge1}\alpha_j\,\Big(\frac{1}{V(2^{j+1}B)}\sum_{x\in 2^{j+1}B}|Tf(x)|^{p_0}\,m(x)\Big)^{\frac{1}{p_0}}, \tag{A.6}$$

*for all* $f \in \mathcal{D}$, *every ball* $B$ *and for some sequence* $\{\alpha_j\}_j$ *with* $\sum_{j\ge1}\alpha_j < \infty$. *Let* $p_0 < p < q_0$ *and* $w \in A_{\frac{p}{p_0}}(\Gamma) \cap RH_{(\frac{q_0}{p})'}(\Gamma)$. *Then for all* $f \in \mathcal{D}$ *we have*

$$\|Tf\|_{L^p(\Gamma,w)} \le C\|Sf\|_{L^p(\Gamma,w)}.$$

We present an adapted weighted Calderón-Zygmund decomposition that extends [6, Proposition 1.15] (see also [4, Lemma 6.6] and [3, Proposition 9.1]).

**Proposition A.4.** *Assume that* ($D$) *and* ($P_1$) *hold. Let* $w \in A_1$ *and take a function* $f$ *on* $\Gamma$ *such that* $\nabla f \in L^1(\Gamma, w)$ *and* $\lambda > 0$*. Then one can find a collection of balls* $(B_i)_{i\in I}$*, functions* $(b_i)_{i\in I}$ *and* $g$ *such that the following properties hold:*

$$f = g + \sum_{i\in I} b_i, \tag{A.7}$$

$$\|\nabla g\|_\infty \le C\lambda, \tag{A.8}$$

$$\operatorname{supp} b_i \subset B_i, \qquad \sum_{x\in 2B_i} \nabla b_i(x)\, w(x)\, m(x) \le C\lambda\, w(B_i), \tag{A.9}$$

$$\sum_{x\in B_i} |b_i(x)|\, w(x)\, m(x) \le C\lambda\, w(B_i)\, r(B_i), \tag{A.10}$$

$$\sum_{i\in I} w(B_i) \le C\lambda^{-1} \sum_{x\in\Gamma} \nabla f(x)\, w(x)\, m(x), \tag{A.11}$$

$$\sum_{i\in I} \chi_{B_i} \le N, \tag{A.12}$$

*where* $C$ *and* $N$ *depend on the constants in* ($D$) *and* ($P_1$) *and* $w \in A_1$.

*Proof.* The proof follows the steps of [6, Proposition 1.15] with $q = p = 1$ with the underlying measure $w(x)\, m(x)$ that is doubling since $w \in A_1$. We need a weighted-Poincaré inequality. We claim that there exists $\tau \ge 1$ such that for every $f \in L^1_{\text{loc}}(\Gamma, w)$ such that $\nabla f \in L^1_{\text{loc}}(\Gamma, w)$ and for every ball $B$ we have

$$\sum_{y\in B} |f(y) - f_{B,w}|\, w(y)\, m(y) \le C\, r(B) \sum_{y\in\tau B} |\nabla f(y)|\, w(y)\, m(y), \tag{$P_1(w)$}$$

where $f_{B,w}$ is the $w$-average of $f$ on $B$. We notice that in [6, Proposition 1.15] the corresponding unweighted Poincaré inequality holds with $\tau = 1$. Here we may have $\tau > 1$, but a closer examination of the proof reveals that this change is harmless: the balls come from a Whitney covering and therefore $\tau B$ on the right hand-side will be handled by passing to a sufficiently large ball that meets the complement of the level set (details are left to the reader). Consequently, one obtains (A.7), (A.8), (A.9), (A.11), (A.12). It remains to show (A.10). Following the notation in [6, Proposition 1.15] and using ($P_1(w)$),

$$\begin{aligned}
\sum_{x\in B_i} |b_i(x)|\, w(x)\, m(x) &= \sum_{x\in B_i} |f(x) - f_{B_i,w}|\, \chi_i(x)\, w(x)\, m(x) \\
&\le \sum_{x\in B_i} |f(x) - f_{B_i,w}|\, w(x)\, m(x) \\
&\lesssim r(B_i) \sum_{x\in\tau \overline{B_i}} |\nabla f(y)|\, w(y)\, m(y) \\
&\le r(B_i)\, w(\tau \overline{B_i})\, M_w(\nabla f)(x_0) \\
&\lesssim r(B_i)\, w(B_i)\, \lambda
\end{aligned}$$

where $x_0 \in \overline{B_i} \cap \Omega^c$, $\Omega = \{x \in \Gamma : M_w(\nabla f)(x) > \lambda\}$ and $M_w$ is the Hardy-Littlewood maximal function for the measure $w(x)\, m(x)$.

To complete the proof we show ($P_1(w)$). We use [11]. From ($P_1$) and $w \in A_1(\Gamma)$

$$\frac{1}{V(B)}\sum_{y\in B}|f(y)-f_B|\,m(y)\le C\,r(B)\,\frac{1}{w(B)}\sum_{y\in B}|\nabla f(y)|\,\frac{w(B)}{V(B)}\,m(y)$$
$$\le C\,r(B)\,\frac{1}{w(B)}\sum_{y\in B}|\nabla f(y)|\,w(y)\,m(y)=a(B).$$

Let $\{B_i\}_i\subset B$ be a family of disjoint balls. Then, $w\in A_1(\Gamma)$ and (1.7) imply

$$\Big(\frac{r(B_i)}{r(B)}\Big)^D\lesssim\frac{V(B_i)}{V(B)}=\frac{1}{w(B)}\sum_{x\in\Gamma}\chi_{B_i}(x)\,\frac{w(B)}{V(B)}\,m(x)$$
$$\lesssim\frac{1}{w(B)}\sum_{x\in\Gamma}\chi_{B_i}(x)\,w(x)\,m(x)=\frac{w(B_i)}{w(B)}$$

We fix $1<r\le\frac{D}{D-1}$ if $D>1$ and $1<r<\infty$ if $D=1$. Then, it is easy to obtain

$$\begin{aligned}\sum_i a(B_i)^r\,w(B_i)&=\sum_i\frac{r(B_i)^r}{w(B_i)^{r-1}}\Big(\sum_{x\in B_i}\nabla f(x)w(x)\,m(x)\Big)^r\\&\lesssim\frac{r(B)^r}{w(B)^{r-1}}\sum_i\Big(\frac{w(B_i)}{w(B)}\Big)^{\frac{r-D\,(r-1)}{D}}\Big(\sum_{x\in B_i}\nabla f(x)w(x)\,m(x)\Big)^r\\&\le\frac{r(B)^r}{w(B)^{r-1}}\Big(\sum_i\sum_{x\in B_i}\nabla f(x)w(x)\,m(x)\Big)^r\\&\le\frac{r(B)^r}{w(B)^{r-1}}\Big(\sum_{x\in B}\nabla f(x)w(x)\,m(x)\Big)^r\\&=a(B)^r\,w(B).\end{aligned}$$

We apply [11, Theorem 2.3] and Kolmogorov's inequality to conclude that

$$\frac{1}{w(B)}\sum_{y\in B}|f(y)-f_{B,w}|\,w(y)\,m(y)\le\frac{2}{w(B)}\sum_{y\in B}|f(y)-f_B|\,w(y)\,m(y)$$
$$\le\|f-f_B\|_{L^{r,\infty}(\Gamma,\frac{w(x)\,m(x)}{w(B)})}\le C\,a(\tau\,B)=C\,r(B)\,\frac{1}{w(\tau\,B)}\sum_{y\in\tau\,B}|\nabla f(y)|\,w(y)\,m(y),$$

and this readily leads to ($P_1(w)$). □

## References


[1] P. Auscher & T. Coulhon, *Riesz transforms on manifolds and Poincaré inequalities*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4** (2005), 1–25.

[2] P. Auscher, T. Coulhon, X.T. Duong & S. Hofmann, *Riesz transforms on manifolds and heat kernel regularity*, Ann. Scient. ENS Paris **37** (2004), no. 6, 911–957.

[3] P. Auscher & J.M. Martell, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part I: General operator theory and weights*, Adv. Math. **212** (2007), no. 1, 225–276.

[4] P. Auscher & J.M. Martell, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part III: Harmonic analysis of elliptic operators*, J. Funct. Anal. **241** (2006), no. 2, 703–746.

[5] P. Auscher & J.M. Martell, *Weighted norm inequalities, off-diagonal estimates and elliptic operators. Part IV: Riesz transforms on manifolds and weights*, Math. Z. **260** (2008), no. 3, 527–539.

[6] N. Badr & E. Russ, *Interpolation of Sobolev spaces, Littlewood-Paley inequalities and Riesz transforms on graphs*, Pub. Mat., **53** (2009) no. 2, 273–328.

[7] F. Bernicot & J. Zhao, *Abstract Hardy Spaces*, J. Funct. Anal. **255** (2008), no. 7, 1761– 1796.

[8] R.R. Coifman & G. Weiss, *Analyse Harmonique non-conmutative sur certains espaces homogènes*, Lecture Notes in Mathematics **242**, Springer-Verlag, 1971.

[9] T. Coulhon & A. Grigor'yan, *Random walks on graphs with regular volume growth*, Geom. and Funct. Anal. **8** (1998), 656–701.

[10] T. Delmotte, *Parabolic Harnack inequality*, Rev. Mat. Iberoamericana **15** (1999),no. 1, 181–232.

[11] B. Franchi, C. Pérez and R. L. Wheeden, *Self-improving properties of John-Nirenberg and Poincaré inequalities on spaces of homogeneous type*, J. Funct. Anal. 153 (1998), 108-146.

[12] V. Gol'dshtein & M. Troyanov, *Axiomatic theory of Sobolev Spaces*, Expo. Math. **19** (2001), 289–336.

[13] P. Hajlasz & P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000), no. 688.

[14] S. Keith & X. Zhong,*The Poincaré inequality is an open ended condition*, Ann. Math **167** (2008), no. 2, 575–599.

[15] J.M. Martell, *Desigualdades con pesos en el Análisis de Fourier: de los espacios de tipo homogéneo a las medidas no doblantes*, Ph.D. Thesis, Universidad Autónoma de Madrid, 2001.

[16] E. Russ, *Riesz transforms on graphs for* $1 \leq p \leq 2$, Math. Scand. **87** (2000), 133–160.

[17] E. Russ, $H^1 - L^1$ *boundedness of Riesz transforms on Riemannian manifolds and on graphs*, Pot. Anal. **14** (2001), 301–330.

[18] J.O. Strömberg & A. Torchinsky, *Weighted Hardy spaces*, Lecture Notes in Mathematics 1381, Springer-Verlag, 1989.

Nadine Badr, Université Claude Bernard, Lyon I et CNRS UMR 5208, Bâtiment Doyen Jean Braconnier, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France

*E-mail address*: badr@math.univ-lyon1.fr

José María Martell, Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, Consejo Superior de Investigaciones Científicas, C/ Serrano 121, E-28006 Madrid, Spain

*E-mail address*: chema.martell@uam.es