Algebraic critical pair lemma

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Outline

I. Introduction : string and linear critical pair lemma

II. Algebraic polygraphs modulo

III. Algebraic critical pair lemma

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Algebraic rewriting and critical branching lemma

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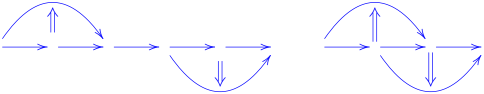

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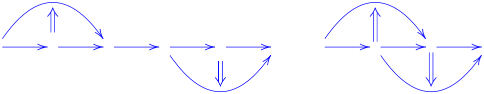

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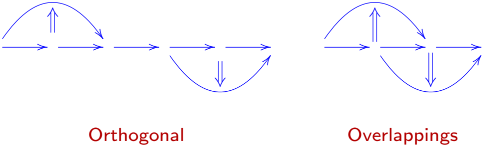

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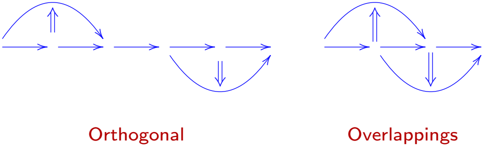

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  - Overlappings are confluent from confluence of critical branchings.
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- **Theorem (String critical pair lemma)** An SRS is locally confluent if and only if all its critical branchings are confluent.

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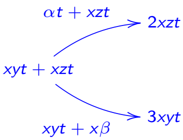

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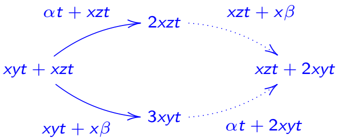

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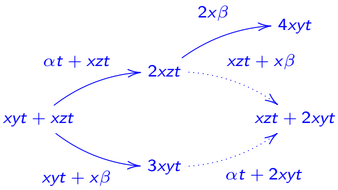

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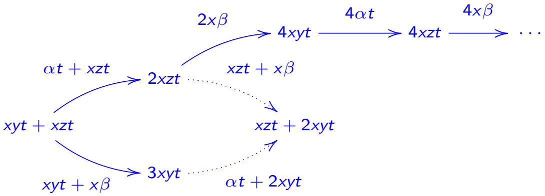

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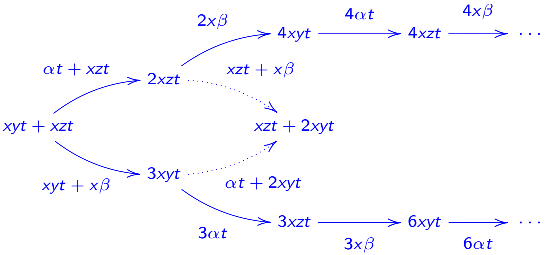

- CBL requires an additional termination assumption to hold.

II. Algebraic polygraphs modulo

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- **Example :** $P = \text{Mon}$, $Q = \{s, \ t\}$ and $R = \{\alpha : \mu(\mu(s,t),s) \Rightarrow \mu(t,\mu(s,t))\}$.

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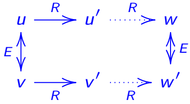

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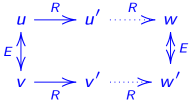

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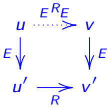

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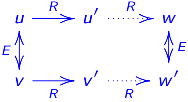

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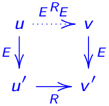

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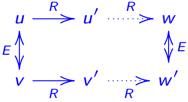

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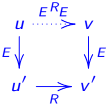

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- Algebraic polygraph modulo : quadruple (P, Q, R, S) where (P, Q, R) is an algebraic polygraph and S is a set of oriented relations such that

$$R \subseteq S \subseteq {}_{P_2\langle Q\rangle} R_{P_2\langle Q\rangle} := {}_P R_P.$$

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- Given an algebraic polygraph modulo (P, Q, R, S), denote by $\pi : P_1\langle Q\rangle \to P_1\langle Q\rangle / P_2\langle Q\rangle$ sending a ground term f on its equivalence class $\overline{f}$ modulo $P_2\langle Q\rangle$.

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- Assume that P is such that $P_2 = P_2' \cup P_2''$, with P_2' confluent modulo P_2''.
  - $\sigma(\overline{f}) = NF(f, P_2' \; mod \; P_2'')$, where $f \in \pi^{-1}(\overline{f})$, the set of normal forms of f for P_2' modulo P_2''.

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  String rewriting systems

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and CMod_2 contains the following generating 2-cells :

$x + 0 \Rightarrow x$	(ring_1)	$x + (-x) \Rightarrow 0$	(ring_2)
$-0 \Rightarrow 0$	(ring_3)	$-(-x) \Rightarrow x$	(ring_4)
$-(x + y) \Rightarrow (-x) + (-y)$	(ring_5)	$x \cdot (y + z) \Rightarrow x \cdot y + x \cdot z$	(ring_6)
$x \cdot 0 \Rightarrow 0$	(ring_7)	$x \cdot (-y) \Rightarrow -(x \cdot y)$	(ring_8)
$1 \cdot x \Rightarrow x$	(ring_9)	$a \oplus 0^{\oplus} \Rightarrow a$	(mod_1)
$x.(y.a) \Rightarrow (x \cdot y).a$	(mod_2)	$1.a \Rightarrow a$	(mod_3)
$x.a \oplus y.a \Rightarrow (x + y).a$	(mod_4)	$x.(a \oplus b) \Rightarrow (x.a) \oplus (y.b)$	(mod_5)
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- **Theorem [Peterson-Stickel, Hullot]** CMod is a presentation of the theory of modules over commutative rings that is confluent modulo AC.

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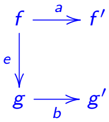

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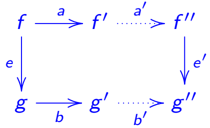

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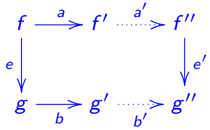

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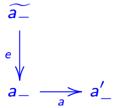

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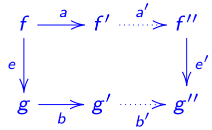

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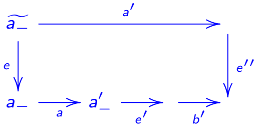

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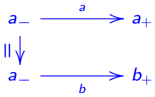

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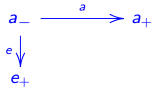

- When ${}_{P_2\langle Q\rangle}R \subseteq S$, property **$b_0$)** is always satisfied.

III. Algebraic critical branching lemma

Algebraic rewriting systems and critical branching lemma

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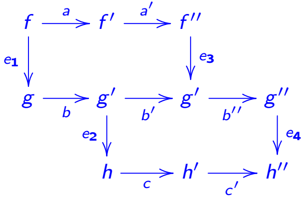

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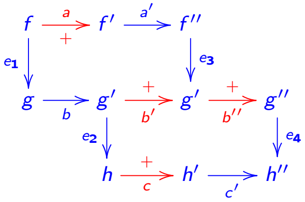

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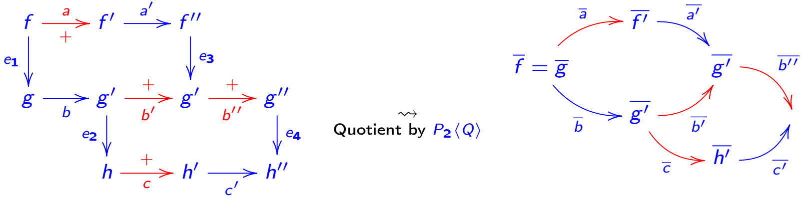

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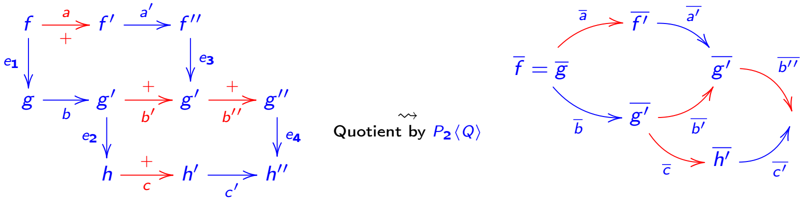

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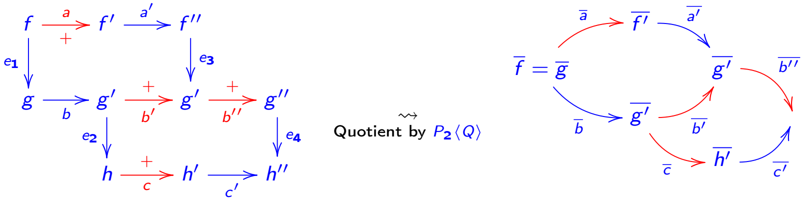

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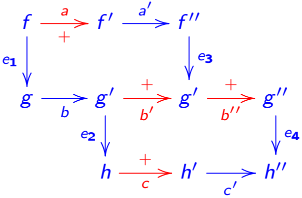
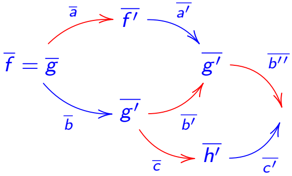


$\leadsto$ **Quotient by** $P_2\langle Q\rangle$



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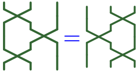
- **Example :** With $P = \mathrm{Mon}$, $Q = \{s, t\}$, $R = \{\alpha \ : \ \mu(\mu(s,t),s) \Rightarrow \mu(t, \mu(s,t))\}$ and σ the full strategy, the AlRS is

$\langle s,\ t \mid sts \to tst \ \rangle$ $\quad s =$ , $\quad t =$ , 

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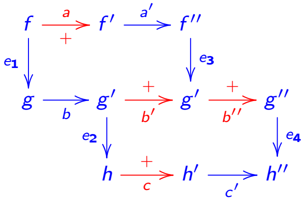

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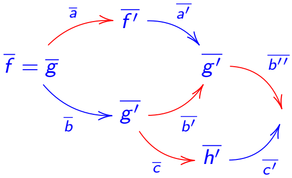

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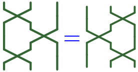
$\langle s,\ t \mid sts \to tst\ \rangle$ $\quad s =$ , $\quad t =$ , 

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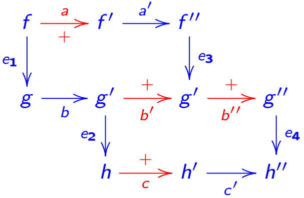

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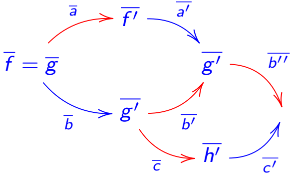

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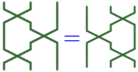

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- **Theorem [Chenavier - D. - Malbos]** Let $\mathcal{P} = (P, Q, R, S)$ be a quasi-terminating and positively σ-confluent APM, and $\mathcal{A}$ be an ARS on $\mathcal{P}$. Then $\mathcal{A}$ is locally confluent if and only if its critical branchings are confluent.

Examples

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  - With the full strategy ($\sigma(\overline{f}) = \pi^{-\mathbf{1}}(\overline{f})$), orthogonal are confluent without quasi-termination.

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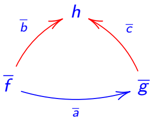
  - ${}_P R_P$ quasi-terminating implies that the quotient AIRS is (quasi)-terminating.
  - The positive σ-confluence implies the following factorization property :



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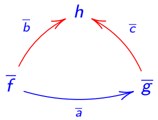

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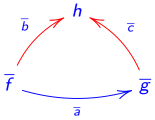

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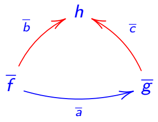

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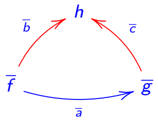

- **Questions :**
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  - How does the critical branching lemma translates if we change the positive strategy ?
- **Conclusion :**
  - This work suggests new tools for rewriting in various algebraic structures.
  - Need a better understanding of how to choose strategies, and ensure positive confluence in general.
  - Develop a critical branching lemma for various algebraic contexts : groups, differential algebras, operads, higher-dimensional categories.

Thank you !