

# Réécriture modulo dans les catégories diagrammatiques.

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Soutenance de thèse de Doctorat

Sous la direction de Philippe Malbos, Stéphane Gaussen et Alistair Savage

20 Novembre 2020



Université Claude Bernard



### I. Introduction

### II. Convergent presentation of the Khovanov-Lauda-Rouquier algebras

### III. Confluence modulo isotopies in the Khovanov-Lauda-Rouquier 2-category

### IV. Conclusion and perspectives

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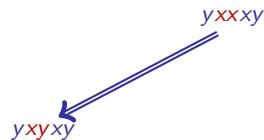
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xxxxy

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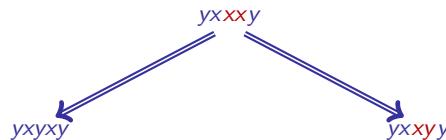
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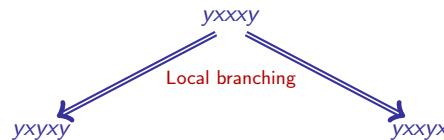
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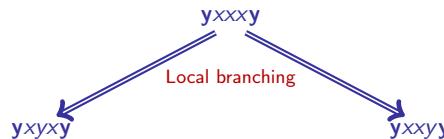
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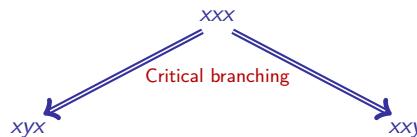
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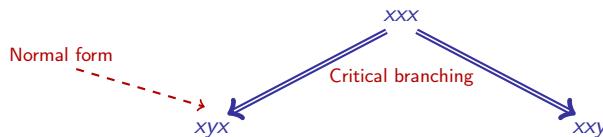
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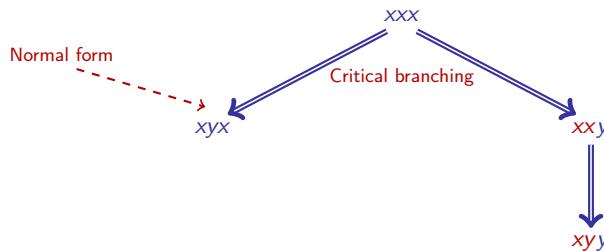
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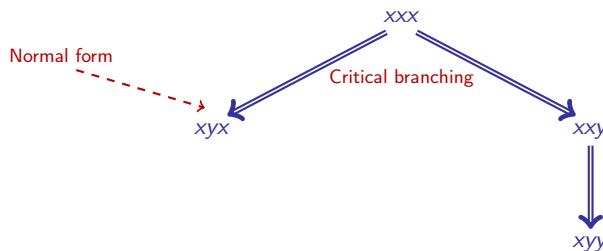
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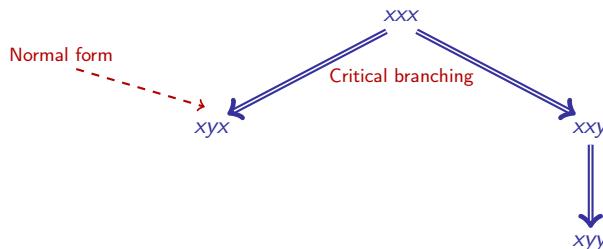
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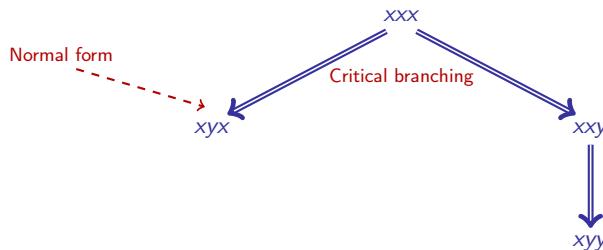


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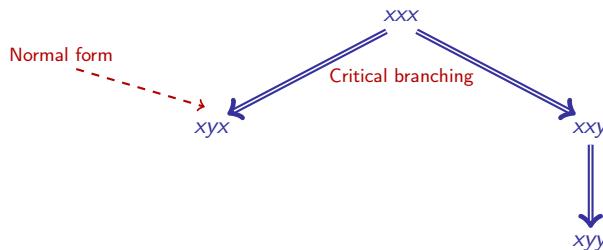


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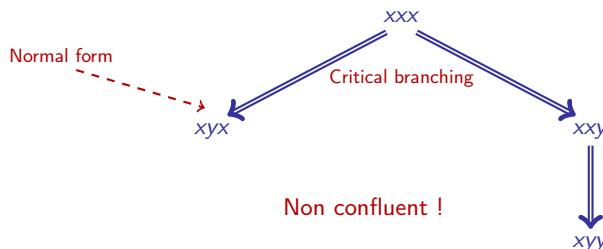


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- i)  $P$  is confluent.
- ii) The vector space  $P_1^\ell := \mathbb{K}[P_1^*]$  admits the direct decomposition

$$P_1^\ell = P_1^{\text{nf}} \oplus I(P)$$

where  $P_1^{\text{nf}}$  is the set of monomials in normal form with respect to  $P_1$ , and  $I(P)$  is the two sided ideal generated by  $\{s_1(\alpha) - t_1(\alpha) \mid \alpha \in P_2\}$ .

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► Several new questions, e.g. extension to rewriting in 2-supercategories (with **M. Ebert** and **A. Lauda**) and explicit proofs of categorification (with **G. Naisse**).

## II. Convergent presentation of the KLR algebras

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$$x_i x_j = x_j x_i$$

$$\tau_i x_j = x_j \tau_i \quad \text{if } |i - j| > 1$$

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$$\tau_i^2 = 0$$

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$$

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$$x_i = \left| \dots \bullet \dots \right|_i, \quad \tau_i = \left| \dots \begin{array}{c} \diagup \\ \diagdown \end{array} \dots \right|_i$$

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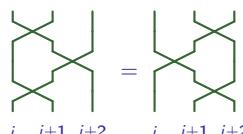
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- ▶ We study these algebras by realizing them as 2-Hom-spaces of linear 2-categories.

## String diagrams

---

- ▶ The 2-cells of a (linear) 2-category can be depicted by a string diagram:

$$\begin{array}{ccc} \begin{array}{c} \text{q} \\ | \cdots | \\ \boxed{f} \\ | \cdots | \\ \text{p} \end{array} & \rightsquigarrow & \begin{array}{c} \text{p} \\ \text{x} \curvearrowright \text{y} \\ \Downarrow f \\ \text{q} \end{array} \end{array}$$

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- Compositions:

$$\begin{array}{c} \text{p} \\ \text{x} \\ \Downarrow f \\ \text{y} \\ \text{q} \end{array} \rightsquigarrow \begin{array}{c} \text{p}' \\ \text{y} \\ \Downarrow g \\ \text{z} \\ \text{q}' \end{array} \rightsquigarrow \begin{array}{c} \text{q} \star_0 \text{q}' \\ \text{x} \\ \Downarrow f \star_0 g \\ \text{z} \\ \text{p}' \star_0 \text{p}' \end{array}$$

$$\begin{array}{c} \text{p} \\ \text{x} \\ \Downarrow f \\ \text{y} \\ \Downarrow g \\ \text{r} \end{array} \rightsquigarrow \begin{array}{c} \text{p} \\ \text{y} \\ \Downarrow f \star_1 g \\ \text{r} \\ \text{y} \end{array}$$

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The diagram shows a box labeled  $f$  with inputs  $q$  and  $p$  and outputs  $x$  and  $y$ . To its right is a commutative diagram with nodes  $X$ ,  $Y$ , and  $f$ . There are two curved arrows: one from  $X$  to  $Y$  and another from  $Y$  to  $X$ . A vertical double-headed arrow labeled  $f$  connects  $X$  and  $Y$ . The labels  $q$  and  $p$  are positioned such that they align with the inputs and outputs of the box  $f$  respectively.

## ► Compositions:

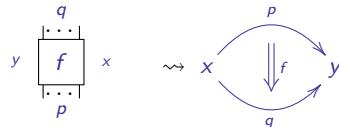
$$\begin{array}{c}
 \text{Diagram showing the equivalence of two commutative diagrams:} \\
 \text{Left: } X \xleftarrow{q} \text{parallel} \xrightarrow{p} y \xrightarrow{f} q \\
 \text{Middle: } y \xleftarrow{q'} \text{parallel} \xrightarrow{p'} z \xrightarrow{g} q' \\
 \text{Right: } X \xleftarrow{p' \star q'} \text{parallel} \xrightarrow{q \star g} z \xrightarrow{f \star g} q' \\
 \text{Equivalence: } X \xleftarrow{q \star g} z \xrightarrow{f \star g} q' \xrightarrow{p' \star q'} X
 \end{array}$$

$$X \xrightarrow{\quad q \quad} \begin{array}{c} P \\ \Downarrow f \\ y \end{array} \rightsquigarrow X \rightsquigarrow \begin{array}{c} P \\ \Downarrow f \star g \\ y \end{array} \xrightarrow{\quad r \quad} Y$$

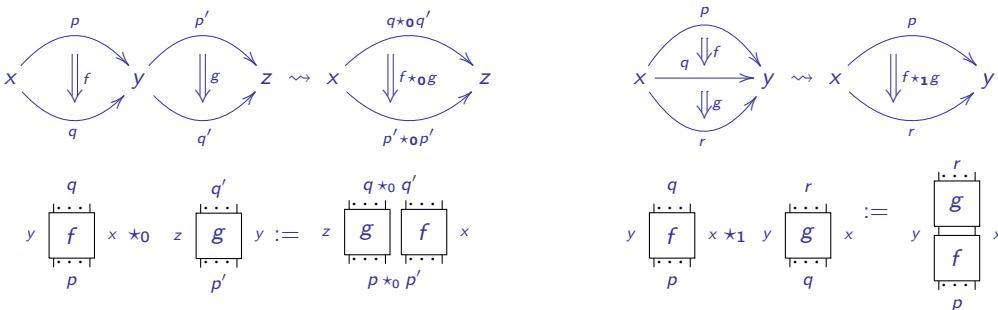
$$y \xrightarrow[p]{f} x \star_1 y \quad r \xrightarrow[q]{g} x \quad := \quad y \xrightarrow[p]{f} r \xrightarrow[q]{g} x$$

## String diagrams

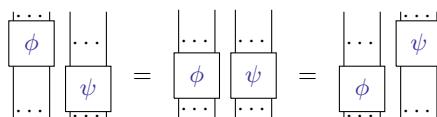
- The 2-cells of a (linear) 2-category can be depicted by a string diagram:



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- These compositions satisfy exchange relations:



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satisfying **globular relations**:  $s_0 s_1 = s_0 t_1, t_0 s_1 = t_0 t_1$ .

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$$\forall x, y \in P_1^* : P_2^\ell(x, y) = \mathbb{K}[P_2^*(x, y)].$$

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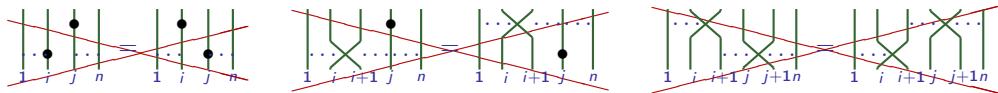
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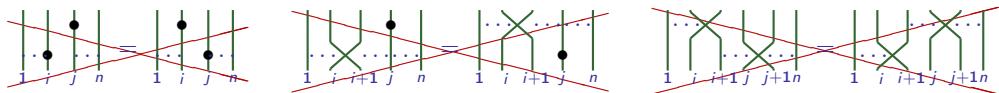
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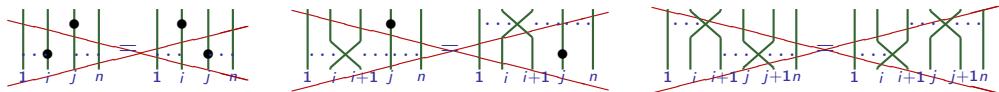


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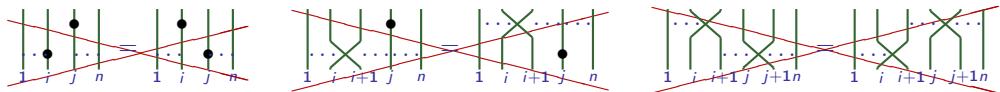
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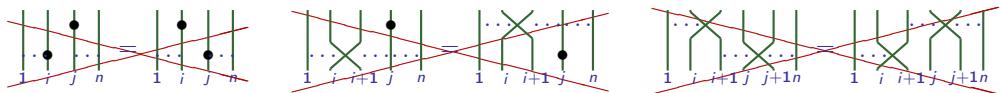
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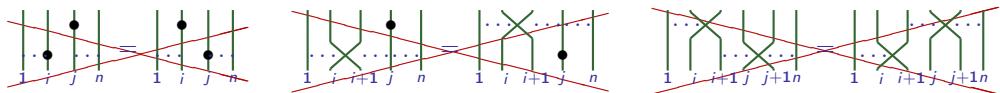
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- ▶ A **rewriting step** of a linear  $(3, 2)$ -polygraph is 3-cell of the form

$$\lambda \begin{array}{c} \dots \\ \boxed{m_1} \\ \dots \\ \boxed{m_2} \boxed{s_2(\alpha)} \boxed{m_3} \\ \dots \\ \boxed{m_4} \\ \dots \end{array} + u \Rightarrow \lambda \begin{array}{c} \dots \\ \boxed{m_1} \\ \dots \\ \boxed{m_2} \boxed{t_2(\alpha)} \boxed{m_3} \\ \dots \\ \boxed{m_4} \\ \dots \end{array} + u$$

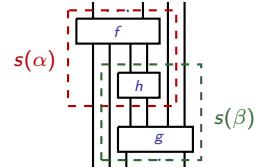
where  $\alpha \in P_3$ , and such that  $m_1 *_1 (m_2 *_0 s_2(\alpha) *_0 m_3) *_1 m_4$  does not appear in the monomial decomposition of  $u$ .

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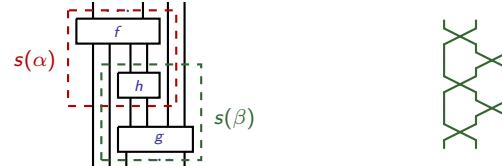
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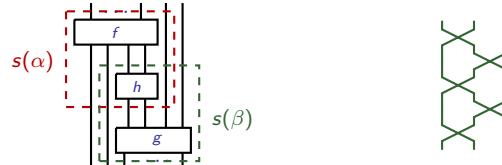
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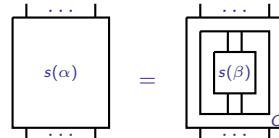


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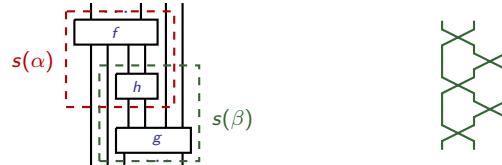


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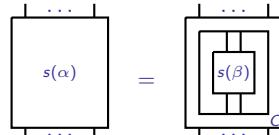


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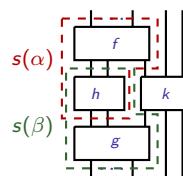
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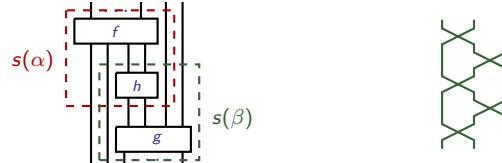


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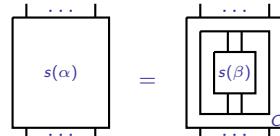


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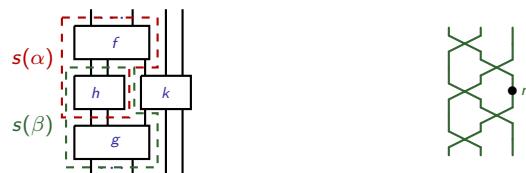
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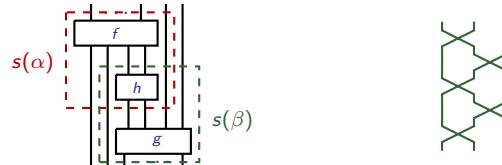


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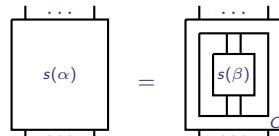


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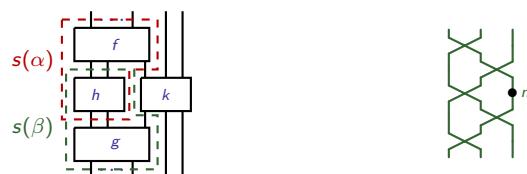
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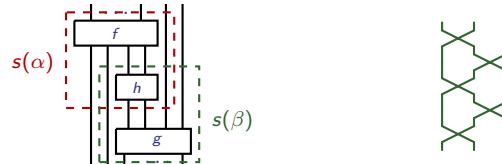
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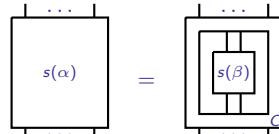
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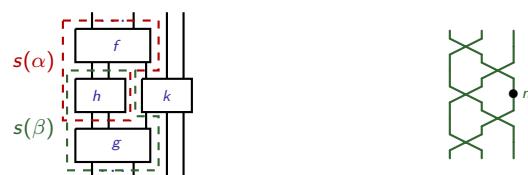
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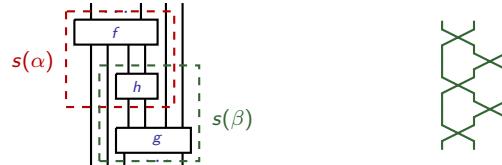
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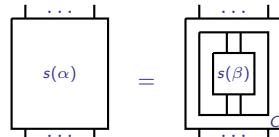
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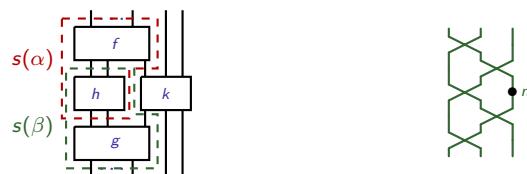
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  - ▶ **Exemple:**  $\text{Seq}(2i + k) = \{ii\bar{k}, \bar{i}ki, k\bar{i}\bar{i}\}$
- ▶ For such an element  $\mathcal{V}$ , we define an algebra  $R(\mathcal{V})$ .

- ▶ **Theorem [Khovanov-Lauda '08]:** If  $R = \bigoplus_{\mathcal{V} \in \mathbb{N}[I]} R(\mathcal{V})$ ,

$$K_0(R - \text{pmod}) \simeq \mathbf{U}_q^-(\mathfrak{g})$$

## Example: Khovanov-Lauda-Rouquier (KLR) algebras

- ▶ These algebras appear in the process of categorifying a quantum group  $\mathbf{U}_q(\mathfrak{g})$  associated with a symmetrizable Kac-Moody algebra  $\mathfrak{g}$ .
- ▶ Let  $\Gamma$  be the Dynkin diagram of  $\mathfrak{g}$ , with set of vertices  $I$ , seen as colors.

$$\Gamma = \begin{array}{ccccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ i & & j & & k & & l \end{array} \quad (\Gamma \text{ simply laced})$$

- ▶ Let  $\mathcal{V} = \sum_{i \in I} \nu_i \cdot i$  be an element of  $\mathbb{N}[I]$ , we consider the set  $\text{Seq}(\mathcal{V})$  of sequels of elements of  $\Gamma$  where  $i$  appears  $\nu_i$  times.

- ▶ **Exemple:**  $\text{Seq}(2i + k) = \{ii\bar{k}, i\bar{k}i, \bar{k}ii\}$

- ▶ For such an element  $\mathcal{V}$ , we define an algebra  $R(\mathcal{V})$ .

- ▶ **Theorem [Khovanov-Lauda '08]:** If  $R = \bigoplus_{\mathcal{V} \in \mathbb{N}[I]} R(\mathcal{V})$ ,

$$K_0(R - \text{pmod}) \simeq \mathbf{U}_q^-(\mathfrak{g})$$

- ▶  $R(\mathcal{V})$  is generated by

$$x_{k,i} = \left| \begin{array}{c|c|c} \dots & \bullet & \dots \\ \hline i_1 & i_k & i_m \end{array} \right| \quad \text{and} \quad \tau_{k,i} = \left| \begin{array}{c|c|c} \dots & \times & \dots \\ \hline i_1 & i_\ell & i_{\ell+1} & i_m \end{array} \right|$$

for any  $i = i_1 \dots i_m \in \text{Seq}(\mathcal{V})$ ,  $1 \leq k \leq m$  and  $1 \leq \ell < m$ .

## Convergent presentation of the KLR algebras

► Relations to realize the algebras  $R(\mathcal{V})$  as 2Hom-spaces of a linear 2-category:  $(\Gamma = \bullet \xrightarrow{i} \bullet \xrightarrow{j} \bullet \xrightarrow{k} \bullet)$

i) Same color:

$$\text{X} = 0$$

$$\text{X} = \text{X} + \text{I} \quad , \quad \text{X} = \text{X} - \text{I}$$

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XX

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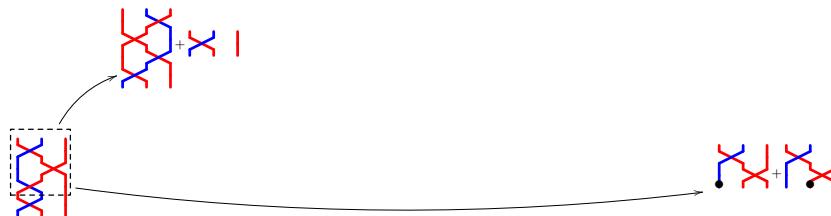
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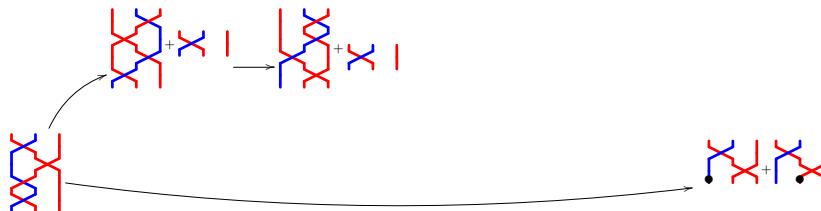
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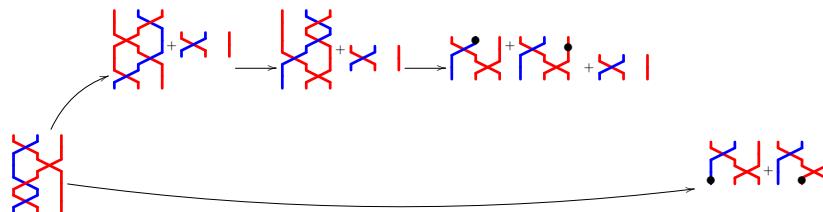
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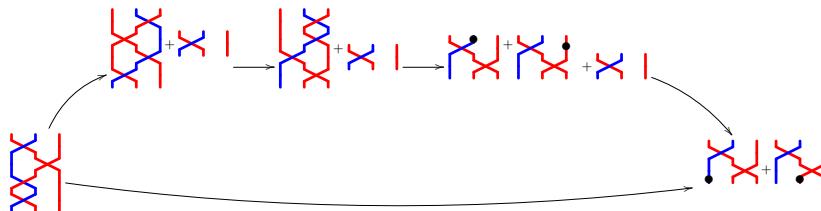
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### III. Confluence modulo in the KLR 2-category

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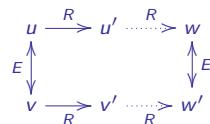
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$$\begin{array}{ccccc} u & \xrightarrow{R} & u' & \xrightarrow{R} & w \\ E \downarrow & & \downarrow E & & \downarrow E \\ v & \xrightarrow{R} & v' & \xrightarrow{R} & w' \end{array}$$

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- We introduce a polygraphic setting for rewriting modulo in diagrammatic linear 2-categories.

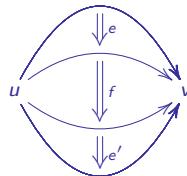
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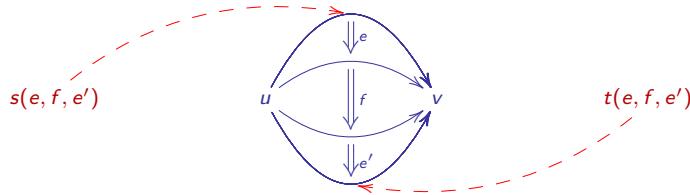
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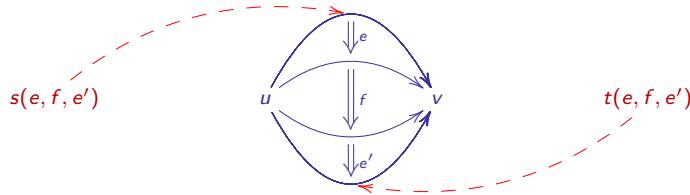


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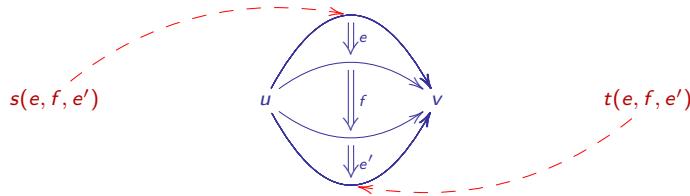
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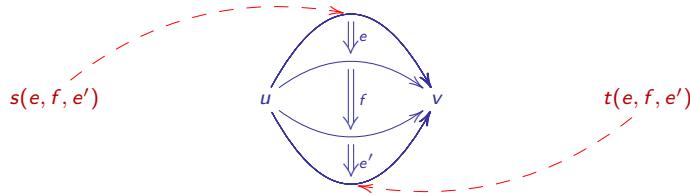
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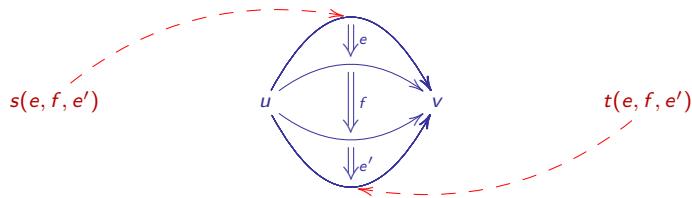


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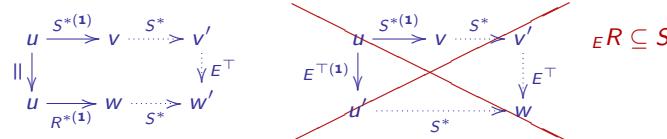
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$$\begin{array}{ccc}
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 u \xrightarrow{S^{*(1)}} v \dots \xrightarrow{S^*} v' \\
 \Downarrow \\
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 \Downarrow E^\top
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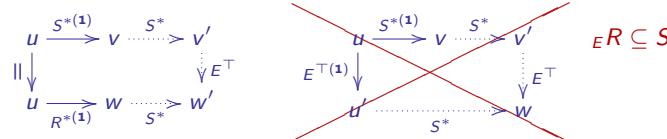
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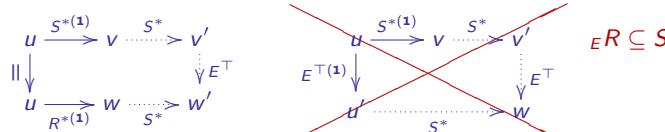
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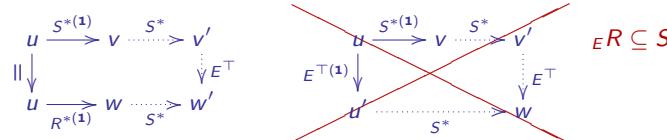
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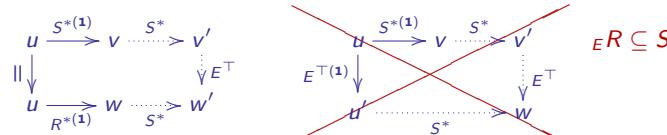
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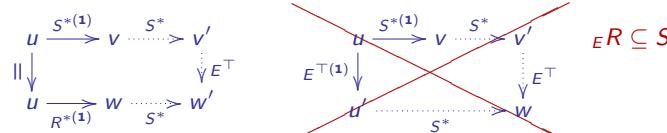


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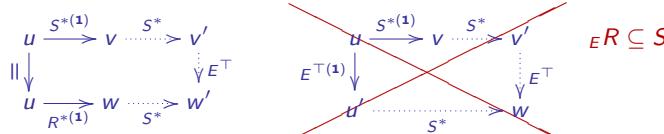
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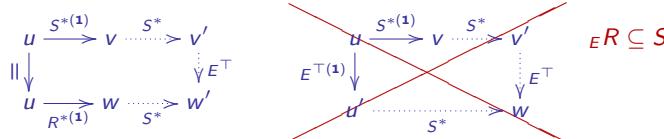
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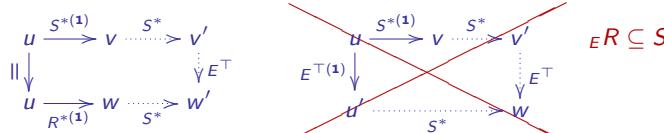
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- Reduce  $u$  into  $\sum \tilde{u}_i$ , where  $\tilde{u}_i$  is a fixed monomial in **quasi normal-form**, that is for every 3-cell  $v \Rightarrow \tilde{u}_i$ , there exists a 3-cell  $\tilde{u}_i \Rightarrow v$ .

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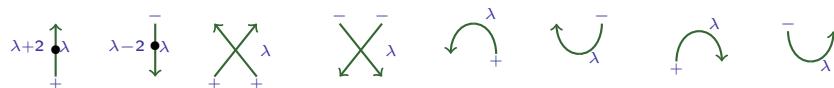
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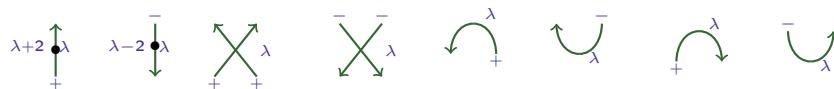
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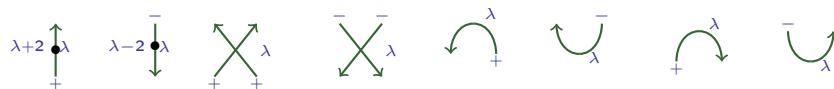
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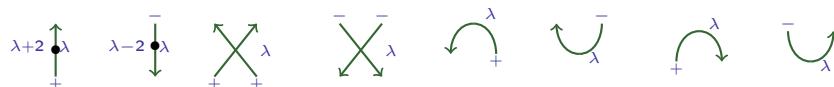


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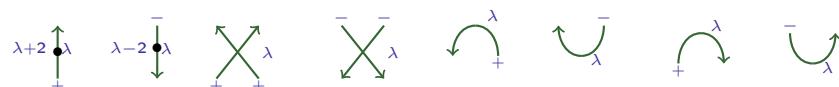
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$$\left. \begin{array}{c} \text{Diagram 1: } \text{U}_{\pm} \Rightarrow \text{U}_{\pm} \Leftarrow \text{U}_{\pm}, \quad \text{Diagram 2: } \text{U}_{\pm} \bullet \text{U}_{\pm} \Rightarrow \bullet \text{U}_{\pm} \Leftarrow \text{U}_{\pm}, \quad \text{Diagram 3: } \bullet \text{U}_{\pm} \Rightarrow \text{U}_{\pm} \bullet \Leftarrow \text{U}_{\pm}, \quad \text{Diagram 4: } \text{U}_{\pm}^{\pm} \Rightarrow \text{U}_{\pm}^{\pm} \Leftarrow \text{U}_{\pm}^{\pm} \end{array} \right\} E$$

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- Bubble relations:

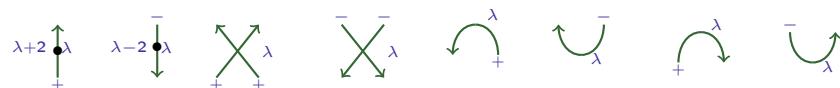
$$n \bullet \text{Diagram 13} \lambda \Rightarrow \begin{cases} 1_{1\lambda} & \text{if } n = \lambda - 1 \\ 0 & \text{if } n < \lambda - 1 \end{cases} ; \quad \lambda \text{Diagram 14} \bullet n \Rightarrow \begin{cases} 1_{1\lambda} & \text{if } n = -\lambda - 1 \\ 0 & \text{if } n < -\lambda - 1 \end{cases}$$

$$\lambda - 1 + \alpha \bullet \text{Diagram 15} \lambda \Rightarrow - \sum_{l=1}^{\alpha} \lambda - 1 + \alpha - l \text{Diagram 16} \lambda \text{Diagram 17} \lambda^{-\lambda - 1 + l} \text{ for all } \lambda \in \mathbb{Z} \text{ and } \alpha > 0 \text{ such that } \lambda - 1 + \alpha \geq 0$$

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- Bubble slide relations of the form

$$\lambda + 1 + \alpha \bullet \text{U}_{\lambda} \Rightarrow \sum_{f=0}^{\alpha} (\alpha + 1 - f) \text{U}_{\alpha - f} \bullet \text{U}_{\lambda}^{\lambda - 1 + f}$$

for any orientations of the bubbles and of the strand.

### Example: the 2-category $\mathcal{U}(\mathfrak{sl}_2)$

## ► Quantum relations:

$$\Rightarrow -\uparrow\downarrow_\lambda + \sum_{n=0}^{\lambda-1} \sum_{r \geq 0} \text{Diagram } r - n - r - 2 ,$$

$$\text{Diagram with green loops and red dots} \Rightarrow -\downarrow\uparrow + \sum_{n=0}^{-\lambda-1} \sum_{r \geq 0} -n-r-2 \text{ (with red dots at } n-r-2, n-r, n \text{)},$$

$$\text{Diagram with a green loop and a blue arrow pointing right} \Rightarrow \sum_{n=0}^{\lambda} \text{Diagram with a green loop and a blue arrow pointing right} \text{,}$$

$$\text{Diagram: } \text{Left: } \text{A green loop with a black dot at the top and a green arrow pointing right. Above it is a green curly brace with a black dot at the top and a green arrow pointing right, labeled } \lambda. \text{ Right: } \text{A green loop with a black dot at the top and a green arrow pointing right, labeled } n. \text{ Between them is a blue double-headed arrow symbol. To the right is a blue summation symbol: } \sum_{n=0}^{-\lambda} \text{ with a green curly brace below it labeled } -n-1. \text{ The brace is positioned such that its right end is at the } -1 \text{ in the summation symbol.}$$

$$\text{Diagram: } \text{Left: } \text{A green loop with a self-intersection and a dot at the top. Right: } \sum_{n=0}^{\lambda} \text{ (green loop with dot at top)} \text{ (green loop with dot at bottom)} \text{ (green loop with dot at top)} \dots$$

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- Quantum relations:

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## Example: the 2-category $\mathcal{U}(\mathfrak{sl}_2)$

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$$\text{Diagram 1} \equiv_{E^\perp} \text{Diagram 2}$$

► **Corollary:** A fixed set of quasi-normal forms containing diagrams with source  $p$  and target  $q$  in normal form with respect to  $E$  and having:

- no loops,
- a minimal number of crossings, and permutation diagrams of through strands are left-adjusted,
- dots placed at the bottom of through strands and to the rightmost interval of arcs,
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gives a linear basis of  $\mathcal{U}(\mathfrak{sl}_2)(p, q)$ .

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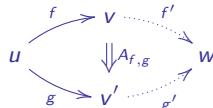
► This holds for any simply-laced Kac-Moody algebra  $\mathfrak{g}$ .

## IV. Conclusion and perspectives

- We developed effective tools based on rewriting modulo to compute in (linear) diagrammatic presentations.

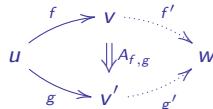
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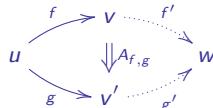
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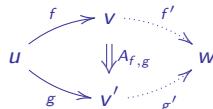


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  - $\text{Irr}(E)$  is  $E$ -normalizing with respect to  $S$ , that is for any  $u$  in  $\text{Irr}(E)$ ,  $NF(u, S) \cap \text{Irr}(E) \neq \emptyset$ .
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- **Long-term project:** Implement computational tools to analyse confluence of diagrammatic presentations.

Thank you for your attention.