Coherence modulo relations and double groupoids

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joint work with Philippe Malbos

Journées LHC

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Plan

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I. Introduction and motivations

Motivations: algebraic context

- This work is part of algebraic rewriting, consisting in applying rewriting methods to study intrinseque properties of algebraic structures presented by generators and relations.

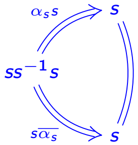
- For instance, computation of syzygies (relations among relations): for the group $\mathbb{Z}^3 = \langle x, y, z \mid [x, y] = 1,\ [y, z] = 1,\ [z, x] = 1 \rangle$, the Jacobi identity

$$[x^y, [y, z]][y^z, [z, x]][z^x, [x, y]] = 1$$

  is such a syzygy, with $[x, y] = xyx^- y^-$ and $x^y = y^- xy$.

- For monoids or categories, Squier's theorem gives a generating family for syzygies from a finite convergent presentation, Guiraud-Malbos '09, Gaussent-Guiraud-Malbos '14, Hage-Malbos '16.

- If a group $G = \langle X \mid R \rangle$ is presented as a monoid $M = \langle X \coprod \overline{X} \mid R \cup \{xx^- \overset{\alpha_x}{\Rightarrow} 1, x^- x \overset{\overline{\alpha_x}}{\Rightarrow} 1\}$, the confluence diagram

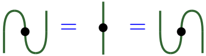


  is an artefact induced by the algebraic structure and should not be considered as a syzygy.

Motivation: objectives

- **Objective:** Study diagrammatic algebras arising in representation theory using algebraic rewriting.
  - Khovanov-Lauda-Rouquier (KLR) algebras for categorification of quantum groups;
  - Temperley-Lieb algebras in statistichal mechanics;
  - Brauer algebras and Birman-Wenzl algebras in knot theory.
- **Main questions**:
  - Coherence theorems; ✓
  - Categorification constructive results;
  - Computation of linear bases for these algebras by rewriting.
- Structural rules of these algebras make the study of local confluence complicated.

  Example: Isotopy relations



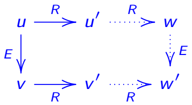
- We use rewriting modulo.
  - Algebraic axioms are not rewriting rules, but taken into account when rewriting.

Three paradigms of rewriting modulo

- Rewriting system R:
  - Usual rewriting theory;
  - Squier's theorem expressed in n-categories.

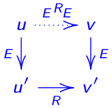
Globular

- In rewriting modulo, we consider a rewriting system R and a set of equations E.
- 3 paradigms of rewriting modulo:
  - Rewriting with R modulo E, Huet '80



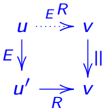
Cubical

  - ${}_E R_E$: Rewriting with R on E-equivalence classes



  - Rewriting with any system S such that $R \subseteq S \subseteq {}_E R_E$, Jouannaud - Kirchner '84.

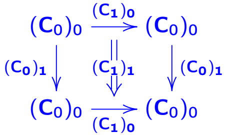
- Many results in rewriting modulo are expressed for ${}_E R$.



II. Double groupoids

Double groupoids

- We introduce a cubical notion of coherence, related to n-categories enriched in double groupoids.
- A double category is an internal category $(\mathbf{C}_1, \mathbf{C}_0, \partial^{\mathbf{C}}_-, \partial^{\mathbf{C}}_+, \circ_{\mathbf{C}}, i_{\mathbf{C}})$ in Cat.

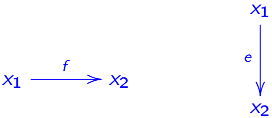


- It gives four related categories

$$\mathbf{C}^{vo} := (\mathbf{C}^v, \mathbf{C}^o, \partial^v_{-,0}, \partial^v_{+,0}, \circ^v, i^v_0), \qquad \mathbf{C}^{ho} := (\mathbf{C}^h, \mathbf{C}^o, \partial^h_{-,0}, \partial^h_{+,0}, \circ^h, i^h_0),$$
$$\mathbf{C}^{sv} := (\mathbf{C}^s, \mathbf{C}^v, \partial^v_{-,1}, \partial^v_{+,1}, \diamond^v, i^v_1), \qquad \mathbf{C}^{sh} := (\mathbf{C}^s, \mathbf{C}^h, \partial^h_{-,1}, \partial^h_{+,1}, \diamond^h, i^h_1),$$

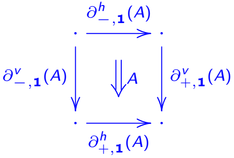
where $\mathbf{C}^{sh}$ is the category $\mathbf{C}_1$ and $\mathbf{C}^{vo}$ is the category $\mathbf{C}_0$.

- Elements of $\mathbf{C}^o$ are called point cells, the elements of $\mathbf{C}^h$ and $\mathbf{C}^v$ are called horizontal cells and vertical cells respectively and pictured by

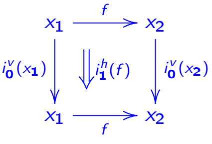


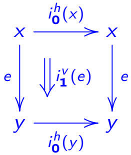
Double groupoids

- Source and target maps make elements of $\mathbf{C}_s$ be square cells

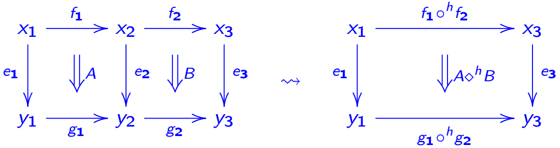


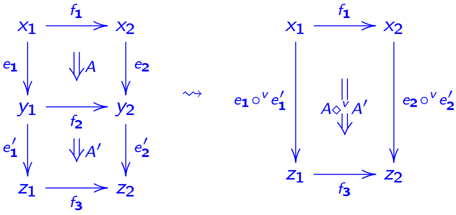
, with identities





- Compositions





for all x_i, y_i, z_i in $\mathbf{C}^o$, f_i in $\mathbf{C}^h$, e_i, e'_i in $\mathbf{C}^v$ and A, A', B in $\mathbf{C}^s$.

Double groupoids

- These compositions satisfy the middle four interchange law:

$$\begin{array}{ccc}
\begin{array}{ccc} x_1 & \xrightarrow{f_1} & x_2 \\ {\scriptstyle e_1}\downarrow & \Downarrow A & \downarrow{\scriptstyle e_2} \\ y_1 & \xrightarrow{g_1} & y_2 \end{array} & & \begin{array}{ccc} x_2 & \xrightarrow{f_2} & x_3 \\ {\scriptstyle e_2}\downarrow & \Downarrow B & \downarrow{\scriptstyle e_3} \\ y_2 & \xrightarrow{g_2} & y_3 \end{array} \\
\diamond^h & \diamond^v & \diamond^h \\
\begin{array}{ccc} y_1 & \xrightarrow{g_1} & y_2 \\ {\scriptstyle e'_1}\downarrow & \Downarrow A' & \downarrow{\scriptstyle e'_2} \\ z_1 & \xrightarrow{h_1} & z_2 \end{array} & & \begin{array}{ccc} y_2 & \xrightarrow{g_2} & y_3 \\ {\scriptstyle e'_2}\downarrow & \Downarrow B' & \downarrow{\scriptstyle e'_3} \\ z_2 & \xrightarrow{h_2} & z_3 \end{array}
\end{array}
=
\begin{array}{ccc}
\begin{array}{ccc} x_1 & \xrightarrow{f_1} & x_2 \\ {\scriptstyle e_1}\downarrow & \Downarrow A & \downarrow{\scriptstyle e_2} \\ y_1 & \xrightarrow{g_1} & y_2 \end{array} & \diamond^v & \begin{array}{ccc} x_2 & \xrightarrow{f_2} & x_3 \\ {\scriptstyle e_2}\downarrow & \Downarrow B & \downarrow{\scriptstyle e_3} \\ y_2 & \xrightarrow{g_2} & y_3 \end{array} \\
& \diamond^h & \\
\begin{array}{ccc} y_1 & \xrightarrow{g_1} & y_2 \\ {\scriptstyle e'_1}\downarrow & \Downarrow A' & \downarrow{\scriptstyle e'_2} \\ z_1 & \xrightarrow{h_1} & z_2 \end{array} & \diamond^v & \begin{array}{ccc} y_2 & \xrightarrow{g_2} & y_3 \\ {\scriptstyle e'_2}\downarrow & \Downarrow B' & \downarrow{\scriptstyle e'_3} \\ z_2 & \xrightarrow{h_2} & z_3 \end{array}
\end{array}$$

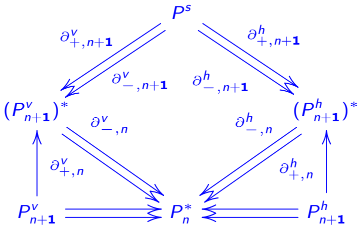
- Double groupoid = double category $(\mathbf{C}_1, \mathbf{C}_0, \partial_-^{\mathbf{C}}, \partial_+^{\mathbf{C}}, \circ_{\mathbf{C}}, i_{\mathbf{C}})$ in which $\mathbf{C}_1$ and C_0 are groupoids.

- n-category enriched in double groupoids = n-category $\mathcal{C}$ such that any homset $\mathcal{C}_n(x, y)$ is a double groupoid.
  - The set $\mathcal{C}_n(x, y)^o$ is the set of n-cells in $\mathcal{C}_n(x, y)$.

- Horizontal $(n+1)$-category is the $(n+1)$-category of rewritings; vertical $(n+1)$-category is the $(n+1)$-category of modulo rules.

Double $(n+2,n)$-polygraphs

- A double n-polygraph is a data (P^v, P^h, P^s) made of:
  - two $(n+1)$-polygraphs P^v and P^h such that $P^v_k = P^h_k$ for $k \leq n$,
  - a 2-square extension P^s of the pair of $(n+1)$-categories $((P^v)^*, (P^h)^*)$, that is a set equipped with four maps $\partial^\mu_{\pm,n}$, with $\mu \in \{v,h\}$, making Γ a 2-cubical set:



- A double $(n+2,n)$-polygraph is a double n-polygraph whose square extension P^s_{n+2} is defined on $((P^v)^\top, (P^h)^\top)$.

- A double n-polygraph (resp. double $(n+2,n)$-polygraph) (P^v, P^h, P^s) generates a free n-category enriched in double categories (resp. in double groupoids), denoted by $(P^v, P^h, P^s)^{\top\!\!\top}$.

Acyclicity

- A 2-square extension P^s of $((P^v)^\top, (P^h)^\top)$ is acyclic if for any square

$$S = \quad \begin{array}{ccc} \cdot & \xrightarrow{(P^h)^\top} & \cdot \\ {\scriptstyle (P^v)^\top}\downarrow & \Downarrow A & \downarrow{\scriptstyle (P^v)^\top} \\ \cdot & \xrightarrow[(P^h)^\top]{} & \cdot \end{array}$$

there exists a square $(n+1)$-cell A in $(P^v, P^h, P^s)^{\top\!\!\top}$ such that $\partial(A) = S$.

- A 2-fold coherent presentation of an n-category $\mathbf{C}$ is a double $(n+2, n)$-polygraph (P^v, P^h, P^s) such that:
  - the $(n+1)$-polygraph $P^v \coprod P^h$ presents $\mathbf{C}$;
  - P^s is acyclic

- **Example:** Let E be a convergent $(n+1)$-polygraph and $\mathbf{C}$ the n-category presented by E. $\mathrm{Cd}(E)$:= square extension of $(E^\top, 1)$ containing squares

$$\begin{array}{ccc} \cdot & \xrightarrow{=} & \cdot \\ {\scriptstyle e_1 \star_{n-1} e_1'}\downarrow & & \downarrow{\scriptstyle e_2 \star_{n-1} e_2'} \\ \cdot & \xrightarrow[=]{} & \cdot \end{array}$$

for a choice of confluence diagram of any critical branching (e_1, e_2) of E.

- From Squier's theorem, $(E, \emptyset, \mathrm{Cd}(E))$ is a 2-fold coherent presentation of $\mathbf{C}$.

III. Polygraphs modulo

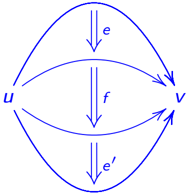
Polygraphs modulo

A *n-polygraph modulo* is a data (R, E, S) made of

- an n-polygraph R of primary rules,
- an n-polygraph E such that $E_k = R_k$ for $k \leq n-2$ and $E_{n-1} \subseteq R_{n-1}$, of modulo rules,
- S is a cellular extension of R^*_{n-1} such that $R \subseteq S \subseteq {}_E R_E$, where the cellular extension ${}_E R_E$ is defined by

$$\gamma^{{}_E R_E} : {}_E R_E \to \mathrm{Sph}_{n-1}(R^*_{n-1})$$

where ${}_E R_E$ is the set of triples (e, f, e') in $E^\top \times R^{*(1)} \times E^\top$ such that



and the map $\gamma^{{}_E R_E}$ is defined by $\gamma^{{}_E R_E}(e, f, e') = (\partial_{-,n-1}(e), \partial_{+,n-1}(e'))$.

Branchings and confluence modulo

- A branching modulo E of the n-polygraph modulo S is a triple (f, e, g) where f and g are n-cells of S^* with f non trivial and e is an n-cell of $E^\top$, such that:

$$\begin{array}{ccc} u & \xrightarrow{f} & u' \\ {\scriptstyle e}\downarrow & & \\ v & \xrightarrow[g]{} & v' \end{array}$$

- It is local if f is an n-cell of $S^{*(1)}$, g is an n-cell of S^* and e an n-cell of $E^\top$ such that $\ell(g) + \ell(e) = 1$.

- It is confluent modulo E if there exists n-cells f', g' in S^* and e' in $E^\top$:

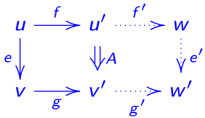
$$\begin{array}{ccccc} u & \xrightarrow{f} & u' & \xrightarrow{f'} & w \\ {\scriptstyle e}\downarrow & & & & \downarrow{\scriptstyle e'} \\ v & \xrightarrow[g]{} & v' & \xrightarrow[g']{} & w' \end{array}$$

- S is said confluent modulo E (resp. locally confluent modulo E) if any branching (resp. local branching) of S modulo E is confluent modulo E.

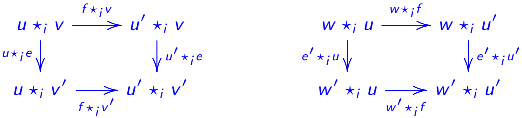
IV. Coherence modulo

Coherent confluence modulo

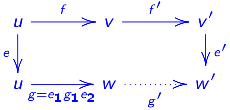
- We consider Γ a 2-square extension of $(E^{\top}, S^{*})$.
- A branching modulo E is Γ-confluent modulo E if there exist n-cells f', g' in S^{*}, e' in $E^{\top}$ and an $(n+1)$-cell A in $(E, S, E \rtimes \Gamma \cup \mathrm{Peiff}(E,S))^{\top\!\!\top, v}$:



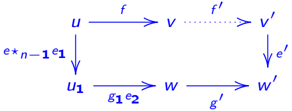
  - $(E, S, -)^{\top\!\!\top, v}$ is the free n-category enriched in double categories generated by $(E, S, -)$, in which all vertical cells are invertible.
  - $\mathrm{Peiff}(E, S)$ is the 2-square extension containing the following squares for all e,$e' \in E^{\top}$ and $f \in S^{*}$.



  - E $\rtimes$ Γ is to avoid "redundant" elements in Γ for different squares corresponding to the same branching of S modulo E:



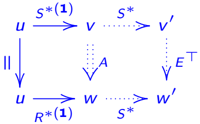
and



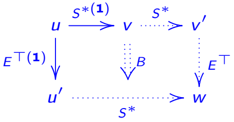
Coherent Newman and critical pair lemmas

- S is Γ-confluent modulo E (resp. locally Γ-confluent modulo E) if any of its branching modulo E (resp. local branching modulo E) is Γ-confluent modulo E.
- **Theorem:** The following assertions are equivalent:
  - S is Γ-confluent modulo E;
  - S is locally Γ-confluent modulo E;
  - S satisfies properties **a)** and **b)**:

a):



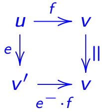
b):



for any local branching of S modulo E.

  - S satisfies properties **a)** and **b)** for any critical branching of S modulo E.

- For $S = {}_E R$, property **b)** is trivially satisfied.



Coherence modulo

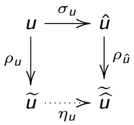
- A set X of $(n-1)$-cells in R^*_{n-1} is E-normalizing with respect to S if for any u in X,*

$$\mathrm{NF}(S,u) \cap \mathrm{Irr}(E) \neq \emptyset.$$

- **Theorem:** Let (R, E, S) be n-polygraph modulo, and Γ be a square extension of the pair of $(n+1, n)$-categories $(E^\top, S^\top)$ such that
  - E is convergent,
  - S is Γ-confluent modulo E,
  - $\mathrm{Irr}(E)$ is E-normalizing with respect to S,
  - ${}_E R_E$ is terminating,

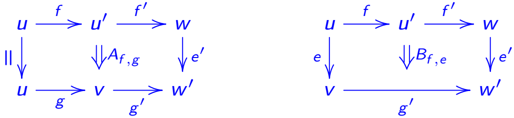
  then $\Gamma \cup \mathrm{Cd}(E)$ is acyclic.

- A normalization strategy for an n-polygraph P is a map σ that sends every $(n-1)$-cell u to an n-cell $\sigma_u : u \to \hat{u}$.
- σ and ρ normalization strategies for S and E weakly commute if:



Coherent extensions

- A coherent completion modulo E of S is a square extension denoted by $\mathcal{C}(S)$ of the pair of $(n+1,n)$-categories $(E^\top, S^\top)$ containing square cells $A_{f,g}$ and $B_{f,e}$:



  for any critical branchings (f,g) and (f,e) of S modulo E.

- **Corollary:** Let (R,E,S) be an n-polygraph modulo such that
  - E is convergent,
  - S is confluent modulo E,
  - $\mathrm{Irr}(E)$ is E-normalizing with respect to S,
  - ${}_E R_E$ is terminating,

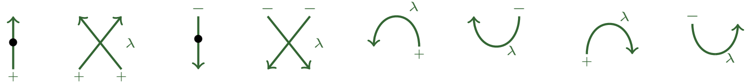
  For any coherent completion Γ of S modulo E, $\Gamma \cup \mathrm{Cd}(E)$ is acyclic.

- **Corollary:** Let R be an n-polygraph.

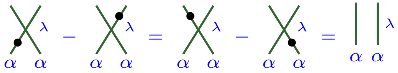
Conclusion

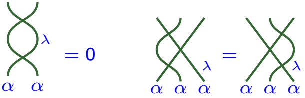
Example: The 2-category $\mathcal{KLR}$

- Let $\mathcal{KLR}$ be the 2-linear category defined by:
  - $\mathcal{KLR}_{\mathbf{0}}$ is a set X corresponding to the weight lattice of a Kac-Moody algebra;
  - $\mathcal{KLR}_{\mathbf{1}} = \{\underline{\varepsilon} = (\varepsilon_{\mathbf{1}}, \dots, \varepsilon_{\ell(\varepsilon)})$ with $\varepsilon_i \in \{-,+\}\}$.
  - $\mathcal{KLR}_{\mathbf{2}}$ admits for generating 2-cells:



- Subject to the following relations:
  - "Nil-Hecke relations" for both orientations of strands:

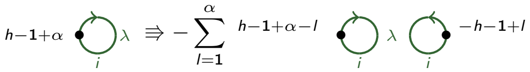




  - Bubble relations:

 $\Rightarrow \begin{cases} 1_{\mathbf{1}_\lambda} & \text{if } n = h-1 \\ 0 & \text{if } n < h-1 \end{cases}$;  $\Rightarrow \begin{cases} 1_{\mathbf{1}_\lambda} & \text{if } n = -h-1 \\ 0 & \text{if } n < -h-1 \end{cases}$

Example: The 2-category $\mathcal{KLR}$



for any $\lambda \in X$ and $\alpha > 0$

- Isotopy relations:

- "Quantum" relations