Explicit bases in representation theory by rewriting

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Plan of this talk

I. Introduction : algebraic rewriting.

II. Presentation of linear $(2, 2)$-categories.

III. Quiver Hecke algebras and Poincaré-Birkhoff-Witt bases.

I. Introduction : algebraic rewriting.

Algebraic rewriting

- Many algebraic properties can be formulated by equations.

  Example.

$$\text{Associativity : } (x \cdot y) \cdot z = x \cdot (y \cdot z)\,;$$

$$\text{Commutativity : } x \cdot y = y \cdot x\,;$$

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- These questions have been studied in an algebraic context for associative and commutative algebras :

  - Gröbner bases to compute with ideals, Gröbner '49 and Buchberger '65 ;
  - Linear rewriting and Koszulness, Guiraud-Hoffbeck-Malbos '2017

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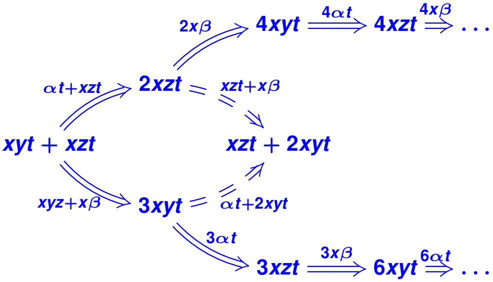
Example. Let $\Delta = \langle x, y, z, t \mid xy \overset{\alpha}{\Longrightarrow} xz \ , \ zt \overset{\beta}{\Longrightarrow} zyt \ \rangle$.



II. Presentation of linear $(2,2)$-categories.

Presentations of linear $(2,2)$-categories

- A linear $(2,2)$-category $\mathcal{C}$ is a 2-category with :
  - 0-cells $\mathcal{C}_0$;
  - 1-cells $\mathcal{C}_1$;
  - 2-cells $\mathcal{C}_2$ such that for every p and q in $\mathcal{C}_1$, the space of 2-cells $\mathcal{C}_2(p,q)$ between p and q is a $\mathbb{K}$-vector space for a field $\mathbb{K}$;

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- *Compute normal forms using the theory of normal forms by rewriting.*

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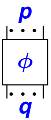
- There are two main difficulties :
  - The analysis of 3-dimensional critical branchings is complicated
  - One has to require termination to prove the critical pair lemma.

Presentations of linear $(2, 2)$-categories

- In these categories, the generating 2-cells have the form of a circuit as follows :



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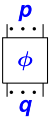

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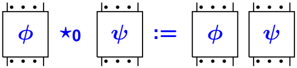
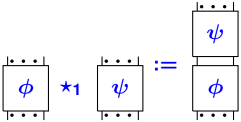
- They can be composed in two ways

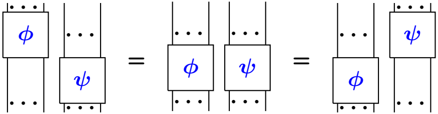
Horizontally



Vertically



- All these compositions are made modulo the exchange law of the 2-category, which is diagrammatically depicted as :



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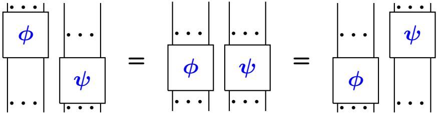

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- We recall that $\star_1 : \mathcal{C}_2(p,q) \otimes \mathcal{C}_2(q,r) \to \mathcal{C}_2(p,r)$ is linear and that all the sources and target maps are compatible with the linear structure.

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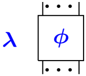

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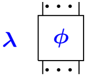

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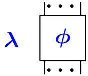

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- The *support* of ϕ is the set of all the ϕ_i in that decomposition.

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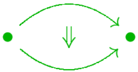

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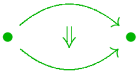

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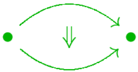

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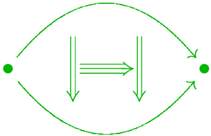

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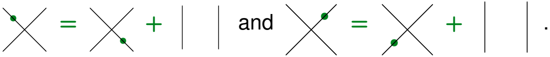

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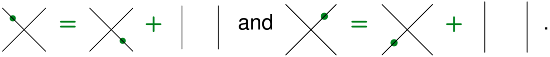
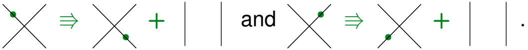
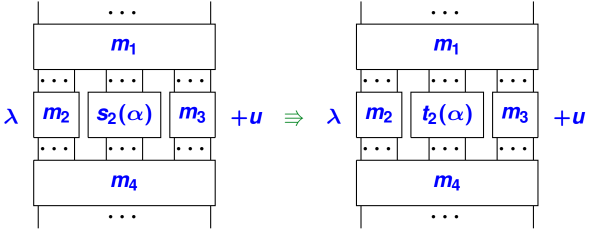


This category can be presented by the linear $(3, 2)$-polygraph defined with the same 0-cell, 1-cell and 2-cells and the relations being oriented in the following way :



Rewriting properties of linear $(3, 2)$-polygraphs

- A *rewriting step* of Σ is a 3-cell of the form



where $s_2(\alpha)$ and $t_2(\alpha)$ are two parallel 2-cells such that the monomial $\lambda m_1 \star_1 (m_2 \star_0 s_2(\alpha) \star_0 m_3) \star_1 m_4$ does not appear in the monomial decomposition of u.

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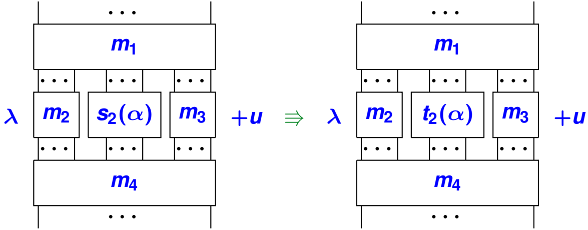

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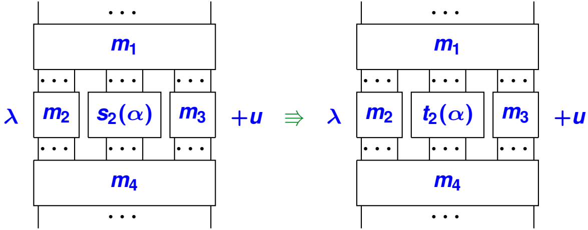

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- A *normal form* is a 2-cell that can't be reduced by any rewriting step.

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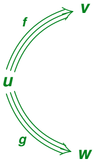
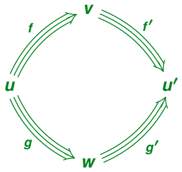
- A *branching* of Σ is a pair of 3-cells with the same 2-source :



Rewriting properties of linear $(3, 2)$-polygraphs

- A *branching* of Σ is a pair of 3-cells with the same 2-source.
- A branching is *confluent* if it can be completed by rewriting sequences f' and g' as follows :



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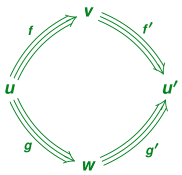

- A *local branching* of Σ is a pair of rewriting steps of Σ with the same 2-source.

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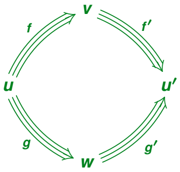

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- Let $\sqsubseteq$ be the order on monomials of Σ such that $f \sqsubseteq g$ if $g = m_1 \star_1 (m_2 \star_0 f \star_0 m_3) \star_1 m_4$ for monomials m_i. A *critical branching* is a branching such that its source is minomal for $\sqsubseteq$.

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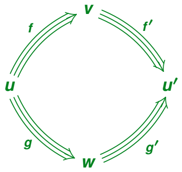

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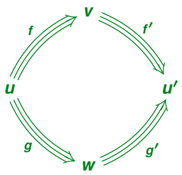
- Let $\sqsubseteq$ be the order on monomials of Σ such that $f \sqsubseteq g$ if $g = m_1 \star_1 (m_2 \star_0 f \star_0 m_3) \star_1 m_4$ for monomials m_i. A *critical branching* is a branching such that its source is minomal for $\sqsubseteq$.
- A linear $(3,2)$-polygraph is :
  - *confluent (resp. locally confluent)* if all its (resp. local) branchings are confluent.
  - *terminating* if it has no infinite rewriting sequence.

Rewriting properties of linear $(3,2)$-polygraphs

- A *branching* of Σ is a pair of 3-cells with the same 2-source
- A branching is *confluent* if it can be completed by rewriting sequences f' and g' as follows :



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  - *terminating* if it has no infinite rewriting sequence.
  - *left monomial* is every source of a 3-cell in Σ is a monomial.

Rewriting results

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Proposition

*A **terminating** linear $(3,2)$-polygraph is confluent if and only if all its critical branchings are confluent.*

Proposition (Alleaume,'16)

Let Σ be a confluent and terminating left-monomial linear $(3,2)$-polygraph and $\mathcal{C}$ be the linear $(2,2)$-category presented by Σ. Then, for any 1-cells u and v of $\mathcal{C}$ with same 0-source and 0-target, the set of monomials of Σ in normal form from u to v gives a basis of $\mathcal{C}(u,v)$.

Critical branchings for linear $(3, 2)$-polygraphs

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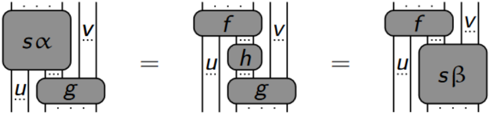

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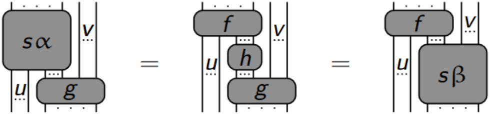

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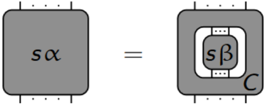

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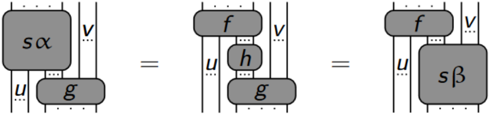

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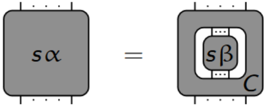
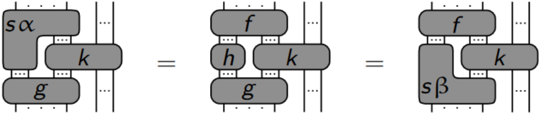


  - Left-indexed (also left-indexed, multi-indexed) critical branchings :



Confluence of indexed critical branchings

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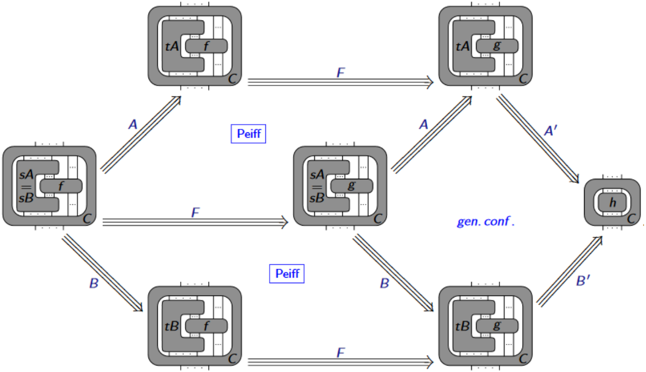

III. Quiver Hecke algebras and Poincaré-Birkhoff-Witt bases

- In higher dimensional representation theory, a natural way to study an algebraic structure is to extend actions of it on categories rather than vector spaces.

KLR algebras for categorification of quantum groups

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- This work was done for a process of categorification of quantum groups associated with symmetrizable Kac-Moody algebras, following Khovanov-Lauda '08 and Rouquier '08.
- The KLR algebras (or quiver Hecke algberas) are a family of algebras that arise naturally in this process.
  - They admit a diagrammatic presentation by generators and relations, Khovanov-Lauda '08
  - They can be seen as linear $(2,2)$-categories.

Poincaré-Birkhoff-Witt bases

- We will explicit linear **(3, 2)**-polygraphs that present the simply-laced KLR algebras and prove that they are convergent.
- We will obtain a rewriting proof of the following algebraic result obtained by Khovanov and Lauda :

Theorem

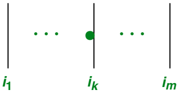
The simply-laced KLR algebras admit Poincaré-Birkhoff-Witt bases

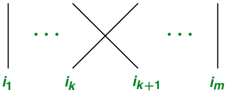
- These PBW bases have interesting algebraic and homological features.
  - They are linear bases.
  - They are build from a monomial order $\preceq$ on a generating set of the algebra.
  - The product of two elements of the basis is greater for $\preceq$ than every element in its monomial decomposition.

Definition of the KLR algebras

- Let Γ be a graph whose set of vertices is denoted by I. We set $\mathcal{V} = \sum_{i \in I} \nu_i . i \in \mathbb{N}[I]$ an element of the free semi-group generated by I.
- Let $\cdot$ be a bilinear form defined on $\mathbb{Z}[I]$ with values in $\mathbb{Z}$ such that $i \cdot j \in \{0, -1\}$ for all $i, j \in I$.
- We put $m := |\mathcal{V}| = \sum \mathcal{V}_i$.
- We consider the set $\mathrm{Seq}(\mathcal{V})$ which consists of all sequences of vertices of Γ with length m in which the vertex i appears exactly $\mathcal{V}_i$ times.
    - For instance, $\mathrm{Seq}(2i + j) = \{iij, iji, jii\}$.
- For $\mathbf{i}$ and $\mathbf{j} \in \mathrm{Seq}(\mathcal{V})$, we define the set ${}_{\mathbf{j}}R(\mathcal{V})_{\mathbf{i}}$ as the set of "braid-like diagrams" from $\mathbf{i}$ to $\mathbf{j}$, that is :
    - Each strand is labelled by a vertex of Γ ;
    - A brand does not intersect with itself ;
    - One has to read $\mathbf{i}$ (resp. $\mathbf{j}$) at the bottom (resp. at the top) of the diagram

Definition of the KLR algebras

- These algebras are given by a diagrammatic presentation by generators and relations.
- For $\mathbf{i} = i_1 \dots i_m \in \mathrm{Seq}(\mathcal{V})$, we represent the generators by :
  - $x_{k,\mathbf{i}}$ = 

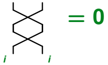
 for $1 \leq k \leq m, \mathbf{i} = i_1 \dots i_m \in \mathrm{Seq}(\mathcal{V})$
  - $\tau_{k,\mathbf{i}}$ = 

 for $1 \leq k \leq m-1, \mathbf{i} = i_1 \dots i_m \in \mathrm{Seq}(\mathcal{V})$
- These algebras can be seen as linear $(2,2)$-categories with :
  - One 0-cell,
  - The 1-cells are the elements of $\mathrm{Seq}(\mathcal{V})$,
  - The 2-cells between two sequences $\mathbf{i}$ and $\mathbf{i}$ are ${}_{\mathbf{j}}R(\mathcal{V})_{\mathbf{i}}$.

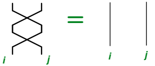
Definition of the KLR algebras

- The local relations are represented by :

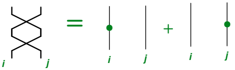
i) For $i \in I$,



ii) For $i, j \in I$ such that $i \cdot j = 0$,



iii) For $i, j \in I$ such that $i \cdot j = -1$,



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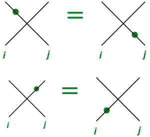

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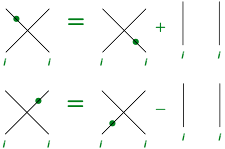
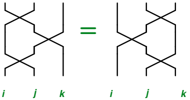


vi) For $i, j, k \in I$, and unless $i = k$ and $i \cdot j = -1$,



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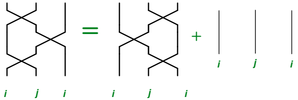
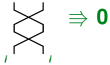


The linear $(3, 2)$-polygraphs KLR

- The $\mathbf{3}$-cells of the linear $(\mathbf{3}, \mathbf{2})$-polygraph KLR are given by :

i) For $i \in I$,



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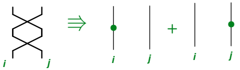

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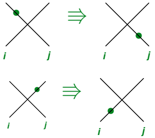

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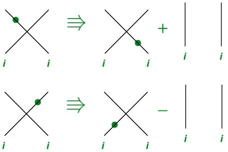

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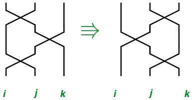

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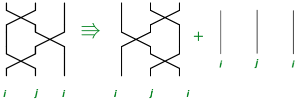

The linear $(\mathbf{3}, \mathbf{2})$-polygraphs KLR

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Theorem

The linear $(\mathbf{3}, \mathbf{2})$-polygraphs KLR are convergent.

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The linear $\mathbf{(3,2)}$-polygraphs $\mathcal{KLR}$ are convergent.

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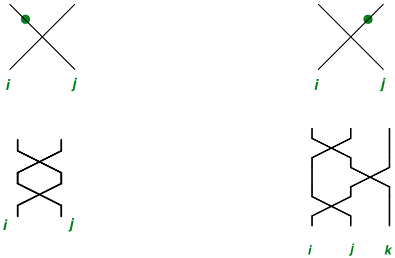
- The monomials in normal form give a basis for each space of $\mathbf{2}$-cells, which provide a basis of the algebra. It turns out to be a Poincaré-Birkhoff-Witt basis.
- There is no exhaustive methods to prove termination in dimension $\mathbf{3}$. However, some techniques exist. We used a theorem of Guiraud-Malbod '09 generalising in a categorical framework an idea of Guiraud '06.

Study of the critical branchings

- We have **4** different forms for the sources of **3**-cells :



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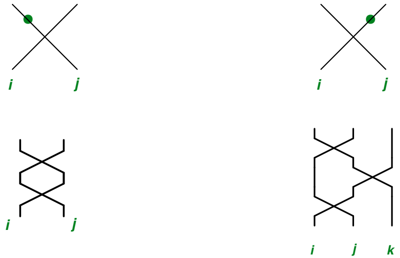
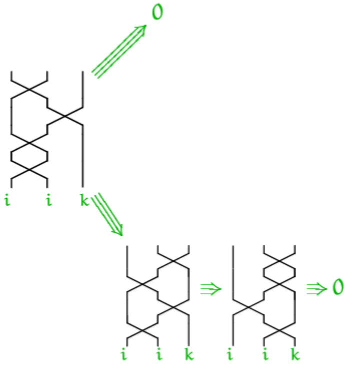


for every ***i***, ***j*** and ***k*** in ***I***.

- They depend on the vertices ***i***, ***j*** and ***k*** at the bottom.
- The critical branchings have to be computed for each sequence of vertices and each values of the bilinear form.

Examples of critical branchings

- **Sequence :** ***iik***
- **Value of · : 0** or **−1**
- **Branching :**



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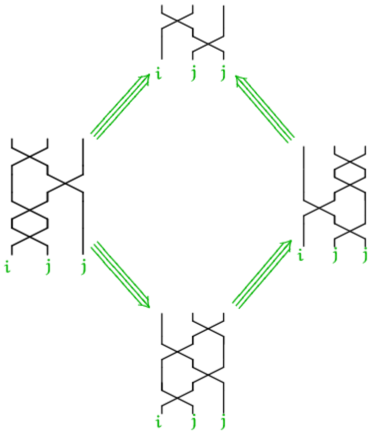

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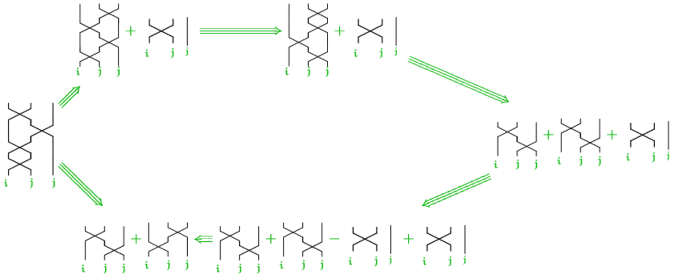

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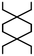
  - **Triple crossings :**  with itself

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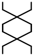
  - **Triple crossings :**  with itself
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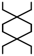
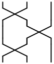
  - **Triple crossings :**  with itself
  - **Double crossings with dots :**  or  with 
  - **Double Yang-Baxter :**  with itself

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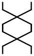
  - **Triple crossings :**  with itself
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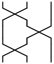
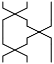
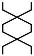
  - **Double Yang-Baxter :**  with itself
  - **Yang-Baxter with crossings :**  with 

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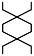
  - **Triple crossings :**  with itself
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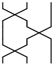
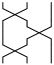
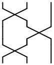
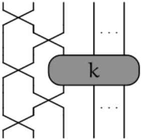
  - **Double Yang-Baxter :**  with itself
  - **Yang-Baxter with crossings :**  with 
  - **Yang-Baxter with dots :**  with  or 

Study of the critical branchings

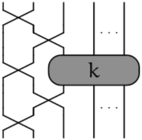
- There is right indexed branchings with the form :



where ***k*** is a diagram that can be plugged in the Yang-Baxter-equation.

Study of the critical branchings

- There is right indexed branchings with the form :



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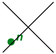
  - 

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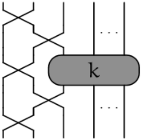

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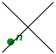
  - 

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- The two families of indexed critical branchings are confluent, so is the linear $(3, 2)$-polygraph KLR.

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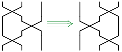
  - The following two **3**-cells :  and  .

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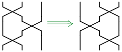
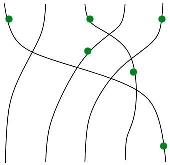
  - The following two **3**-cells :  and  .

- The set of normal forms of this polygraph correspond to braid diagrams with a minimal number of crossings, that is they are given by permutations whose length is minimal for the Coxeter presentation of $\mathcal{S}_m$.

Study of the normal forms : dots

- Starting from a diagram



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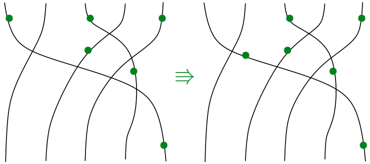

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Study of the normal forms : dots

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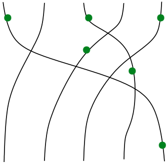
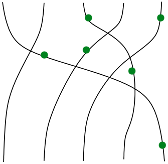
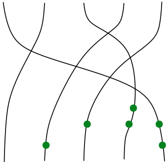
 $\Rrightarrow$  $(\pm \boldsymbol{D}_1^{\text{low}}) \Rrightarrow \dots \Rrightarrow$  $(\pm \sum \boldsymbol{D}_i^{\text{low}})$

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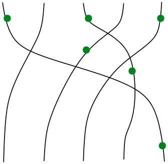
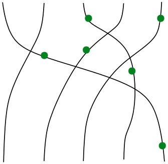
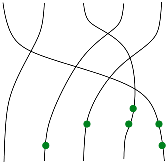
 $\Rrightarrow$  $(\pm \boldsymbol{D}_1^{\text{low}}) \Rrightarrow \dots \Rrightarrow$  $(\pm \sum \boldsymbol{D}_i^{\text{low}})$

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- The monomial in normal forms are exactly the diagrams which have a minimal number of crossings and all dots placed at the bottom of the strands.

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- Using linear rewriting techniques, we proved that the simply-laced KLR algebra admits bases of type Poincaré Birkhoff-Witt.

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- To recover all the bases for general spaces of **2**-cells, there are adjunction morphisms (cup and cap) that appear and the rewriting techniques become more complicated.

THANKS FOR YOUR ATTENTION.