The topological conjugacy relation for Toeplitz subshifts

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Definition

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Definition

A countable equivalence relation is called *hyperfinite* if it induced by a Borel action of $\mathbb{Z}$.

Given an equivalence relation E on X and a function $f : E \to \mathbb{R}$, for $x \in X$ denote by $f_x : [x]_E \to \mathbb{R}$ the function $f_x(y) = f(x, y)$.

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Definition

Suppose E is a countable Borel equivalence relation. E is *amenable* if there exist positive Borel functions $\lambda^n : E \to \mathbb{R}$ such that

- $\lambda^n_x \in \ell^1([x]_E)$ and $||\lambda^n_x||_1 = 1$,
- $\lim_{n\to\infty} ||\lambda^n_x - \lambda^n_y||_1 = 0$ for $(x, y) \in E$.

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Theorem (Connes–Feldman–Weiss, Kechris–Miller)

If μ is any Borel probability measure on X and E is a.e. amenable, then E is a.e. hyperfinite.

Suppose G is a group. A natural action of G on 2^G is given by *left-shifts*:

$$(g \cdot s)(h) = s(g^{-1}h).$$

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Definition

Two G-subshifts $T, S \subseteq 2^G$ are *topologically conjugate* if there exists a homeomorphism $f : S \to T$ which commutes with the left actions.

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Definition

A G-subshift S is *free* if the left action on S is free, i.e. for every $x \in S$: if $g \cdot x = x$, then $g = 1$.

Definition

Recall that a group G is *residually finite* if for every $g \in G$ with $g \neq 1$ there exists a finite-index (normal) subgroup N such that $g \notin N$.

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Definition

Given a residually finite group G, the *profinite topology* on G is the one with basis at 1 consisting of finite-index subgroups.

Definition (Toeplitz, Krieger)

A word $x \in 2^G$ is called *Toeplitz* if x is continuous in the profinite topology.

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Note

In case $G = \mathbb{Z}$, equivalently a word $x \in 2^{\mathbb{Z}}$ is Toeplitz if for every $k \in \mathbb{Z}$ there exists $p > 0$ such that k *has period* p *in* x, i.e.

$$x(k + ip) = x(k) \quad \text{for all } i \in \mathbb{Z}$$

Definition

A subshift $S \subseteq 2^G$ is Toeplitz if it is generated by a Toeplitz word, i.e. there exists a Toeplitz $x \in 2^G$ such that $S = \mathrm{cl}(G \cdot x)$.

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Theorem (folklore for $\mathbb{Z}$, Krieger for arbitrary G)

Every Toeplitz subshift is minimal.

It turns out that for any countable group G the topological conjugacy relation of G subshifts is a countable Borel equivalence relation.

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A block code induces a G-invariant function $\hat{\sigma} : 2^G \to 2^G$:

$$\hat{\sigma}(x)(g) = \sigma(g^{-1} \cdot x \restriction A).$$

Theorem (Curtis–Hedlund–Lyndon)

Any G-invariant homeomorphism of G-subshifts is given by a block code.

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In particular, as there are only countably many block codes, the topological conjugacy relation is a countable Borel equivalence relation.

Question (Gao–Jackson–Seward)

Given a countable group G, what is the complexity of topological conjugacy of **minimal** (or even free minimal) G-subshifts?

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Theorem (Gao–Jackson–Seward)

For any infinite countable group G the topological conjugacy of free minimal G-subshifts is not smooth.

Definition

A group G is *locally finite* if any finitely generated subgroup of G is finite.

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Theorem (Gao–Jackson–Seward)

If G is locally finite, then the topological conjugacy of free minimal G-subshifts is hyperfinite.

Definition

Note that any countable group G admits a natural *right action* on the set of its free minimal G-subshifts: $S \cdot g = \{x \cdot g : x \in S\}$, where

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Note

It is not difficult to see that S and $S \cdot g$ are topologically conjugate for any $g \in G$.

Theorem (S.–Tsankov)

For any residually finite countable groups G that there exists a probability measure μ on the set of free Toeplitz G-subshifts such that

- μ is invariant under the right action of G
- the stabilizers of points in this action are a.e. amenable.

Theorem (folklore)

If a countable group G acts on a probability space preserving the measure and so that

- the induced equivalence relation is amenable,
- a.e. stabilizers are amenable,

then the group G is amenable.

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Corollary

For any residually finite non-amenable group G the topological conjugacy relation is not hyperfinite.

Definition

Given a $\mathbb{Z}$-subshift $T \subseteq 2^{\mathbb{Z}}$, its *topological full group* $[[T]]$ consists of all homeomorphisms $f : T \to T$ such that $f(x)$ belongs to the same $\mathbb{Z}$-orbit as x, for all $x \in T$ and there is a continuous function $n : T \to \mathbb{Z}$ such that $f(x) = S^{n(x)}(x)$.

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Theorem (Giordano–Putnam–Skau)

If T, T' are minimal $\mathbb{Z}$-subshifts, then the following are equivalent:

- $[[T]]$ and $[[T']]$ are isomorphic (as groups)
- T is topologically conjugate to T' or to the inverse shift on T'.

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- $[[T]]$ and $[[T']]$ are isomorphic (as groups)
- T is topologically conjugate to T' or to the inverse shift on T'.

Theorem (Juschenko–Monod)

If T is a minimal $\mathbb{Z}$-subshift, then $[[T]]$ is amenable.

In terms of Borel-reducibility the two previous theorems show that the topological conjugacy of minimal $\mathbb{Z}$-subshifts is (almost) Borel reducible to the isomorphism of countable amenable groups.

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Question (Thomas)

What is the complexity of the topological conjugacy of minimal $\mathbb{Z}$-subshifts?

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Question (Thomas)

What is the complexity of the topological conjugacy of minimal $\mathbb{Z}$-subshifts?

Theorem (Clemens)

The topological conjugacy of (arbitrary, not neccessarily minimal) $\mathbb{Z}$-subshifts is a universal countable Borel equivalence relation.

Note

Recall that a word $x \in 2^{\mathbb{Z}}$ is Toeplitz if for every $k \in \mathbb{Z}$ there exists $p > 0$ such that k *has period* p *in* x, i.e.

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Notation

Given $x \in 2^{\mathbb{Z}}$ Toeplitz write

$$\mathrm{Per}_p(x) = \{k \in \mathbb{Z} : k \text{ has period } p \text{ in } x\}.$$

Write also

$$H_p(x) = \{0, \ldots, p-1\} \setminus \mathrm{Per}_p(x).$$

Definition

A Toeplitz word $x \in 2^{\mathbb{Z}}$ is said to have *separated holes* if

$$\lim_{p\to\infty} \min\{|i-j| : i,j \in H_p(x), i \neq j\} = \infty.$$

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Definition

A subshift $S \subseteq 2^{\mathbb{Z}}$ *has separated holes* if it is generated by a Toeplitz word which has separated holes.

Theorem (S.–Tsankov)

The topological conjugacy relation of $\mathbb{Z}$-Toeplitz subshifts with separated holes is amenable.

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Question

Is it true that the conjugacy relation of all $\mathbb{Z}$-Toeplitz subshifts is hyperfinite?