The group of linear isometries of the Gurarij space is extremely amenable

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When Topological Dynamics meets Model Theory
July 1

The first author was supported by the grants FAPESP 2013/14458-9
and FAPESP 2014/12405-8.
(EA) Extreme amenability

- and a connection to Ramsey theory.
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   - group of linear isometries
   - approximate Ramsey property for finite dimensional normed spaces
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(S) Poulsen simplex
  - new characterization
  - group of affine homeomorphisms
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(M) Miscellaneous
  - Hilbert cube
  - Pseudoarc
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**Examples (Pestov)**

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- $\text{Aut}(\mathbb{Q}, <)$

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“Exotic groups” (Herrer–Christensen)
Extremely amenable groups

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**FIRST**
“Exotic groups” (Herrer–Christensen)
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**Lemma (Bodirsky–Pinsker–Tsankov)**
*Open subgroup of an extremely amenable group is extremely amenable.*
A (countable) structure $\mathcal{A}$ is ultrahomogeneous $\iff$ every partial finite isomorphism can be extended to an automorphism of $\mathcal{A}$. 

Theorem (KPT; NvT) $\text{Aut}(\mathcal{A})$ is extremely amenable $\iff$ finitely-generated substructures of $\mathcal{A}$ satisfy the Ramsey property and are rigid.

Examples:
- $\left(\mathbb{Q},<\right)$
- $\left(\mathbb{R},<\right)$
- $\left(\mathbb{C},\cdot\right)$
- Finite linear orders (Ramsey)
- Finite linearly ordered graphs (NR; AH)
- Finite Boolean algebras (GR)
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For $M$ approximately ultrahomogeneous, $\text{Iso}(M)$ is extremely amenable $\iff$ finitely-generated substructures satisfy the approximate Ramsey property.
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Examples (B-LA-M)

- $G$
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Examples (B-LA-M)

- $\mathbb{G}$: finitely-dimensional normed spaces
- $(P, p)$: finite-dimensional simplexes

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Extreme amenability of linear isometries of $\mathbb{G}$
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Gurarij space $\mathbb{G}$

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3. for every $E$ finite dimensional, $i : E \hookrightarrow \mathbb{G}$ isometric embedding and $\varepsilon > 0$ there is a linear isometry $f : \mathbb{G} \rightarrow \mathbb{G}$

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Conditions (1), (2), (3) uniquely define $\mathcal{G}$ up to a linear isometry.
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Conditions (1),(2),(3) uniquely define $G$ up to a linear isometry.

**KUBIŚ-SOLECKI; HENSON**
Simple proof - metric Fraïssé theory.
Group of linear isometries

$\text{Iso}_l(G) + \text{point-wise convergence topology} = \text{Polish group}$
Group of linear isometries

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- \( E \) - finite dimensional subspace of \( \mathbb{G} \)
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- $\varepsilon > 0$

$$V_\varepsilon(E) = \{ g \in \text{Iso}(G) : \| g \upharpoonright E - \text{id} \upharpoonright E \| < \varepsilon \}$$
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Isoₙ(𝐺) is a universal Polish group.
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Katětov construction
Approximate Ramsey property for $l^n_\infty$'s

Theorem (B-LA-M)

$d \leq m$

$r$ - number of colours

$\varepsilon > 0$

$\exists n$ for every colouring $c$

$\text{Emb}(l^d_\infty, l^n_\infty) \rightarrow \{0, 1, \ldots, r-1\}$

there is $i \in \text{Emb}(l^m_\infty, l^n_\infty)$ and $\alpha < r$

$i \circ \text{Emb}(l^d_\infty, l^m_\infty) \subset (c^{-1}(\alpha))$

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$\text{Iso}(G)$ is extremely amenable.

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**Theorem (B-LA-M)**

Iso($G$) is extremely amenable.
Pestov’s characterization of extreme amenability

$G$ - topological group
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$f : G \rightarrow \mathbb{R}$ is **finitely oscillation stable** if

$$\forall X \subset G \text{ finite and } \varepsilon > 0 \exists g \in G \text{ such that } \text{osc}(f|_{gX}) < \varepsilon.$$
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Lemma (Pestov)

TFAE

- $G$ is extremely amenable,
- every $f : G \to \mathbb{R}$ bounded left-uniformly continuous is finite oscillation stable.
Pestov’s characterization of extreme amenability

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**TFAE**

- $G$ is extremely amenable,
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Theorem (Graham and Rothschild)

For every $k \leq m$ and $r \geq 2$, there exists $n$ such that for every colouring of the $k$-element partitions of $n$ by $r$-many colours there is an $m$-element partition $X$ of $n$ such that all $k$-element coarsenings of $X$ have the same colour.
Approximate Ramsey property for finite-dimensional normed spaces

\( E, F \) - finite dimensional spaces
\( \theta \geq 1 \)

\[ \text{Emb}_\theta(E, F) = \{ T : E \rightarrow F : T \text{ embedding} \& \|T\| \|T^{-1}\| \leq \theta \} \]
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**Theorem (B-LA-M)**

\( r \) - number of colours, \( \varepsilon > 0 \quad \exists H \text{ f.d. with } \text{Emb}(F, H) \neq \emptyset \) such that for every

\[ c : \text{Emb}_\theta(E, H) \rightarrow \{0, 1, \ldots, r - 1\} \]

\[ \exists i \in \text{Emb}_\theta(F, H) \text{ and } \alpha < r \text{ such that} \]

\[ i \circ \text{Emb}_\theta(E, F) \subset (c^{-1}(\alpha))_{\theta-1+\varepsilon} \]
Theorem

Finite metric spaces satisfy the approximate Ramsey property.
Theorem

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Corollary (Pestov)

*Iso(\(U\)) is extremely amenable.*
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Corollary (Pestov)

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Theorem (Nešetřil)

*Linearly ordered finite metric spaces satisfy the (exact) Ramsey property.*
Poulsen simplex $P$

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Properties (1),(2) and (3) uniquely determine $P$ up to an affine homeomorphism.
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The set of extreme points of $P$ is dense in $P$. 
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FACT
\( T : \{0, 1\}^\mathbb{Z} \rightarrow \{0, 1\}^\mathbb{Z} \) the shift \( \Rightarrow T \)-invariant probability measures form \( P \)
A projective characterization of $P$

$S_n :=$ positive part of the unit ball of $l_1^n$ – finite-dimensional simplex with $n + 1$ extreme points
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$\text{Epi}(S_n, S_m) :=$ continuous affine surjections $S_n \rightarrow S_m$
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$AH(P) := \text{group of affine homeomorphisms of } P + \text{compact-open topology}$
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(\text{U}) $\forall n \exists \phi : P \rightarrow S_n$ – continuous affine surjection

(\text{APU}) $\forall \varepsilon > 0 \forall n \forall \phi_1, \phi_2 : P \rightarrow S_n \exists f \in \text{AH}(P)$ with $d(\phi_1, \phi_2 \circ f) < \varepsilon$
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Theorem (B-LA-M)

$(U) + (APU)$ characterize $P$ among non-trivial metrizable simplexes up to affine homeomorphism.
Approximate Ramsey property for $P$

$\text{Epi}_0(S_n, S_m)$ - continuous affine surjections preserving 0
Approximate Ramsey property for $P$

$\text{Epi}_0(S_n, S_m)$ - continuous affine surjections preserving 0

**Theorem (B-LA-M)**

If $d \leq m$ and $r$ natural numbers and $\varepsilon > 0$ given, then there exists $n$ such that for every colouring

$$c : \text{Epi}_0(S_n, S_d) \to \{0, 1, \ldots, r\}$$

there is $\pi \in \text{Epi}_0(S_n, S_m)$ and $\alpha < r$ such that

$$\text{Epi}_0(S_m, S_d) \circ \pi \subset (c^{-1}(\alpha))_\varepsilon$$
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$p$ - extreme point of $P$

\[ AH_p(P) = \{ f \in AH(P) : f(p) = p \} \]
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**Theorem (B-LA-M)**

$d \leq m$ and $r$ natural numbers and $\varepsilon > 0$ given $\Rightarrow \exists n$ such that for every colouring

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$$AH_p(P) = \{f \in AH(P) : f(p) = p\}$$

**Theorem (B-LA-M)**

$AH_p(P)$ is extremely amenable.
Universal minimal flows

\[ G = \text{Aut}(\mathcal{A}) \simeq \mathcal{A} \text{ ultrahomogeneous} \]
Universal minimal flows

\[ G = \text{Aut}(\mathcal{A}) - \mathcal{A} \text{ ultrahomogeneous} \]
\[ G^* = \text{Aut}(\mathcal{A}^*) - \mathcal{A}^* \text{ ultrahomogeneous expansion of } \mathcal{A} \]
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**OFTEN** \( M(G) \cong \hat{G}/G^* \)
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<td>( \mathbb{N} )</td>
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Theorem (B-LA-M)

\[ M(AH(P)) \cong \widehat{AH(P)}/AH_p(P) \cong P \]
PROBLEM

What is the universal minimal flow of Homeo(\(\mathcal{Q}\))?
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What is the universal minimal flow of Homeo(\(Q\))? 

\(Q\) is homeomorphic to \(P\).

Theorem (B-LA-M)
Homeo(\(Q\)) admits a closed subgroup with the universal minimal flow being the natural action on \(Q\).
Hilbert cube $\mathcal{Q} = [-1, 1]^\mathbb{N}$

**PROBLEM**
What is the universal minimal flow of $\text{Homeo}(\mathcal{Q})$?

$\mathcal{Q}$ is homeomorphic to $P$.

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$\text{Homeo}(\mathcal{Q})$ admits a closed subgroup with the universal minimal flow being the natural action on $\mathcal{Q}$.

$\mathcal{Q}$ with its natural convex structure.
Hilbert cube $Q = [-1, 1]^\mathbb{N}$

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What is the universal minimal flow of Homeo($Q$)?

$Q$ is homeomorphic to $P$.

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Dana Bartošová

Extreme amenability of linear isometries of $G$
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IRWIN-SOLECKI

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Lemma (B-Kwiatkowska; Solecki)

\(M(\text{Aut}(\mathbb{P}))\) is not metrizable.
Lionel’s conjecture

Oligomorphic automorphism groups of countable structures have metrizable universal minimal flows.
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Oligomorphic automorphism groups of countable structures have metrizable universal minimal flows.

Good example

$\text{Aut}(\mathbb{P})$ is NOT oligomorphic.
Theorem (Veech)

Locally compact groups have non-metrizable universal minimal flows.
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Locally compact groups have non-metrizable universal minimal flows.

Good example

Aut(\mathbb{P}) is NOT locally compact.
OBRIGADA