

# Termination in linear $(2, 2)$ -categories with braidings and duals

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Dupont Benjamin

Higher Dimensional Rewriting and Algebra

Oxford, 7 July 2018

## Motivation: diagrammatic rewriting

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- ▶ **Diagrammatic rewriting:** 3-dimensional linear rewriting systems on diagrams
  - ▶ The two essential properties to study are **termination** and **confluence**.

## Motivations: termination issues

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- ▶ Consider a diagrammatic algebra  $A$  admitting relations of the form

$$\text{Diagrammatic relation} = \text{Diagrammatic relation} - \uparrow \downarrow + \sum_{n=0}^h \sum_{r \geq 0} \text{Diagrammatic term}.$$

The diagrammatic terms are as follows:

- Leftmost term: A diagram with two vertical lines and two horizontal lines forming an 'X' shape.
- Second term: A diagram with two vertical lines and two horizontal lines forming an 'X' shape, with a blue equals sign.
- Third term: A diagram with two vertical lines and two horizontal lines forming an 'X' shape, with a blue minus sign.
- Fourth term: A diagram with two vertical lines and two horizontal lines forming an 'X' shape, with a blue plus sign.
- Summation term: A diagram with two vertical lines and two horizontal lines forming an 'X' shape, with a blue summation symbol  $\sum$  above it, and a blue  $n=0$  below it.
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- Final term: A diagram with two vertical lines and two horizontal lines forming an 'X' shape, with a blue plus sign above it.

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The diagrammatic terms involve various configurations of lines and loops. The first term is a sum of two diagrams: one with two vertical lines and two diagonal lines crossing, and another with two vertical lines and two diagonal lines crossing in a different orientation. The second term is a sum of diagrams involving loops and lines, with labels  $n$  and  $r$ . The third term is a sum of diagrams involving loops and lines, with labels  $n$  and  $r$ .

one naturally asks:

- ▶ How should we orient the rules to obtain a terminating presentation of  $A$  ?
- ▶ In this work, we construct heuristics to prove termination of some diagrammatic rewriting systems.
- ▶ **Main problem:** A diagrammatic rewriting system does not always admit a monomial (total and well-founded) termination order.
- ▶ We will define termination orders similar to monomial orders, counting the generators in the diagrams, stable by contexts and well-founded, but that are not required to be total.

- I. Linear  $(2, 2)$ -categories, braidings and duals
- II. Decreasing order operators
- III. Termination heuristics in particular linear  $(2, 2)$  categories
- IV. Illustration on the diagrammatic rewriting system  $\mathcal{KLR}$

# I. Linear $(2, 2)$ -categories, braidings and duals

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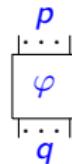
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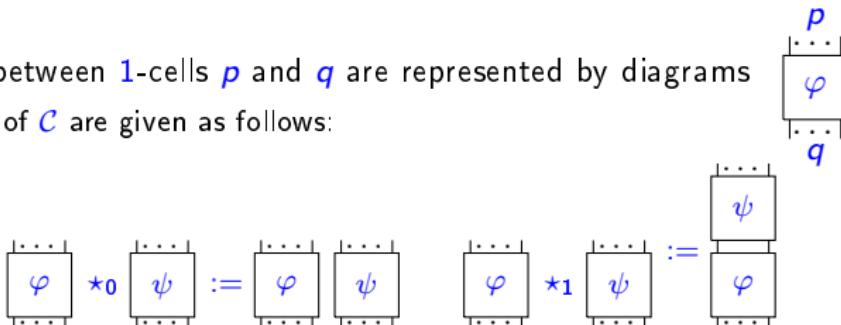
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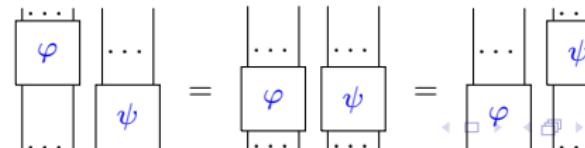
The image shows two diagrams illustrating the composition of 2-cells. The top diagram shows a 2-cell  $\varphi$  (represented as a rectangle with labels  $p$  and  $q$  at the top and bottom) composed with a 2-cell  $\psi$  (represented as a rectangle with labels  $q$  and  $r$  at the top and bottom). The result is a 2-cell  $\psi \circ \varphi$  (represented as a rectangle with labels  $p$  and  $r$  at the top and bottom). The bottom diagram shows the composition  $\varphi \circ \psi$  (represented as a rectangle with labels  $p$  and  $r$  at the top and bottom), which is defined as the composition of  $\varphi$  and  $\psi$  via the map  $\star_1$ .

$$\begin{array}{c} \varphi \circ \psi := \varphi \star_1 \psi = \psi \circ \varphi \end{array}$$

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- ▶ modulo the exchange law of  $\mathcal{C}$ , diagrammatically depicted as



# Diagrammatic rewriting systems

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- We consider diagrammatic algebras interpreted as linear  $(2, 2)$ -categories, admitting a diagrammatic presentation by generators and relations with:
  - a set  $\mathcal{C}_1^g$  of generating 1-cells wrt the  $\star_0$ -composition;
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- ▶ A **diagrammatic rewriting system** (DRS) is a **linear  $(3, 2)$ -polygraph**. Explicitly, a  $\text{DRS}\Sigma$  presenting  $\mathcal{C}$  is a quadruple  $(\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3)$  with:

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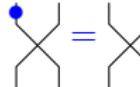
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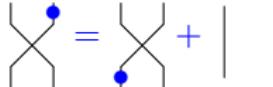
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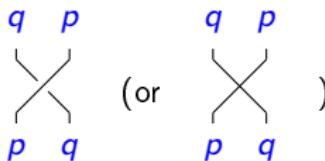
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## Braidings and adjunctions

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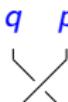
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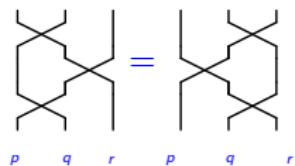
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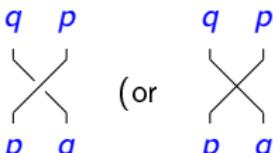
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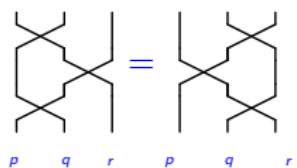


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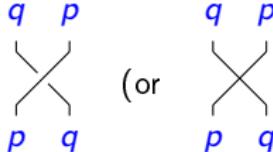

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- ▶ Let  $p : x \rightarrow y$  be a 1-cell of  $\mathcal{C}$ . We say that a 1-cell  $q : y \rightarrow x$  is a **left-adjoint** of  $p$ , denoted by  $q = \hat{p}$  if there exists 2-cells  $\varepsilon : p \star_0 \hat{p} \Rightarrow 1_y$  and  $\eta : 1_x \Rightarrow \hat{p} \star_0 p$  respectively represented by

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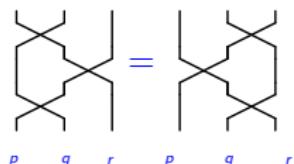
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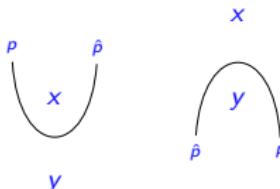
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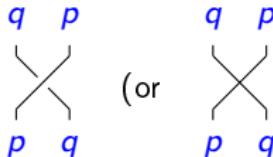
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# Braidings and adjunctions

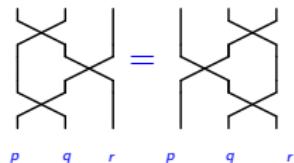
- A **braiding** on a linear  $(2, 2)$ -category  $\mathcal{C}$  is a family of 2-cells  $\sigma_{p,q} : p \star_0 q \rightarrow q \star_0 p$

for any  $p, q$  in  $\mathcal{C}_1^g$ , represented by



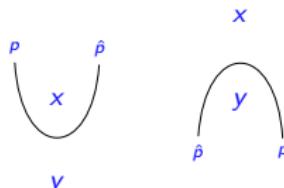
(or ) satisfying some naturality

axioms yielding to the **Yang-Baxter** equation:

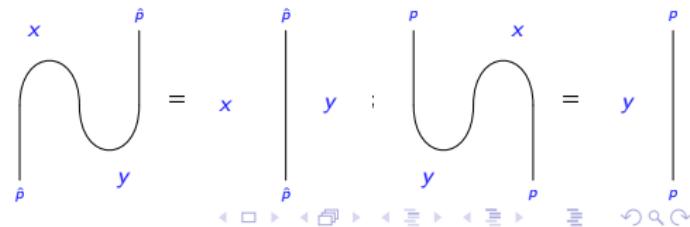


for any  $p, q, r$  in  $\mathcal{C}_1^g$

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and satisfying



## Cyclic 2-cells

---

- Given a pair of 1-cells  $p, q : x \rightarrow y$  in  $\mathcal{C}$  with chosen biadjoints  $(\hat{p}, \eta_p, \hat{\eta}_p, \varepsilon_p, \hat{\varepsilon}_p)$  and  $(\hat{q}, \eta_q, \hat{\eta}_q, \varepsilon_q, \hat{\varepsilon}_q)$ , then for any 2-cell  $\alpha : p \Rightarrow q$ , we construct two duals  ${}^*\alpha$  and  $\alpha* : \hat{q} \Rightarrow \hat{p}$  as follows:

$$\begin{array}{ccc} {}^*\alpha := & \begin{array}{c} \text{Diagram of } {}^*\alpha \\ \text{A 2-cell } \alpha \text{ in the middle, with } \varepsilon_q \text{ above and } \eta_p \text{ below.} \end{array} & \alpha^* := \\ & \begin{array}{c} \text{Diagram of } \alpha^* \\ \text{A 2-cell } \alpha \text{ in the middle, with } \hat{\eta}_q \text{ above and } \hat{\varepsilon}_p \text{ below.} \end{array} & \end{array}$$

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$$\begin{aligned} {}^*\alpha &:= \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \\ \alpha^* &:= \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \end{aligned}$$

Diagrams illustrating the construction of dual 2-cells:

- ${}^*\alpha$  (left): A 2-cell  $\alpha$  from  $p$  to  $q$  is dualized to a 2-cell  ${}^*\alpha$  from  $\hat{q}$  to  $\hat{p}$ . The diagram shows a vertical line from  $x$  to  $y$  with a curved arrow from  $\hat{q}$  to  $\hat{p}$ . The curved arrow is labeled  $\varepsilon_q$  above and  $\eta_p$  below. The vertical line is labeled  $\alpha$  at its midpoint.
- $\alpha^*$  (right): A 2-cell  $\alpha$  from  $p$  to  $q$  is dualized to a 2-cell  $\alpha^*$  from  $x$  to  $y$ . The diagram shows a vertical line from  $x$  to  $y$  with a curved arrow from  $\hat{p}$  to  $\hat{q}$ . The curved arrow is labeled  $\hat{\eta}_q$  above and  $\hat{\varepsilon}_p$  below. The vertical line is labeled  $\alpha$  at its midpoint.

- A 2-cell  $\alpha : p \Rightarrow q$  is said **cyclic** if the equation  ${}^*\alpha = \alpha^*$  is satisfied, or either of the equivalent conditions  ${}^{**}\alpha = \alpha$  or  $\alpha^{**} = \alpha$  are satisfied, yielding relations of the form

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (1)$$

Diagrams illustrating cyclic 2-cell relations:

- Left relation: A 2-cell  $\alpha$  from  $p$  to  $q$  is equated to its dual  ${}^*\alpha$  from  $\hat{q}$  to  $\hat{p}$ . The diagram shows two vertical lines from  $x$  to  $y$  with curved arrows from  $p$  to  $q$  and  $\hat{q}$  to  $\hat{p}$ . The curved arrows are labeled  $\eta_q$  and  $\eta_p$  respectively.
- Right relation: A 2-cell  $\alpha$  from  $p$  to  $q$  is equated to its double dual  $\alpha^{**}$  from  $x$  to  $y$ . The diagram shows two vertical lines from  $x$  to  $y$  with curved arrows from  $\hat{q}$  to  $\hat{p}$  and  $p$  to  $q$ . The curved arrows are labeled  $\hat{\varepsilon}_q$  and  $\hat{\varepsilon}_p$  respectively.

## Cyclic 2-cells

- Given a pair of 1-cells  $p, q : x \rightarrow y$  in  $\mathcal{C}$  with chosen biadjoints  $(\hat{p}, \eta_p, \hat{\eta}_p, \varepsilon_p, \hat{\varepsilon}_p)$  and  $(\hat{q}, \eta_q, \hat{\eta}_q, \varepsilon_q, \hat{\varepsilon}_q)$ , then for any 2-cell  $\alpha : p \Rightarrow q$ , we construct two duals  ${}^*\alpha$  and  $\alpha* : \hat{q} \Rightarrow \hat{p}$  as follows:

$${}^*\alpha := \begin{array}{c} x \\ \curvearrowright \\ \hat{q} \end{array} \quad \begin{array}{c} \hat{p} \\ \curvearrowright \\ y \end{array}$$

$$\alpha^* := \begin{array}{c} \hat{p} \\ \curvearrowright \\ x \end{array} \quad \begin{array}{c} \hat{\eta}_q \\ \curvearrowright \\ \alpha \\ \curvearrowright \\ \hat{\varepsilon}_p \\ y \end{array}$$

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$$\begin{array}{c} p \\ \curvearrowright \\ \alpha \\ \curvearrowright \\ \eta_q \\ y \end{array} = \begin{array}{c} p \\ \curvearrowright \\ \eta_p \\ \curvearrowright \\ \hat{q} \\ y \end{array} \quad \begin{array}{c} \hat{q} \\ \curvearrowright \\ \alpha \\ \curvearrowright \\ \hat{\varepsilon}_q \\ x \end{array} = \begin{array}{c} \hat{q} \\ \curvearrowright \\ \hat{\varepsilon}_p \\ \curvearrowright \\ p \\ x \end{array} \quad (1)$$

- A linear  $(2, 2)$ -category  $\mathcal{C}$  in which any 2-cell  $\alpha$  is cyclic is called a **pivotal category**.

## II. Decreasing order operators

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- ▶ Example.   $\leadsto$  number of crossings.

- ▶ Given a DRS  $\Sigma$ , one defines a **decreasing order operator** (DOO) for  $\Sigma$  as a family of functions  $\Phi_{p,q} : \Sigma_2(p, q) \rightarrow \mathbb{N}^{m(p,q)} \times \mathbb{Z}$  indexed by 1-cells  $p$  and  $q$ , satisfying:

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  - For any 3-cell  $\alpha : D_1 \Rightarrow D_2$  with  $D_1, D_2$  in  $\Sigma_2(p, q)$ , the function  $\Phi_{p,q}$  satisfy

$$\Phi_{p,q}(D_1) > \Phi_{p,q}(D')$$

where  $>$  is the lexicographic order on  $\mathbb{N}^{m(p,q)} \times \mathbb{Z}$  and  $D'$  is a monomial in  $D_2$ . We denote this by  $D_1 >_{\text{lex}} D_2$ .

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- The  $\Phi_{p,q}$  are stable by context: for any  $D_1$  and  $D_2$  in  $\Sigma_2(p, q)$  and any context  $C$  of  $\Sigma$ , if  $D_1 >_{\text{lex}} D_2$ , then  $C[D_1] >_{\text{lex}} C[D_2]$ .

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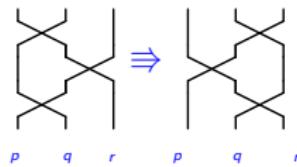
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- The  $\Phi_{p,q}$  are stable by exchange law.

### III. Termination heuristics in particular linear (2, 2) categories

## Termination with braid relations

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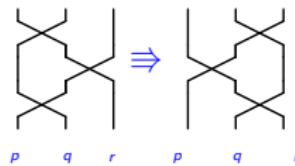
- Let  $\mathbf{Crs}$  be the DRS having: only one 0-cell, a set of generating 1-cells  $\mathbf{Crs}_1$ , for 2-cells the braidings  $\sigma_{p,q}$  for each  $p$  and  $q$  in  $\mathbf{Crs}_1$ , and 3-cells as follows:



## Termination with braid relations

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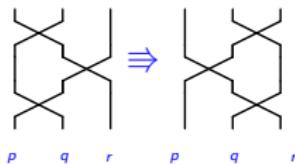
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- $\mathbf{Crs}$  is terminating by the DOO  $\Phi_{p,q}$  counting the number  $\text{yb}(D)$  of occurrences of 2-cells  $\sigma_{p,q} *_0 \text{id}_r$  in a diagram  $D$ , for  $p, q$  and  $r$  in  $\mathbf{Crs}_1$ .

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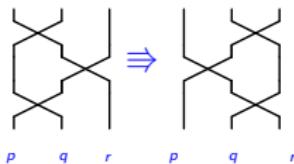


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- Let  $\mathbf{Crs}'$  be the DRS defined by

$$\mathbf{Crs}' = \mathbf{Crs} \cup \left\{ \begin{array}{ccc} \text{braiding diagram} & \Rightarrow & \text{two strands} \\ p \qquad \qquad \qquad q & & p \qquad \qquad \qquad q \end{array} \right\}$$

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- We add as first component to the  $\Phi_{p,q}$  defined for  $\mathbf{Crs}$  a component counting the number of crossings of the diagrams.

## Termination with braid relations and additional 2-cells

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- Let  $\mathbf{Crs}^{\text{add}}$  be a DRS defined by

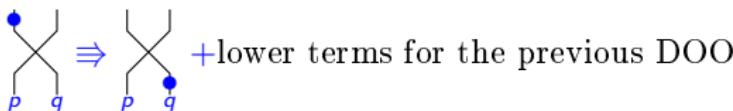
$$\mathbf{Crs}^{\text{add}} = (\mathbf{Crs}'_0, \mathbf{Crs}'_1, \mathbf{Crs}'_2 \cup \left\{ \begin{array}{c} q \\ \bullet \alpha \\ q \end{array} \quad \text{for } q \text{ in } \mathbf{Crs}'_1 \right\}, \mathbf{Crs}'_3)$$

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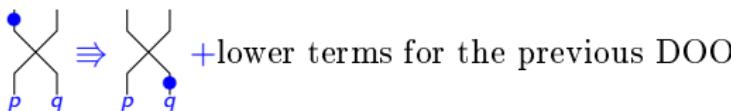


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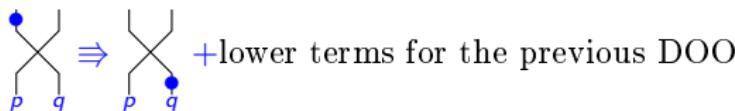
- We define a new DOO as follows: for  $p, q \in \mathbf{Crs}'_1$ , we set  $m := \max(\ell(p), \ell(q))$ . We add to  $\Phi p, q$  the components  $(c_k(D))_{1 \leq k \leq m}$  defined by:

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+ lower terms for the previous DOO

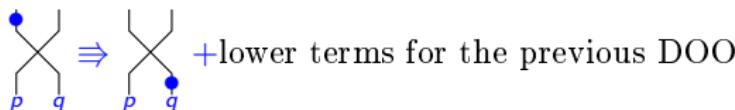
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  - 0 if there is no  $\bullet$  on the  $k$ -th strand and if the  $k$ -strand is not a **through strand**, but this can not occur with only braidings.

# Termination with braid relations and additional 2-cells

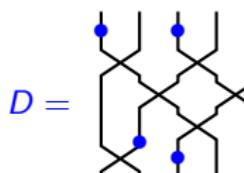
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  - 0 if there is no  $\bullet$  on the  $k$ -th strand and if the  $k$ -strand is not a **through strand**, but this can not occur with only braidings.
  - the number of crossings below the upper dot of the  $k$ -th strand.
- Example. For



## Example: the nil Hecke algebra

---

- ▶ For  $n \in \mathbb{N}$ , let us consider the *Nil-Hecke algebra*  $\mathcal{NH}_n^0$  which is a  $\mathbb{K}$ -algebra for a field  $\mathbb{K}$  defined by:

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- relations:

$$\text{Diagram: } \begin{array}{c} \text{X} \\ \text{X} \end{array} = 0,$$

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$$\text{Diagram: } \begin{array}{c} \bullet \\ \text{X} \end{array} = \begin{array}{c} \text{X} \\ \bullet \end{array} + \begin{array}{c} | \\ | \end{array}$$

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- relations:

$$\text{Diagram: } \text{Braid relation } 1 = 0,$$

$$\text{Diagram: } \text{Braid relation } 2 = \text{Diagram: } \text{Braid relation } 3,$$

$$\text{Diagram: } \text{Braid relation } 4 = \text{Diagram: } \text{Braid relation } 5 + \text{Diagram: } \text{Braid relation } 6,$$

$$\text{Diagram: } \text{Braid relation } 7 = \text{Diagram: } \text{Braid relation } 8 - \text{Diagram: } \text{Braid relation } 9,$$

- $\coprod_{n \in \mathbb{N}^*} \mathcal{NH}_n^0$  form a linear  $(2, 2)$ -category with only one 0-cell, the 1-cells are permutations and 2-cells are braiding diagrams.

## Example: the nil Hecke algebra

- For  $n \in \mathbb{N}$ , let us consider the *Nil-Hecke algebra*  $\mathcal{NH}_n^0$  which is a  $\mathbb{K}$ -algebra for a field  $\mathbb{K}$  defined by:

- generators  $\xi_i = \begin{array}{c|c|c|c|c} & \dots & \bullet & \dots & \\ \hline 1 & & i & & n \end{array}$  for  $1 \leq i \leq n$  and  $\partial_i = \begin{array}{c|c|c|c|c} & \dots & \times & \dots & \\ \hline 1 & & i & & n \end{array}$  for  $1 \leq i < n$ ;

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- relations:

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- We prove that  $\Sigma$  is terminating using the following DOO: for a given diagram  $D$  in  $\mathcal{NH}_n^0(\sigma, \tau)$ ,

$$\Phi_{\sigma, \tau}(D) = (c(D), \text{yb}(D), c_1(D), \dots, c_n(D)).$$

## Termination with adjunctions

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- Let  $\mathcal{C}$  be a linear  $(2, 2)$ -category whose 1-cells are equipped with biadjunctions, yielding isotopy relations of the form

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- The DRS given by these orientations is not confluent: the first Knuth-Bendix step imposes to add the following relations:

$$\text{N} = \text{N} \bullet, \quad \text{U} = \text{U} \bullet.$$

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---

- ▶ Let **Pearl** be the DRS defined by:

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$$\text{l-dot} \left( \underset{n}{\bullet} \cap \right) = \text{l-dot} \left( \underset{n}{\bullet} \cup \right) := n$$

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- ▶ Adding a  $\star_0$  and  $\star_1$ -context to  $D$ , we add a constant number of cups and caps, and  $\text{l-dot}(D)$  can not increase since a dot cannot move from right of a cap/cup to its left even by adding a context

## Termination or quasi-termination ?

---

- ▶ If we choose different orientation for the dot move relations, we create rewriting cycles

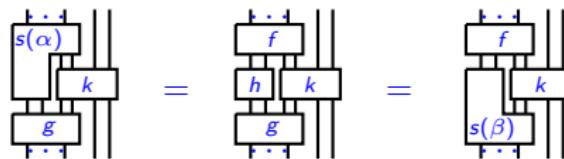
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- In a DRS  $\Sigma$ , there are **indexed critical branchings** of the following form:



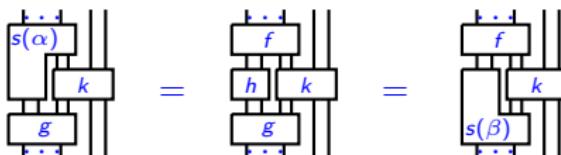
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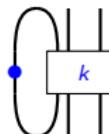


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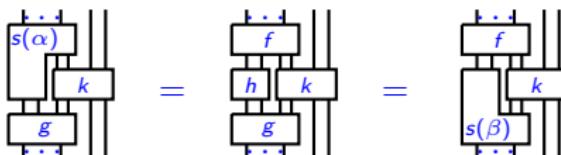
**Example.** If  $\Sigma$  contains braiding 2-cells  $\sigma_{p,q}$  for any  $p, q$  in  $\Sigma_1$ , there are infinitely many indexed critical branchings in  $\Sigma$  for each  $n \in \mathbb{N}$ .

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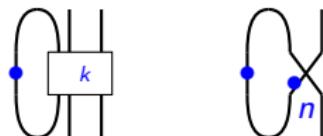


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## Quasi-termination

- ▶ A DRS  $\Sigma$  is **quasi-terminating** if for each rewriting sequence  $(u_n)_{n \in \mathbb{N}}$  of 2-cells of  $\Sigma$ , it contains an infinite number of occurrences of the same 2-cell.
- ▶ Let  $\Sigma$  be a DRS containing the following 3-cells:



$\Sigma$  is not terminating, one wants to study its quasi-termination.

- ▶ A **quasi-reduced** monomial in  $\Sigma$  is a monomial on which we can only apply the rules



- ▶ We may prove that  $\Sigma$  is quasi-terminating by constructing a DOO on the sets  $Q\text{-red}(\Sigma_2(p, q))$  of quasi-reduced monomials between two 1-cells  $p$  and  $q$ .
  - ▶ This DOO does not take into account the number of left-dotted cups and caps.
  - ▶ It ensures that there is no other obstruction to termination than the bubble cycles.

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  - ▶ (in a  $\mathbb{Z}$ -graded context, some relations making the degree decrease with a lower bound on the degree under which all diagrams are zero).
- ▶ **Proposition.** There is a DRS  $\Sigma$  presenting  $\mathcal{C}$  in which the relations are oriented in such a way that  $\Phi_{p,q}(s(\alpha)) > \Phi_{p,q}(t(\alpha))$  for a DOO of  $\Sigma$  constructed as above and any  $3$ -cell  $\alpha$ , and thus  $\Sigma$  is terminating.

## IV. Illustration on the linear $(3, 2)$ -polygraph $\mathcal{KLR}$

## The linear $(3, 2)$ -polygraph $\mathcal{KLR}$

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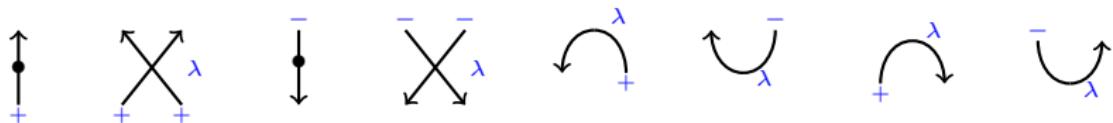
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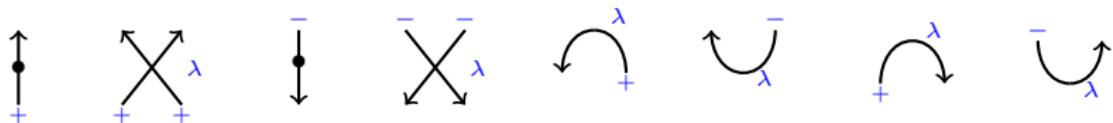
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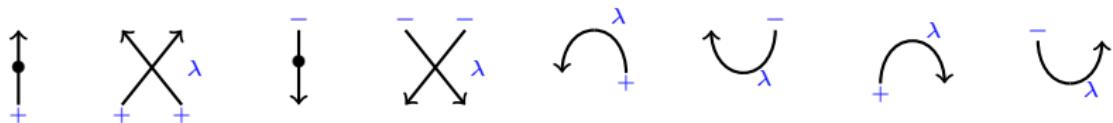
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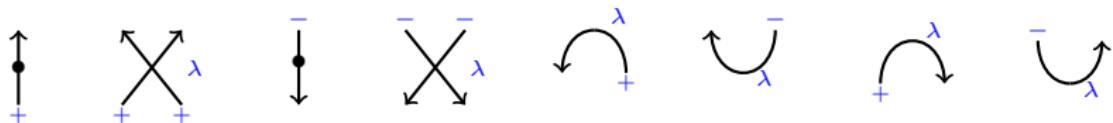
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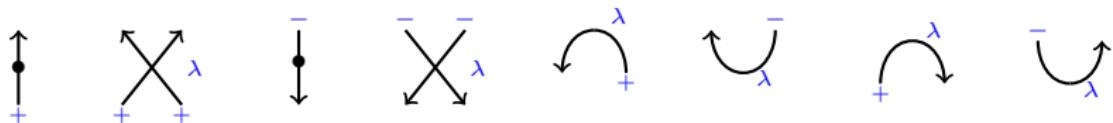
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  - The isotopy 3-cells;
  - Some bubble conditions 3-cells:

$$n \circlearrowleft \lambda \Rightarrow \begin{cases} 1_{1_\lambda} & \text{if } n = h-1 \\ 0 & \text{if } n < h-1 \end{cases}$$

$$\lambda \circlearrowleft n \Rightarrow \begin{cases} 1_{1_\lambda} & \text{if } n = -h-1 \\ 0 & \text{if } n < -h-1 \end{cases}$$

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# The linear $(3, 2)$ -polygraph $\mathcal{KLR}$

- The 3-cells in  $\mathcal{KLR}_3$  are given by:

- the infinite Grassmannian relation: for any  $\lambda \in X$  and  $\alpha > 0$ ,

$$\text{Diagram: } \text{A circular arrow with a dot at the top-left labeled } h-1+\alpha \text{ and } i \text{ at the bottom. A blue } \lambda \text{ is placed near the arrow.} \Rightarrow - \sum_{l=1}^{\alpha} \text{Diagram: } \text{A circular arrow with a dot at the top-left labeled } h-1+\alpha-l \text{ and } i \text{ at the bottom. A blue } \lambda \text{ is placed near the arrow.} \text{ and } \text{Diagram: } \text{A circular arrow with a dot at the top-left labeled } -h-1+l \text{ and } i \text{ at the bottom. A blue } \lambda \text{ is placed near the arrow.}$$

where  $h$  is a number given by the Kac-Moody algebra.

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- Some invertibility 3-cells:

$$\text{Diagram: } \text{A complex 3-cell with label } \lambda \xrightarrow{\lambda} -\uparrow\downarrow + \sum_{n=0}^{h-1} \sum_{r \geq 0} \text{ (Diagram with label } \lambda\text{)}^{h-1-n-r-2} , \text{ (Diagram with label } \lambda\text{)}^{h-1-n-r-2}$$

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### ► Some invertibility 3-cells:

$$\begin{aligned}
 \text{Diagram 1: } & \Rightarrow -\uparrow\downarrow + \sum_{n=0}^{h-1} \sum_{r \geq 0} \text{Diagram 2} \\
 \text{Diagram 2: } & \Rightarrow -\downarrow\uparrow + \sum_{n=0}^{-h-1} \sum_{r \geq 0} \text{Diagram 3}
 \end{aligned}$$

### ► Some " $\mathfrak{sl}_2$ " 3-cells:

$$\text{Diagram 1: } \text{Left: } \text{A loop with a self-intersection and a crossing point labeled } \lambda. \text{ Right: } \text{A sum } \sum_{n=0}^h \text{ with } n \text{ terms. Each term } n \text{ shows a loop with a self-intersection and a crossing point labeled } \lambda, \text{ with a dot at the crossing point labeled } -n-1. \text{ The label } \lambda \text{ is placed below the crossing point.} \\ \text{Diagram 2: } \text{Left: } \text{A loop with a self-intersection and a crossing point labeled } \lambda. \text{ Right: } \text{A sum } -\sum_{n=0}^{-h} \text{ with } -n-1 \text{ terms. Each term } -n-1 \text{ shows a loop with a self-intersection and a crossing point labeled } \lambda, \text{ with a dot at the crossing point labeled } n. \text{ The label } \lambda \text{ is placed below the crossing point.}$$

$$\text{Diagram 1: } \text{Left: } \text{A loop with two cusps and a self-intersection point. Right: } \Rightarrow - \sum_{n=0}^{-h} \text{ }_{-n-1} \text{ }_{n} \text{ ; Diagram 2: } \text{Left: } \text{A loop with two cusps and a self-intersection point. Right: } \Rightarrow \sum_{n=0}^h \text{ }_{n} \text{ }_{-n-1} \text{ .}$$

# The linear $(3, 2)$ -polygraph $\mathcal{KLR}$

- $\mathcal{KLR}$  is terminating using the following DDO:

$$\begin{array}{ccc} \Phi_{\varepsilon, \varepsilon'} : \mathcal{KLR}_2(\varepsilon, \varepsilon') & \rightarrow & \mathbb{N}^{m+4} \times \mathbb{Z} \\ D & \mapsto & (c(D), c_1(D), \dots, c_m(D), \text{ybg}(D), I(D), \text{l-dot}(D), \deg_b(D)) \end{array}$$

with:

- $c(D)$  is the number of crossings between strands in  $D$ ;
- for  $1 \leq k \leq m$ ,  $c_k(D)$  is defined as above;
- $\text{ybg}(D)$  defined as above;
- $I(D)$  corresponds to the number of rightward caps and leftward cups that appear in  $D$ ;
- $\text{l-dot}(D)$  corresponds to the number of positively leftward dotted caps and cups as described above.
- 

$$\deg_b(D) := \begin{cases} \#\{\text{bubbles in } D\} + \sum_{\pi \text{ clockwise bubble in } D} \deg(\pi) & \text{if } D \text{ is a diagram with bubbles} \\ 0 & \text{if } D \text{ is a diagram without bubbles} \\ -\infty & \text{if } D = 0. \end{cases}$$

- ▶ We presented heuristics to prove termination of some DRS presenting diagrammatic algebras coming from representation theory.
- ▶ The next question to study is confluence of these DRS.
  - ▶ The diagrammatic structure yield a combinatorial explosion for computation of critical pairs, as for instance isotopy relations.
  - ▶ Isotopy should not be considered as rewrite rules, but as equations we have to take into account when rewriting.
  - ▶ Develop a context of **rewriting modulo isotopy**, and obtain linear bases and coherence results in that setting.