Introduction to model theory

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Overview

(*Syntax*/Semantics)

1. *Choose a signature σ: a list of basic symbols.* Look at σ-structures: sets and relations interpreting σ.
2. *Build a language $\mathcal{L}$: well-formed formulas using σ.* Look at the definable sets on the structures.
3. *Choose axioms (a theory, T): a set of statements from $\mathcal{L}$.* Restrict to models of T (how many are there?).
4. *Look at consistent sets of formulas.* Finitely satisfiable conditions: types.
5. Invoke a monster (a structure realizing most types).
6. Look at definable groups and/or automorphism groups.

Signature

A *signature* is a list of

- *relation symbols (basic predicates)*
- and *function symbols*,

each with a prescribed *arity* (a natural number). Function symbols of arity 0 are called *constants*.

In continuous logic (CL), a *modulus of uniform continuity* is also prescribed.

Examples 1.

- $\sigma_{\text{rings}} = \{+, -, \cdot, 0, 1\}$, where $+$, $-$, $\cdot$ are binary function symbols and 0, 1 are constants.
- $\sigma_{\text{graphs}} = \{R\}$, where R is a binary predicate.
- $\sigma_{\text{MALG}} = \{\mu, \Delta, \cap, \cdot^c, 0, 1\}$, where μ is a 1-Lipschitz unary predicate, Δ, $\cap$ are binary function symbols, $\cdot^c$ is a unary function symbol and 0, 1 are constants.

Structures

Fix a signature σ. A (classical) *σ-structure* M is a set (which we will also denote by M) together with interpretations for the symbols in σ:

- each n-ary basic predicate P is interpreted as a relation $P^M \subset M^n$;
- each n-ary function symbol f is interpreted as a function $f^M : M^n \to M$.

In CL: a metric σ-structure M is a *bounded complete metric space*; an n-ary predicate P is interpreted as a continuous function $P^M : M^n \to [0,1]$. Moreover, P^M and f^M must respect the given moduli of uniform continuity.

Examples 2.
- Every ring or field is naturally a σ_{rings}-structure.
- A measure algebra (with the distance given by the measure of the symmetric difference) is naturally a σ_{MALG}-structure.
- Any complete bounded metric space is a structure over $\sigma = \emptyset$.

The first-order language

First-order formulas are well-formed expressions using the symbols of σ and the *logical symbols*: the equality relation, connectives, variables and quantifiers.

More formally, one starts by defining *terms*:

- every constant or variable is a term;
- if f is an n-ary function symbol and $t_0, \dots t_{n-1}$ are terms, then $f(t_0, \dots, t_{n-1})$ is a term.

Examples 3.
- $x^2 + 2x - 1$ is a term in σ_{rings} (more formally, replace x^2 by $\cdot(x, x)$, 2 by $+(1, 1)$, etc).
- $x \cap y^c$ is a term in σ_{MALG}.

The first-order language

Then one defines *basic formulas*:

- if t and t' are terms, $t = t'$ is a basic formula;
- if P is an n-ary basic predicate and t is an n-tuple of terms, $P(t)$ is a basic formula.

In CL, $t = t'$ is replaced by $d(t, t')$.

Finally, the set $\mathcal{L}_\sigma$ of *formulas* is given as follows:

- basic formulas are formulas;
- if φ and ψ are formulas, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, $\neg\varphi$;
- if φ is a formula and x is a variable, then $\forall x\varphi$ and $\exists x\varphi$ are formulas.

In CL, connectives are replaced by any continuous combinations $[0,1]^n \to [0,1]$. Quantifiers are suprema and infima: $\sup_x \varphi$, $\inf_x \varphi$. One also considers *forced limits* of sequences of formulas.

The first-order language

Remark: Formulas may or may not have *free* variables (i.e. not quantified). Intuitively, in the first case they express properties, in the second they express statements.

Respectively, in CL, they express functions or statements of a numerical nature.

Examples 4.

- $x^2+2x-1=0$ ("x is a root of the polynomial x^2+2x-1").
- $\exists x\ x^2+2x-1=0$ ("the polynomial x^2+2x-1 has a root").
- $\forall y_0 \forall y_1 \exists x\ x^2+y_1 x+y_0=0$ ("every monic quadratic polynomial has a root").
- $\frac{1}{4}(\|x+y\|^2-\|x-y\|^2)$ (the inner product in a real Hilbert space, in the language of Banach spaces).
- $\sup_x \inf_y |\mu(x\cap y)-\mu(x\cap y^c)|$ (the measure of the largest atom in a measure algebra).

Intepretation of formulas

Let φ be a σ-formula. We usually write $\varphi(x)$ to indicate that the free variables of φ are contained in x (a tuple of distinct variables).

Let M be a σ-structure and let $a\in M^{|x|}$. We write

$$\varphi^M(a)$$

for the truth value of $\varphi(x)$ on M when x is interpreted to denote the tuple a. Of course, quantifiers are interpreted as ranging over *elements of M*.

We omit the formal (recursive, natural) definition.

In CL, $\varphi^M(a)$ is a real number.

Satisfaction, definable sets

We write

$$M\models\varphi(a)$$

to say that $\varphi^M(a)$ is true.

A subset $D\subset M^n$ is *definable* if there is a formula $\varphi(x)$, $|x|=n$, such that

$$D=\{a\in M^n : M\models\varphi(a)\}.$$

Sometimes this set is denoted by $\varphi(M)$.

In CL one can think of truth as given by the value zero, then write

$$M\models\varphi(a)$$

to mean that $\varphi^M(a)=0$. A function $P:M^n\to[0,1]$ is a *definable predicate* if there is a formula $\varphi(x)$ such that $\varphi^M=P$ as functions on M^n.

Definability with parameters

It is useful to admit parameters: if M is a σ-structure and $B \subset M$ is any subset, then $D \subset M^n$ is *B-definable* if there is a formula $\varphi(x, y)$ and a tuple $b \in B^m$ such that

$$D = \{a \in M^n : M \models \varphi(a, b)\}.$$

Equivalently: D is definable in the σ_B-structure M_B, where we have expanded σ to a signature σ_B with constants c_b for each $b \in B$, and M_B is just M with the obvious interpretation of this constants.

We denote the set of σ_B-formulas by $\mathcal{L}_\sigma(B)$. Thus, with a small abuse of notation, $\varphi(x, b) \in \mathcal{L}_\sigma(B)$.

Theories

A *theory* (on a given signature) is a set of statements (formulas with no free variables). A structure M is a *model* of a theory T, denoted $M \models T$, if each $\varphi \in T$ is true in M.

A theory T *implies* a statement φ if φ is true in every model of T:

$$\text{if } M \models T, \text{ then } M \models \varphi.$$

Each structure M induces a theory,

$$\mathrm{Th}(M) = \{\varphi : M \models \varphi\},$$

which is *complete* in the sense that, for every statement φ, either $T \models \varphi$ or $T \models \neg\varphi$.

Theories

Examples 5.

- The theory of infinite sets is axiomatized by the statements

$$\varphi_n : \exists x_0 \ldots \exists x_{n-1} \bigwedge_{0 \le i < j < n} x_i \neq x_j.$$

- The usual axioms of fields can be written in the first-order language of σ_{rings}.
- By adding the (infinitely many) axioms saying that 0 is different from 1, $1+1$, $1+1+1$, etc, and that every monic polynomial of degree $n \geq 2$ has a root, we obtain the theory of algebraically closed fields of characteristic 0, denoted by ACF_0.
- The theory of measure algebras is also first-order axiomatizable. Moreover, we have

$$M \models \sup_x \inf_y |\mu(x \cap y) - \mu(x \cap y^c)|$$

if and only if M is atomless.

Elementary extensions

Let M and N be two σ-structures such that $M \subset N$ as sets. Then N is an *extension* of M (or M is a *substructure* of N) if we have

$$P^M(a) = P^N(a),\ \ f^M(a) = f^N(a)$$

for every basic predicate P and function symbol f, and every tuple a *from* M.

In CL, M must be a metric subspace of N.

If moreover

$$\varphi^M(a) = \varphi^N(a)$$

for every $\varphi(x) \in \mathcal{L}_\sigma$, then N is an *elementary extension* of M, denoted $M \prec N$. In particular, if $M \prec N$ then $\mathrm{Th}(M) = \mathrm{Th}(N)$.

E.g.: as linear orders, $\mathbb{Q}$ is an extension of $\mathbb{Z}$ but $\mathbb{Z} \not\prec \mathbb{Q}$. Instead, $\mathbb{Q} \prec \mathbb{R}$.

Compactness

Let $\Gamma(x)$ be a set of σ-formulas with free variables from x.

$\Gamma(x)$ is *satisfiable* if there is an x-tuple a in some σ-structure M such that

$$M \models \Gamma(a).$$

We also say that $\Gamma(x)$ is *realized* by a.

$\Gamma(x)$ is *finitely realized (in M)* if every finite $\Delta(x) \subset \Gamma(x)$ is realized (by some tuple of M).

Theorem 6. *If $\Gamma(x)$ is finitely realized (in M) then it is satisfiable (realized in some elementary extension of M, e.g. in an ultrapower of M).*

Compactness

In CL, the same definition says that $\Gamma(x)$ is *satisfiable* (or *realized in M*) if for some a in some structure (resp., in M) we have $\varphi(a) = 0$ for every $\varphi \in \Gamma(x)$.

$\Gamma(x)$ is *approximately finitely realized (in M)* if for any $\epsilon > 0$ and finitely many formulas $\varphi_i(x) \in \Gamma(x)$, $i < n$, there is a tuple a (in M) such that

$$|\varphi_i(a)| < \epsilon$$

for every $i < n$.

(*Equivalently:* the closed ideal generated by $\{\varphi^M : \varphi \in \Gamma(x)\}$ in the space of real-valued continuous bounded functions $C(M^{|x|})$ is proper.)

Theorem 7. *If $\Gamma(x)$ is approximately finitely realized (in M) then it is satisfiable (realized in some elementary extension of M, e.g. in an ultrapower of M).*

Types

Fix a σ-structure M, $B \subset M$. A *(partial) type in x over B in M* is a set $\pi(x) \subset \mathcal{L}_\sigma(B)$ that is (approximately) finitely realized in M. When $|x| = n$, $\pi(x)$ is also called an *n-type* over B.

Given an x-tuple a in M, we define the *type of a over B* by

$$\mathrm{tp}(a/B) = \{\varphi \in \mathcal{L}_\sigma(B) : M \models \varphi(a)\}.$$

These are *complete* types: maximal for inclusion. That is, complete types over A are ultrafilters in the algebra of B-definable sets.

If $B \subset M \prec N$, then any set $\Gamma(x) \subset \mathcal{L}_\sigma(B)$ is (app.) finitely realized in M if and only if it is (app.) finitely realized in N. In particular, types over B in M or in N coincide.

By the compactness theorem, every type over $B \subset M$ is realized in some elementary extension of M.

Types, quantifier elimination

A theory has *quantifier elimination* if $\mathrm{tp}(a/B)$ is determined by the *basic formulas* in $\mathcal{L}_\sigma(B)$ satisfied by a. Using that this is true for dense linear orders and for pure sets, we see that:

Examples 8.
- There is only one 1-type over $\emptyset$ in $(\mathbb{Q}, <)$, only three 2-types over $\emptyset$, etc: a type over $\emptyset$ is determined by the order isomorphism type of a tuple that realizes it.
- The type $\{x \neq b : b \in \mathbb{N}\}$ is the only non-realized 1-type over $B = \mathbb{N}$ in the pure set $M = \mathbb{N}$.
- There as many non-realized 1-types over $B = \mathbb{Q}$ in $(\mathbb{Q}, <)$ as there are partitions $\mathbb{Q} = C \sqcup D$ with $c < d$ for every $c \in C$, $d \in D$.

Space of types

Fix $B \subset M$ as before. We denote by $S_x(B)$ the space of all complete types over B in the variable x, or alternatively $S_n(B)$ if $|x| = n$. It is a compact Hausdorff totally disconnected space with basic clopen sets

$$[\varphi] = \{p \in S_x(B) : \varphi \in p\}$$

for each $\varphi(x) \in \mathcal{L}_\sigma(B)$.

In CL, the space of complete types $S_x(B)$ can be seen as the maximal ideal space of the algebra of B-definable predicates on M, with its usual Gelfand topology (of course, here it need not be totally disconnected). In other words, $S_x(B)$ is the minimal compactification of M^n through which every function φ^M ($\varphi(x) \in \mathcal{L}_\sigma(B)$) factors.

Saturation

Let κ be an infinite cardinal. A structure M is *κ-saturated* if, for any $B \subset M$ of cardinality $|B| < \kappa$, every type in $S_1(B)$ is *realized in M* (*equivalently:* any n-type over B).

Examples 9.
1. Every $\aleph_0$-categorical structure is $\aleph_0$-saturated.
2. A model of ACF_0 is $\aleph_0$-saturated if and only if it has infinite transcendence degree.

The monster

Fix a theory T. It is usual and convenient to work inside a fixed very saturated, homogeneous model of T containing all models of interest as elementary substructures.

More precisely, for an arbitrarily large cardinal κ one can find a model $\mathbb{M}$ (a *monster model*) such that:

- (call a set B *small* if $|B| < \kappa$)
- all small models of T are elementary embeddable in $\mathbb{M}$;
- every type over a small subset of $\mathbb{M}$ is realized in $\mathbb{M}$;
- every *elementary map* between small subsets of $\mathbb{M}$ can be extended to an automorphism of $\mathbb{M}$.

Definable groups

Let M be a structure. A *definable group* in M is given by definable sets $G \subset M^n$ and $\cdot \subset M^n \times M^n \times M^n$ such that

$$M \models \text{“}(G, \cdot) \text{ is a group”}.$$

We may abuse notation and identify G and $\cdot$ with the formulas defining them.

Then for any elementary extension $M \prec N$ we have that $(G^N, \cdot^N)$ is also a group. In fact it contains $(G, \cdot)$ as a subgroup, since for any $a, b, c \in G$ we have

$$M \models a \cdot b = c \ \text{ if and only if } \ N \models a \cdot b = c.$$

Definable groups

Now let $S_G(M)$ be the space of types over M containing the formula G. That is, the closure of the image of the set G in the natural embedding $\mathrm{tp} : M^n \to S_n(M)$. By saturation we have

$$S_G(M) = \{\mathrm{tp}(\tilde{g}/M) : \tilde{g} \in G^{\mathbb{M}}\}.$$

But G is a subgroup of $G^{\mathbb{M}}$, and this induces an action of G on $S_G(M)$:

$$g.\mathrm{tp}(\tilde{g}/M) = \mathrm{tp}(g \cdot \tilde{g}/M)$$

for $g \in G \subset M^n$ and $\tilde{g} \in G^{\mathbb{M}} \subset \mathbb{M}^n$. Since the product is definable, this is a well-defined action by homeomorphisms. That is, $S_G(M)$ is a point-transitive G-flow.

Automorphism groups

Let M be a structure. We denote by $\mathrm{Aut}(M)$ the group of automorphisms of M. Then $\mathrm{Aut}(M)$ is a topological group under the topology of *pointwise convergence*. If M is countable (separable) then $\mathrm{Aut}(M)$ is a Polish group.

In fact, automorphism groups of classical countable structures are precisely the closed subgroups of S_∞: if $G \leq S(X)$, one can define basic predicates on X to turn it into a structure with $G = \mathrm{Aut}(X)$.

Similarly, any Polish group can be seen as the automorphism group of a separable metric structure: one chooses a left-invariant metric on G, takes $X = \widehat{G_L}$ its completion and defines appropriate predicates on X to turn it into a metric structure with $G = \mathrm{Aut}(M)$.

Automorphism groups

$\mathrm{Aut}(M)$ acts continuously (by isometries) on M. It also acts continuously on $S_x(M)$. If $g \in \mathrm{Aut}(M)$, $p \in S_x(M)$ then gp is defined by

$$\varphi(x, m)^{gp} = \varphi(x, g^{-1}m)^p,$$

where $\varphi(x, y)$ ranges over σ-formulas, $m \in M^{|y|}$, and $\varphi(x, b)^q$ denotes the value of $\varphi(a, b)$ for any a realizing $q \in S_x(M)$.

Categoricity

Let κ be a cardinal. A theory T is *κ-categorical* if there is only one model of cardinal κ up to isomorphism.

In CL: if there is only one model of density character κ.

Examples 10.

- The theory of infinite sets is κ-categorical for every infinite κ.
- ACF_0 is κ-categorical for every $\kappa \geq \aleph_1$ but not for $\kappa = \aleph_0$.
- $\mathrm{Th}(\mathbb{Q}, <)$ is κ-categorical for $\kappa = \aleph_0$ but not for any $\kappa \geq \aleph_1$.
- The theory of infinite dimensional Hilbert spaces is categorical in every infinite cardinal.
- The theory of atomless measure algebras is $\aleph_0$-categorical but not κ-categorical for larger κ.

$\aleph_0$-categorical structures

Theorem 11. *Let T be a complete theory in a countable signature. The following are equivalent.*

1. *T is $\aleph_0$-categorical.*

2. *$S_n(\emptyset)$ is finite for every n.*

Theorem 12. *Let M be a countable structure such that $Th(M)$ is $\aleph_0$-categorical. Then:*

- *M is homogeneous: if a, b are finite tuples with $tp(a/\emptyset) = tp(b/\emptyset)$ then there is $g \in Aut(M)$ with $ga = b$.*

- *A set $D \subset M^n$ is definable if and only if it is $Aut(M)$-invariant.*

- *It follows that $S_n(\emptyset)$ can be identified with M^n/G.*

Hence the theory of M is $\aleph_0$-categorical if and only if the action of $\mathrm{Aut}(M)$ on M is oligomorphic.

$\aleph_0$-categorical structures

Analogous continuous/approximate statements hold for $\aleph_0$-categorical structures in CL. Among them:

- $S_n(T)$ can be identified with the metric quotient $M^n \mathbin{/\!\!/} \mathrm{Aut}(M)$ (in particular these quotients are compact for all n, and this is equivalent to $\aleph_0$-categoricity).
- A predicate $P : M^n \to \mathbb{R}$ is definable if and only if it is uniformly continuous and $\mathrm{Aut}(M)$-invariant.

Suppose M is $\aleph_0$-categorical and denote by E the set of endomorphisms of M, which is a topological semigroup under the topology of pointwise convergence. Then by (approximate) homogeneity we have the following:

Theorem 13. *E is exactly the pointwise closure of G in M^M, and it can be identified with the left-completion $\widehat{G_L}$.*

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