Diagrammatic rewriting modulo isotopies

Benjamin Dupont

Institut Camille Jordan, Université Lyon 1

joint work with Philippe Malbos

SYCO 2

Glasgow, 18 December 2018

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Plan

I. Introduction and motivations

II. Double groupoids

III. Polygraphs modulo

IV. Coherence modulo

I. Introduction and motivations

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  Exemple. For the group $\mathbb{Z}^3 = \langle x, y, z \mid [x, y] = 1,\ [y, z] = 1,\ [z, x] = 1 \rangle$, the Jacobi identity

$$[x^y, [y, z]][y^z, [z, x]][z^x, [x, y]] = 1$$

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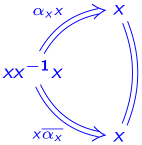
- If a group $G = \langle X \mid R \rangle$ is presented as a monoid $M = \langle X \coprod \overline{X} \mid R \cup \{xx^- \overset{\alpha_x}{\Rightarrow} 1, x^- x \overset{\overline{\alpha_x}}{\Rightarrow} 1\}$, the confluence diagram



  is an artefact induced by the algebraic structure and should not be considered as a syzygy.

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  Example: Isotopy relations

- We use rewriting modulo.
  - Algebraic axioms are not rewriting rules, but taken into account when rewriting.

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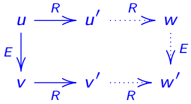

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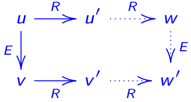

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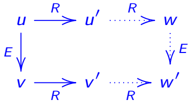

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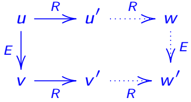

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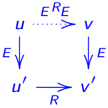

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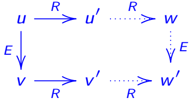

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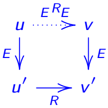

  - $_ER_E$: Rewriting with R on E-equivalence classes
  - Rewriting with any system S such that $R \subseteq S \subseteq {}_ER_E$, Jouannaud - Kirchner '84.

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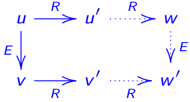

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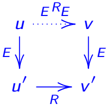
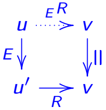


  - Rewriting with any system S such that $R \subseteq S \subseteq {}_E R_E$, Jouannaud - Kirchner '84.
- Main interest and results for ${}_E R$.



II. Double groupoids

Double groupoids

- We introduce a cubical notion of coherence, related to n-categories enriched in double groupoids.

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\begin{array}{ccc}
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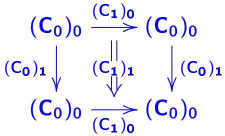
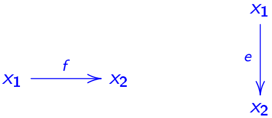


- There are point cells, horizontal cells and vertical cells respectively pictured by



Double groupoids

- There are square cells

$$\begin{array}{ccc} \cdot & \xrightarrow{\partial^h_{-,\mathbf{1}}(A)} & \cdot \\ {\scriptstyle \partial^v_{-,\mathbf{1}}(A)}\Big\downarrow & \Downarrow A & \Big\downarrow{\scriptstyle \partial^v_{+,\mathbf{1}}(A)} \\ \cdot & \xrightarrow[\partial^h_{+,\mathbf{1}}(A)]{} & \cdot \end{array}$$

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\cdot & \xrightarrow[\partial^h_{+,\mathbf{1}}(A)]{} & \cdot
\end{array}
\quad \text{, with identities} \quad
\begin{array}{ccc}
x_1 & \xrightarrow{f} & x_2 \\
{\scriptstyle i^v_{\mathbf{0}}(x_1)}\big\downarrow & \Downarrow i^h_{\mathbf{1}}(f) & \big\downarrow{\scriptstyle i^v_{\mathbf{0}}(x_2)} \\
x_1 & \xrightarrow[f]{} & x_2
\end{array}
\qquad
\begin{array}{ccc}
x & \xrightarrow{i^h_{\mathbf{0}}(x)} & x \\
{\scriptstyle e}\big\downarrow & \Downarrow i^v_{\mathbf{1}}(e) & \big\downarrow{\scriptstyle e} \\
y & \xrightarrow[i^h_{\mathbf{0}}(y)]{} & y
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$$\begin{array}{ccc} \cdot & \xrightarrow{\partial^h_{-,1}(A)} & \cdot \\ {\scriptstyle \partial^v_{-,1}(A)}\downarrow & \Downarrow A & \downarrow{\scriptstyle \partial^v_{+,1}(A)} \\ \cdot & \xrightarrow[\partial^h_{+,1}(A)]{} & \cdot \end{array} \text{, with identities } \begin{array}{ccc} x_1 & \xrightarrow{f} & x_2 \\ {\scriptstyle i^v_0(x_1)}\downarrow & \Downarrow i^h_1(f) & \downarrow{\scriptstyle i^v_0(x_2)} \\ x_1 & \xrightarrow[f]{} & x_2 \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{i^h_0(x)} & x \\ {\scriptstyle e}\downarrow & \Downarrow i^v_1(e) & \downarrow{\scriptstyle e} \\ y & \xrightarrow[i^h_0(y)]{} & y \end{array}$$

- Compositions

$$\begin{array}{ccccc} x_1 & \xrightarrow{f_1} & x_2 & \xrightarrow{f_2} & x_3 \\ {\scriptstyle e_1}\downarrow & \Downarrow A & \downarrow{\scriptstyle e_2} & \Downarrow B & \downarrow{\scriptstyle e_3} \\ y_1 & \xrightarrow[g_1]{} & y_2 & \xrightarrow[g_2]{} & y_3 \end{array} \rightsquigarrow \begin{array}{ccc} x_1 & \xrightarrow{f_1 f_2} & x_3 \\ {\scriptstyle e_1}\downarrow & \Downarrow A \diamond^v B & \downarrow{\scriptstyle e_3} \\ y_1 & \xrightarrow[g_1 g_2]{} & y_3 \end{array}$$

for all x_i, y_i, z_i point cells, f_i, g_i horizontal cells, e_i, e_i' vertical cells and A, A', B square cells.

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\end{array}
$$

$$\diamond^h$$

$$
\begin{array}{ccc}
y_1 & \xrightarrow{g_1} & y_2 \\
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 & \diamond^v & \qquad \diamond^h \\
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\end{array} \\
\diamond^h \\
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\end{array}
\right)
\diamond^v
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- Horizontal $(n+1)$-category will be the $(n+1)$-category of rewritings; vertical $(n+1)$-category is the $(n+1)$-category of modulo rules.

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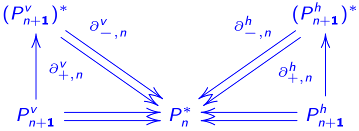

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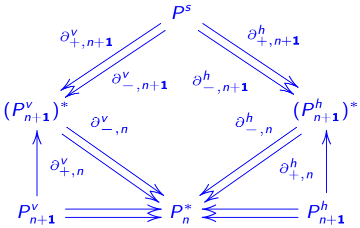

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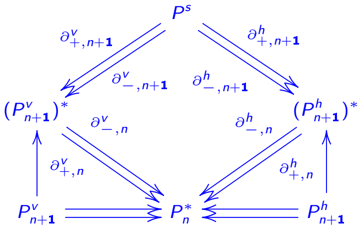

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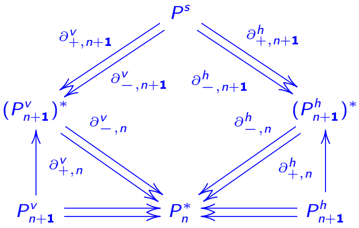

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- A double n-polygraph (resp. double $(n+2, n)$-polygraph) (P^v, P^h, P^s) generates a free $(n-1)$-category enriched in double categories (resp. in double groupoids), denoted by $(P^v, P^h, P^s)^{\top\!\!\top}$.

Acyclicity

- A 2-square extension P^s of $((P^v)^\top, (P^h)^\top)$ is acyclic if for any square

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  $\mathrm{Cd}(E)$:= square extension of $(E^\top, 1)$ containing squares

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- From Squier's theorem, $(E, \emptyset, \mathrm{Cd}(E))$ is a 2-fold coherent presentation of $\mathbf{C}$.

III. Polygraphs modulo

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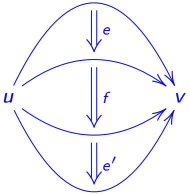
where ${}_E R_E$ is the set of triples (e, f, e') in $E^\top \times R^{*(1)} \times E^\top$ such that



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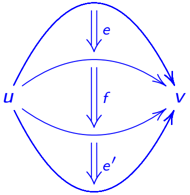

and the map $\gamma^{{}_E R_E}$ is defined by $\gamma^{{}_E R_E}(e, f, e') = (\partial_{-,n-1}(e), \partial_{+,n-1}(e'))$.

Branchings and confluence modulo

- A branching modulo E of the n-polygraph modulo S is a triple (f, e, g) where f and g are n-cells of S^* with f non trivial and e is an n-cell of $E^\top$, such that:

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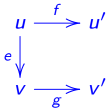

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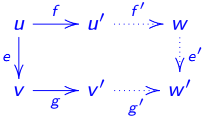

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- S is said confluent modulo E (resp. locally confluent modulo E) if any branching (resp. local branching) of S modulo E is confluent modulo E.

IV. Coherence modulo

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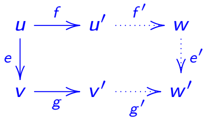

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- We consider Γ a 2-square extension of $(E^{\top}, S^{*})$.
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  - $(E, S, -)^{\top\!\!\top, v}$ is the free n-category enriched in double categories generated by $(E, S, -)$, in which all vertical cells are invertible.

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  - $\text{Peiff}(E,S)$ is the 2-square extension containing the following squares for all $e, e' \in E^{\top}$ and $f \in S^{*}$.

$$\begin{array}{ccc} u \star_i v & \xrightarrow{f \star_i v} & u' \star_i v \\ {\scriptstyle u \star_i e}\downarrow & & \downarrow{\scriptstyle u' \star_i e} \\ u \star_i v' & \xrightarrow[f \star_i v']{} & u' \star_i v' \end{array} \qquad \begin{array}{ccc} w \star_i u & \xrightarrow{w \star_i f} & w \star_i u' \\ {\scriptstyle e' \star_i u}\downarrow & & \downarrow{\scriptstyle e' \star_i u'} \\ w' \star_i u & \xrightarrow[w' \star_i f]{} & w' \star_i u' \end{array}$$

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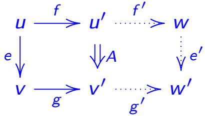

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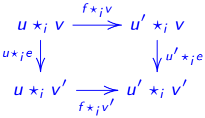
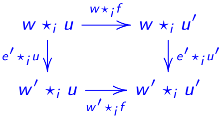
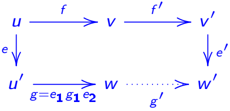




  - $E \rtimes \Gamma$ is to avoid "redundant" elements in Γ for different squares corresponding to the same branching of S modulo E:



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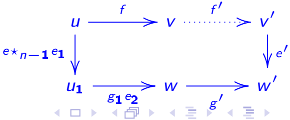

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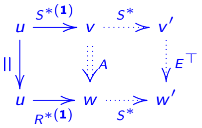

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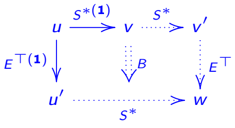

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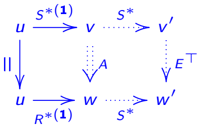

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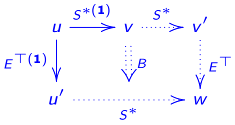

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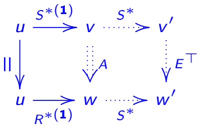
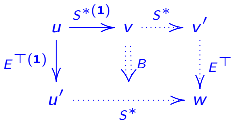
    a): 

    b): 

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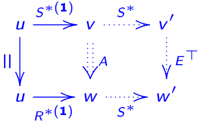

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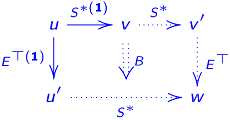

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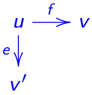

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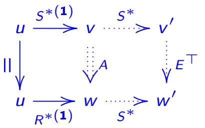
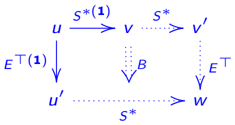
a): 

b): 

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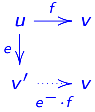

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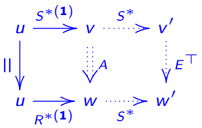

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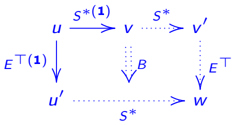

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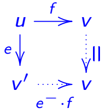

Coherence modulo

- A set X of $(n-1)$-cells in R^*_{n-1} is E-normalizing with respect to S if for any u in X,*

$$\mathrm{NF}(S,u) \cap \mathrm{Irr}(E) \neq \emptyset.$$

- **Theorem.** [D.-Malbos '18] Let (R, E, S) be n-polygraph modulo, and Γ be a square extension of the pair of $(n+1, n)$-categories $(E^\top, S^\top)$ such that
  - E is convergent,
  - S is Γ-confluent modulo E,
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  then $\Gamma \cup \mathrm{Cd}(E)$ is acyclic.

Coherent extensions

- A coherent completion modulo E of S is a square extension denoted by $\mathcal{C}(S)$ of the pair of $(n+1,n)$-categories $(E^\top, S^\top)$ containing square cells $A_{f,g}$ and $B_{f,e}$:

$$
\begin{array}{ccccc}
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\Vert\downarrow & & \Downarrow A_{f,g} & & \downarrow e' \\
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\end{array}
\qquad\qquad
\begin{array}{ccccc}
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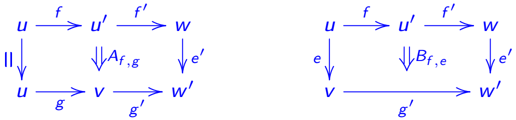

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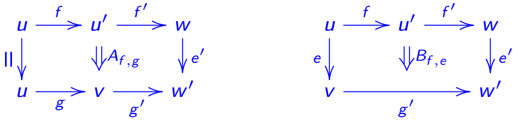

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- **Corollary:** Usual Squier's theorem. $(E = \emptyset)$

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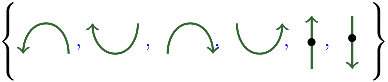
  - $E_2 =$ 

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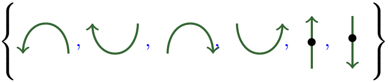
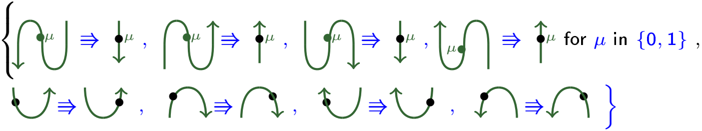
  - $E_2 =$ 
  - $E_3 =$ 

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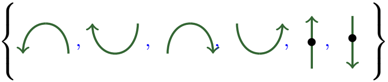
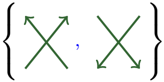
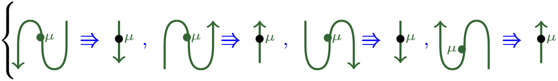
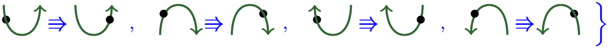
  - $E_2 =$  $R_2 = E_2 \coprod$ 
  - $E_3 =$  for μ in $\{0, 1\}$,



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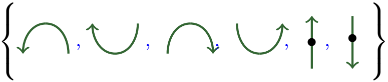
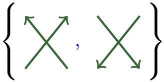
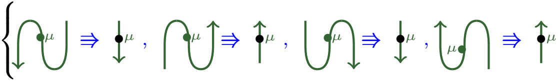
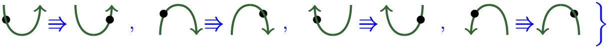
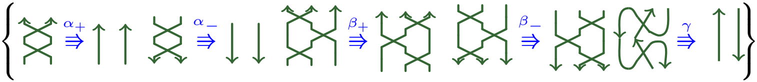
  - $E_2 =$  $R_2 = E_2 \coprod$ 
  - $E_3 =$  for μ in $\{0, 1\}$,
    
  - $R_3 =$ 

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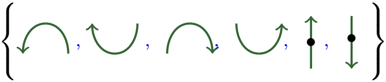
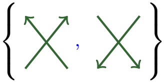
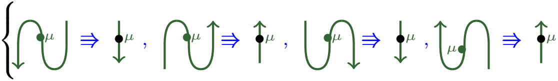
  - $E_2 =$  $R_2 = E_2 \coprod$ 
  - $E_3 =$ 

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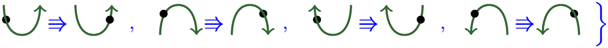
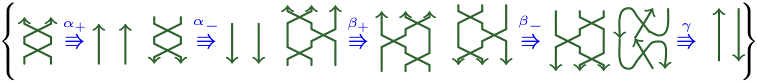


  - $R_3 =$ 

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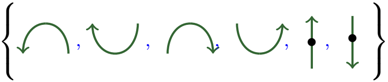
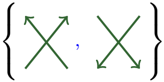
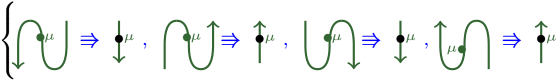
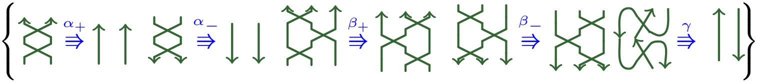
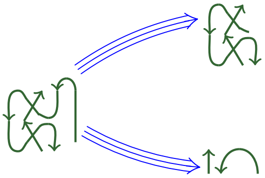
  - $E_2 =$  $R_2 = E_2 \coprod$ 
  - $E_3 =$  for μ in $\{0, 1\}$,
    
  - $R_3 =$ 

- **Fact:** E is convergent.

Toy example: Diagrammatic rewriting modulo isotopy

- If no rewriting modulo:



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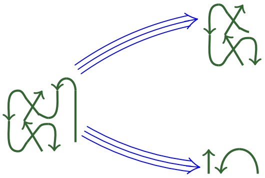

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- If rewriting modulo:
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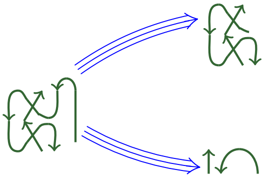

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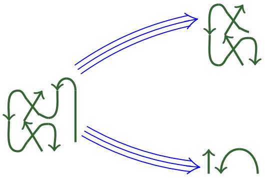

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- $\mathrm{Irr}(E)$ is E-normalizing.
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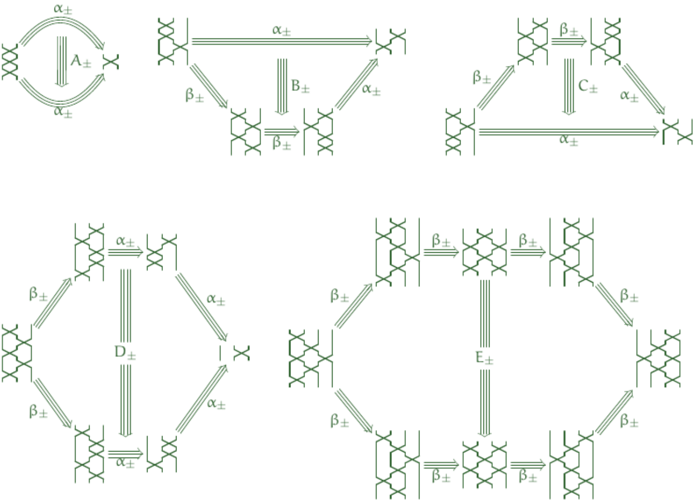

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THANK YOU FOR YOUR ATTENTION.