

CONCERNING THE EXISTENCE OF AMPLE GENERICS

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ABSTRACT.

1. REPROVING [KR07, Theorem 3.4]

We start by giving an alternative proof to [KR07, Theorem 3.4], which is simplified in the sense that all reference to turbulence is removed. Later on we shall prove a metric generalisation of the same result, which is somewhat more involved, so a simpler proof going along the same lines may help rendering the later proof more accessible.

Throughout, \mathcal{K} denotes a Fraïssé class, \mathfrak{M} its limit, and $G = \text{Aut}(\mathfrak{M})$. We explicitly allow empty structures, so modulo AP, JEP is equivalent to there being a unique empty structure in \mathcal{K} , denoted \emptyset (or yet equivalently, each zero-ary predicate symbol has the same truth value in all members of \mathcal{K}).

Definition 1.1. Let \mathcal{K} be a Fraïssé class. We define $\mathcal{K}_{a,m}$ to consist of all $(\mathfrak{A}, \bar{\psi})$ where $\mathfrak{A} \in \mathcal{K}$ and $\bar{\psi}$ is an m -tuple of partial isomorphisms $\psi_i: \mathfrak{A} \dashrightarrow \mathfrak{A}$. We say that $(\mathfrak{A}, \bar{\psi}) \subseteq (\mathfrak{B}, \bar{\varphi})$ if $\mathfrak{A} \subseteq \mathfrak{B}$ and $\psi_i \subseteq \varphi_i$ for all $i < m$.

- Definition 1.2.**
- (i) We say that $\mathcal{K}_{a,m}$ has *JEP* if every two members thereof embed into a third one.
 - (ii) Let $(\mathfrak{B}, \bar{\varphi}) \in \mathcal{K}_{a,m}$ and $A \subseteq B$. We say that $(\mathfrak{B}, \bar{\varphi})$ *isolates an existential type* for A if whenever $(\mathfrak{B}, \bar{\varphi})$ embeds into two members of $\mathcal{K}_{a,m}$, these can be amalgamated in $\mathcal{K}_{a,m}$ over A (although not necessarily over B).
 - (iii) We say that $\mathcal{K}_{a,m}$ is *existentially atomic* (in the terminology of [KR07], has *WAP*) if every $(\mathfrak{A}, \bar{\psi}) \in \mathcal{K}_{a,p}$ embeds in some $(\mathfrak{B}, \bar{\varphi}) \in \mathcal{K}_{a,m}$ which isolates an existential type for A .

Notice that when $\mathcal{K}_{a,m}$ is existentially atomic, JEP is equivalent to $(\emptyset, \bar{\emptyset})$ isolating an existential type for itself.

Fact 1.3. *The class $\mathcal{K}_{a,m}$ has JEP if and only if G^m admits a dense orbit under conjugation.*

Proof. Kechris and Rosendal [KR07, Theorem 2.1]. See also Theorem 3.2 below, generalising this. ■_{1.3}

Lemma 1.4. *Assume G^m admits a generic orbit under conjugation. Then $\mathcal{K}_{a,m}$ is existentially atomic.*

Proof. Assume that $\bar{f} \in G^m$ is generic, so in particular has a dense orbit. Let $(\mathfrak{A}, \bar{\psi}) \in \mathcal{K}_a$. We may assume that $\mathfrak{A} \subseteq \mathfrak{M}$, in which case $\bar{\psi}$ defines a non empty open set $[\bar{\psi}] \subseteq G^m$. Since \bar{f} has a dense orbit, up to conjugation, i.e., up to choosing another embedding for $\mathfrak{A} \subseteq \mathfrak{M}$, we may assume that $\bar{\psi} \subseteq \bar{f}$. Since $\bar{f}^G = \bigcup_{g \in G/G_A} (\bar{f}^{G_A})^g$ and $[G : G_A] \leq \aleph_0$, \bar{f}^{G_A} cannot be meagre. Thus $U = (\bar{f}^{G_A})^\circ$ is non empty, so $U \cap \bar{f}^{G_A} \neq \emptyset$, and by G_A -invariance $\bar{f} \in U$. Let then $[\bar{\varphi}]$ be a basic open set with $\bar{f} \in [\bar{\varphi}] \subseteq U$, and we may also assume that $\bar{\psi} \subseteq \bar{\varphi}$. Let $\mathfrak{B} \subseteq \mathfrak{M}$ be finite such that $\mathfrak{A} \subseteq \mathfrak{B}$ and $\bar{\varphi}$ is defined on \mathfrak{B} , and

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we claim that $(\mathfrak{B}, \bar{\varphi})$ isolates an existential type for A . Indeed, assume that $(\mathfrak{B}, \bar{\varphi}) \subseteq (\mathfrak{C}, \bar{\rho}) \in \mathcal{K}_{a,m}$. Embedding \mathfrak{C} in \mathfrak{M} over B we have $[\bar{\rho}] \subseteq [\bar{\varphi}]$, so $\bar{f}_A^G \cap [\bar{\rho}] \neq \emptyset$, and so for some $g \in G_A$ we have $\bar{\rho}^g \subseteq \bar{f}$. The same can be done for any other embedding $(\mathfrak{B}, \bar{\varphi}) \subseteq (\mathfrak{D}, \bar{\chi}) \in \mathcal{K}_{a,m}$, say with $\bar{\chi}^h \subseteq \bar{f}$, and then $(\mathfrak{C} \cup \mathfrak{D}, \bar{\rho}^g \cup \bar{\chi}^h)$ is the desired amalgam. $\blacksquare_{1.4}$

Definition 1.5. We say that (\mathfrak{M}, \bar{f}) , with $\bar{f} \in G^m$, is an *existentially atomic limit* of $\mathcal{K}_{a,m}$ if

- (i) For every finite $A \subseteq M$ there is a finite substructure $(\mathfrak{B}, \bar{\varphi}) \subseteq (\mathfrak{M}, \bar{f})$ which isolates an existential type for A (and $A \subseteq B$).
- (ii) If $(\mathfrak{B}, \bar{\varphi}) \subseteq (\mathfrak{M}, \bar{f})$ is finite, isolating an existential type for some $A \subseteq B$, and $(\mathfrak{B}, \bar{\varphi}) \subseteq (\mathfrak{C}, \bar{\rho}) \in \mathcal{K}_{a,m}$, then $(\mathfrak{C}, \bar{\rho})$ embeds in (\mathfrak{M}, \bar{f}) over A .

Lemma 1.6. *Assume that $\mathcal{K}_{a,m}$ has JEP and is existentially atomic, and let $X \subseteq G^m$ denote the set of \bar{f} such that (\mathfrak{M}, \bar{f}) is an existentially atomic limit of $\mathcal{K}_{a,m}$. Then X is a an orbit under conjugation and a dense G_δ set.*

Proof. Assume that $\bar{f}, \bar{g} \in X$, and let us show that they are conjugate. It will be convenient to use the following notation: if $h \in G$ and $C \subseteq M$, then $h \upharpoonright_C = h \upharpoonright_{C \cap h^{-1}C} : C \dashrightarrow C$. We may also view $h \upharpoonright_C$ as $h^{i^{-1}} = i^{-1}hi$ where $i : C \rightarrow M$ is the inclusion. Let $M = \{a_n\}_n$. We construct an increasing sequence of finite partial isomorphisms $\theta_n : \mathfrak{M} \dashrightarrow \mathfrak{M}$, say $\theta_n : \mathfrak{A}_n \cong \mathfrak{B}_n$, as well as a sequence of finite partial isomorphisms (which need not be increasing) $\zeta_n : \mathfrak{M} \dashrightarrow \mathfrak{M}$, $\zeta_n \supseteq \theta_n$, such that:

- (i) If n is even, then, with $\mathfrak{D} = \text{img } \zeta_n$, we have that $\bar{f} \supseteq \bar{g}^{\zeta_n^{-1}}$, $(\mathfrak{D}, \bar{g} \upharpoonright_{\mathfrak{D}})$ isolates an existential type for B_n , and $B_n \supseteq \{a_m\}_{m < n}$.
- (ii) If n is odd, then, with $\mathfrak{C} = \text{dom } \zeta_n$, we have that $\bar{g} \supseteq \bar{f}^{\zeta_n}$, $(\mathfrak{C}, \bar{f} \upharpoonright_{\mathfrak{C}})$ isolates an existential type for A_n , and $A_n \supseteq \{a_m\}_{m < n}$.

We may start with $\zeta_0 = \theta_0 = \emptyset$. For n even, we define $A_{n+1} = A_n \cup \{a_m\}_{m \leq n}$ and find $A_{n+1} \subseteq C \subseteq M$ such that $(\mathfrak{C}, \bar{f} \upharpoonright_C)$ isolates an existential type for A_{n+1} . Then $\zeta_n^{-1} : (\mathfrak{D}, \bar{g} \upharpoonright_{\mathfrak{D}}) \rightarrow (\mathfrak{C}, \bar{f} \upharpoonright_C)$ is an embedding, which means that there is $\zeta_{n+1} : (\mathfrak{C}, \bar{f} \upharpoonright_C) \rightarrow (\mathfrak{M}, \bar{g})$ extending θ_n . Then $\theta_{n+1} = \zeta_{n+1} \upharpoonright_{A_{n+1}}$ will do. The opposite direction is handled identically. At the limit we obtain $\theta = \bigcup \theta_n \in G$ and $\bar{g} = \bar{f}^\theta$.

Let Ξ denote the collection of all triplets $(A, \mathfrak{B}, \bar{\varphi})$ where $A \subseteq M \cap B$ and $(\mathfrak{B}, \bar{\varphi}) \in \mathcal{K}_{a,m}$ isolates an existential type for A . Let also $\Xi_A = \{(\mathfrak{B}, \bar{\varphi}) : (A, \mathfrak{B}, \bar{\varphi}) \in \Xi\} \subseteq \mathcal{K}_{a,m}$. For $(A, \mathfrak{B}, \bar{\varphi}) \in \Xi$ define $U_{(A, \mathfrak{B}, \bar{\varphi})} \subseteq G^m$ to consist of all those \bar{f} such that either $(\mathfrak{B}, \bar{\varphi})$ embeds in (\mathfrak{M}, \bar{f}) over A , or some other $(\mathfrak{C}, \bar{\rho}) \in \Xi_A$, which is incompatible with $(\mathfrak{B}, \bar{\varphi})$ over A , embeds there over A . Clearly $U_{(A, \mathfrak{B}, \bar{\varphi})}$ is open, and since $\mathcal{K}_{a,m}$ is existentially atomic, it is dense. In addition there are only countably many possibilities for $U_{(A, \mathfrak{B}, \bar{\varphi})}$, so $\bigcap_{\Xi} U_{(A, \mathfrak{B}, \bar{\varphi})}$ is a dense G_δ set, and it is not difficult to check that it is exactly X . $\blacksquare_{1.6}$

Putting together,

Theorem 1.7 ([KR07, Theorem 3.4]). *Let \mathcal{K} be a Fraïssé class, $\mathfrak{M} = \lim \mathcal{K}$, $G = \text{Aut}(\mathfrak{M})$. Then the following are equivalent:*

- (i) *The class $\mathcal{K}_{a,m}$ has JEP and is existentially atomic.*
- (ii) *The action of G on G^m by conjugation admits a co-meagre (dense G_δ) orbit.*
- (iii) *The class $\mathcal{K}_{a,m}$ has JEP and admits an existentially atomic limit.*

Proof. (i) \implies (ii). By Lemma 1.6.

(ii) \implies (i). By Lemma 1.4.

(i) \implies (iii). By Lemma 1.6.

(iii) \implies (i). Immediate. $\blacksquare_{1.7}$

2. METRIC FRAÏSSÉ LIMITS VIA APPROXIMATE MAPS

The first step towards generalising the criterion of the previous section to automorphism groups of metric structures is to consider metric Fraïssé classes. These were first treated in the PhD thesis of Konstantinos Schoretsanitis [Sch07]. We shall give here a complete treatment of some the results contained therein, following a somewhat different path.

Definition 2.1. Let \mathcal{L} denote a countable relational metric language. A *Fraïssé class* (in \mathcal{L}) is a class \mathcal{K} of finite structures having

- *HP*: every structure embedding in a member of \mathcal{K} is in \mathcal{K} .
- *JEP*: Every two members of \mathcal{K} embed in a third one.
- *NAP (near amalgamation property)*: Say $\mathfrak{A} \in \mathcal{K}$ embeds both in $\mathfrak{B} \in \mathcal{K}$ and in $\mathfrak{C} \in \mathcal{K}$. Then one can embed \mathfrak{B} and \mathfrak{C} in some $\mathfrak{D} \in \mathcal{K}$ such that the images of \mathfrak{A} under the two embeddings are arbitrarily close.
- *SC*: For each n , let \mathcal{K}_n be the class of all enumerations of members of \mathcal{K} of size n , and for $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}_n$, define $d^{\mathcal{K}}(\mathfrak{A}, \mathfrak{B})$ to be the infimum of all possible $d(A, B) = \max_i d(a_i, b_i)$ under embeddings of \mathfrak{A} and \mathfrak{B} into some $\mathfrak{C} \in \mathcal{K}$. Then $d^{\mathcal{K}}$ is separable and complete.

Notice that modulo NAP, JEP is equivalent to there being a unique empty structure in \mathcal{K} .

We are now going to introduce a new formalism which will allow us to “hide”, in some sense, some small errors (e.g., those arising from NAP). The advantage of (strictly) approximate maps, compared with the current practice in metric model theory of using exact maps, should already become apparent in the treatment of metric Fraïssé limits below. In later sections, the simplifications made possible through the use of approximate maps are going to be virtually indispensable.

One obvious simplification is that many limit arguments simply disappear. In fact, the unique limit construction lies in the back-and-forth argument of Lemma 2.8, which is as good as we could wish for – even in the discrete setting, a back-and-forth argument *is* a limit argument – and there are none of those horrific arguments along the lines of “choose a finite isomorphism, then change it by ε , then change it by $\varepsilon/2$ more, and so on”. Had we worked with exact maps and taken some variant of the statement of Corollary 2.10 as the *definition* of a Fraïssé limit, the existence proof would look more like those of Facts 1.4 and 1.5 of [BU07] (which could be accordingly shortened using the formalism introduced here).

Definition 2.2. Let X, Y and Z denote metric spaces.

- (i) Given any $\psi: X \times Y \rightarrow [0, \infty)$ and $\varphi: Y \times Z \rightarrow [0, \infty)$ we define a composition $\varphi\psi: X \times Z \rightarrow [0, \infty)$ and an inverse $\psi^*: Y \times X \rightarrow [0, \infty)$ by

$$\varphi\psi(x, z) = \inf_{y \in Y} \psi(x, y) + \varphi(y, z), \quad \psi^*(y, x) = \psi(x, y).$$

We observe that $\psi \leq \psi d_X$ (respectively, $\psi \leq d_Y \psi$) if and only if ψ is 1-Lipschitz in the first (respectively, second) argument. We say that ψ is *Katětov* in the first (respectively, second) argument if $\psi \leq \psi d_X$ and $d_X \leq \psi^* \psi$ (respectively, if $\psi \leq d_Y \psi$ and $d_Y \leq \psi \psi^*$).

- (ii) We say that $\psi: X \times Y \rightarrow [0, \infty)$ is an *approximate isometry* from X to Y , and write $\psi: X \rightsquigarrow Y$, if it Katětov in both arguments (or *bi-Katětov*, see Uspenskij [Usp08, Section 6]). As an exception, we also allow $\psi = \infty$ identically as the *empty approximate isometry*, observing that this is a destructive element for composition.
- (iii) We identify an ordinary isometry $f: X \rightarrow Y$ with $\psi_f(x, y) = d(fx, y)$. More generally, we identify a partial isometry $f: X \dashrightarrow Y$ with $\psi_{f'} \psi_i^*$ where $f': \text{dom } f \rightarrow Y$ is f viewed as a total map on its domain, and $i: \text{dom } f \rightarrow X$ is the inclusion. In particular, id_X is identified with d_X , the neutral element for composition.
- (iv) Let $\psi: X \rightsquigarrow X$ and $\theta: X \rightsquigarrow Y$ be approximate isometries. Then we define, as for ordinary maps, $\psi^\theta = \theta \psi \theta^*: Y \rightsquigarrow Y$.

- (v) Let $\psi: X \rightsquigarrow Y$ be an approximate isometry and $i: X \subseteq X', j: Y \subseteq Y'$ isometric inclusions, or embeddings. Then $j\psi i^*: X' \rightsquigarrow Y'$ is called the *trivial extension* of ψ to $X' \rightsquigarrow Y'$ (compare with the analogous notion for ordinary partial maps).
- (vi) If $\psi, \varphi: X \rightsquigarrow Y$ are approximate isometries, we say that ψ *approximates* φ , or that φ *refines* ψ , if $\psi \geq \varphi$. We write $\psi > \varphi$ if (and only if) there finite $X_0 \subseteq X, Y_0 \subseteq Y$, and $\chi: X_0 \rightsquigarrow Y_0$ such that $\psi \geq \chi \geq \varphi + \varepsilon$ for some $\varepsilon > 0$ (which is equivalent, by finiteness of X_0 and Y_0 , to $\chi > \varphi$ point by point). We then say that ψ *strictly approximates* φ , which *strictly refines* ψ . Notice that the empty approximate isometry strictly approximates all approximate isometries, including itself.

We observe that

- (i) Let $\psi: X \times Y \rightarrow [0, \infty)$ be given and let $Z = X \amalg Y$, with $d(x, y) = d(y, x) = \psi(x, y)$. Then ψ is bi-Katětov if and only if d is a pseudo-distance on Z .
- (ii) $(\psi^*)^* = \psi, (\varphi\psi)^* = \psi^*\varphi^*$.
- (iii) Inversion is compatible with the identification of partial isometries with approximate ones. Similarly for composition $\psi_g\psi_f = \psi_{gf}$ when $\text{dom } g \supseteq \text{img } f$ or $\text{dom } g \subseteq \text{img } f$.
- (iv) If $\psi > \varphi$ then $\psi\rho > \varphi\rho$ for all ρ (some argument is actually required here, but is intentionally left to the reader).
- (v) Assume that $\psi: X \rightsquigarrow Y$ is an approximate isometry, with both $X, Y \subseteq Z$, and let us identify ψ with its trivial extension to $\psi: Z \rightsquigarrow Z$. Then ψ either (strictly) approximates all of id_X, id_Y and id_Z (all viewed as partial isometries $Z \rightsquigarrow Z$) or none. We shall therefore allow ourselves to say in such situations that “ ψ approximates id ” without further precision. For indeed, if $\psi > \text{id}_X$ then in fact $\psi > \text{id}_X \geq \text{id}_Z$, and if $\psi > \text{id}_Z$ then $\psi = \psi \text{id}_X > \text{id}_Z \text{id}_X = \text{id}_X$, and similarly with Y in place of X .

Definition 2.3. Let \mathcal{K} be a class of finite structures, $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$. By an *approximate (\mathcal{K} -)isomorphism* between structures $\psi: \mathfrak{A} \rightsquigarrow \mathfrak{B}$ we mean an approximate isometry such that for all $\varepsilon > 0$ there are $\mathfrak{C} \in \mathcal{K}$ and embeddings $i: \mathfrak{A} \rightarrow \mathfrak{C}, j: \mathfrak{B} \rightarrow \mathfrak{C}$ such that $j\psi i^* + \varepsilon > \text{id}$, or equivalently $\psi + \varepsilon > j^*i$. If the same can be done for $\varepsilon = 0$ then ψ is a *strictly approximate isomorphism*. The family of strictly approximate isomorphisms from \mathfrak{A} to \mathfrak{B} will be denoted $\text{Str}_{\mathcal{K}}(\mathfrak{A}, \mathfrak{B})$, with \mathcal{K} usually dropped.

If $\psi \in \text{Str}(\mathfrak{A}, \mathfrak{B})$ then there exists $\delta > 0$ such that $\psi - \delta \in \text{Str}(\mathfrak{A}, \mathfrak{B})$, and we define $\Delta(\psi)$ to be the supremum of all such δ .

It is clear from the definition that if ψ is an approximate isomorphism then so is ψ^* . If \mathcal{K} is closed under isomorphism then this is equivalent to: $\psi + \varepsilon$ approximates an embedding of \mathfrak{A} in some $\mathfrak{C} \supseteq \mathfrak{B}$.

Lemma 2.4. *The following are equivalent for a class \mathcal{K} :*

- (i) *The class \mathcal{K} has NAP.*
- (ii) *The composition of any two strictly approximate isomorphisms in \mathcal{K} is one as well.*
- (iii) *The composition of any two approximate isomorphisms in \mathcal{K} is one as well.*
- (iv) *Every partial isomorphism, in the classical sense, between members of \mathcal{K} , is an approximate isomorphism.*

Proof. (i) \implies (ii). Easy.

(ii) \implies (iii). Since, first, ψ is an approximate isomorphism id and only if $\varphi + \varepsilon$ is a strictly approximate isomorphism for all $\varepsilon > 0$, and second, $(\varphi + \varepsilon)(\psi + \varepsilon) = \varphi\psi + 2\varepsilon$.

(iii) \implies (iv). Since an embedding is an approximate isomorphism.

(iv) \implies (i). Immediate. ■_{2.4}

Definition 2.5. Let \mathcal{K} be a Fraïssé class. By a *limit* of \mathcal{K} we mean a separable structure \mathfrak{M} such that

- (i) $\text{Age}(\mathfrak{M}) \subseteq \mathcal{K}$.

- (ii) For every $\mathfrak{B} \in \mathcal{K}$, strictly approximate isomorphism $\psi: \mathfrak{M} \rightsquigarrow \mathfrak{B}$ (i.e., the trivial extension of some strictly approximate isomorphism $\psi: \mathfrak{A} \rightsquigarrow \mathfrak{B}$, where $\mathfrak{A} \subseteq \mathfrak{M}$ is finite) and $\varepsilon > 0$ there exists a strictly approximate embedding $\varphi: \mathfrak{B} \rightsquigarrow \mathfrak{M}$ such that $\varphi^* \varphi < \text{id}_B + \varepsilon$ and $\varphi \psi > \text{id}$.

Lemma 2.6. *Let \mathcal{K} be a Fraïssé class, and \mathfrak{M} a structure with $\text{Age}(\mathfrak{M}) \subseteq \mathcal{K}$. Let $M_0 \subseteq M$ be dense and countable, and for each n let $\mathcal{K}_{n,0} \subseteq \mathcal{K}_n$ be $d^{\mathcal{K}}$ -dense. Assume furthermore that the second property of Definition 2.5 holds whenever $A \subseteq M_0$ and $\mathfrak{B} \in \mathcal{K}_{n,0}$ for some n . Then \mathfrak{M} is a limit of \mathcal{K} .*

Proof. Let $\mathfrak{A} \subseteq \mathfrak{M}$ be finite, $\mathfrak{B} \in \mathcal{K}$ (say $\mathfrak{B} \in \mathcal{K}_n$), $\psi: \mathfrak{A} \rightsquigarrow \mathfrak{B}$ a strictly approximate isomorphism and $\varepsilon > 0$. Let $\delta = \frac{1}{3} \min \varepsilon, \Delta(\psi)$, and choose $A' \subseteq M_0$ of the same length as A with $d(A, A') < \delta$, as well as $\mathfrak{B}' \in \mathcal{K}_{n,0}$ such that $d^{\mathcal{K}}(\mathfrak{B}, \mathfrak{B}') < \delta$. Let $\chi = [A \mapsto A'] + \delta \in \text{Str}(\mathfrak{A}, \mathfrak{A}')$, $\rho = [B \mapsto B'] + \delta \in \text{Str}(\mathfrak{B}, \mathfrak{B}')$ and $\psi' = \rho \psi \chi^* - 2\delta \in \text{Str}(\mathfrak{A}', \mathfrak{B}')$. By assumption there exists $\varphi': \mathfrak{B}' \rightsquigarrow \mathfrak{M}$ such that $\varphi'^* \varphi' < \text{id}_{B'} + \varepsilon - 2\delta$ and $\varphi' \psi' > \text{id}$. Let $\varphi = \varphi' \rho: \mathfrak{B} \rightsquigarrow \mathfrak{M}$. Then $\varphi^* \varphi > \rho^* \rho + \varepsilon - 2\delta > \text{id}_B + \varepsilon$ and $\varphi \psi > \varphi' \rho \psi \chi^* \chi - 2\delta = \varphi' \psi' \chi > \chi > \text{id}$, as desired. $\blacksquare_{2.6}$

Lemma 2.7. *Let \mathcal{K} be a Fraïssé class and $\mathfrak{A} \in \mathcal{K}$. Then \mathcal{K} admits a limit in which \mathfrak{A} embeds.*

Proof. We construct an increasing chain of $\mathfrak{A}_n \in \mathcal{K}$, starting with $\mathfrak{A}_0 = \mathfrak{A}$. For each n we fix a countable $d^{\mathcal{K}}$ -dense subset of \mathcal{K}_n , call it $\mathcal{K}_{n,0}$. For each \mathfrak{A}_n and each $\mathfrak{B} \in \mathcal{K}_{n,0}$ we choose a countable dense family of strictly approximate isomorphisms between \mathfrak{A}_n and \mathfrak{B} . We then make sure that for each of the countably many approximate isomorphisms $\psi: \mathfrak{A}_n \rightsquigarrow \mathfrak{B}$ chosen in this fashion there is some m and an embedding $\varphi: \mathfrak{B} \rightarrow \mathfrak{A}_m$ such that $\varphi \psi$ approximates id . We let $\mathfrak{M}_0 = \bigcup \mathfrak{A}_n$ be the (incomplete) set-theoretic union, $\mathfrak{M} = \widehat{\mathfrak{M}_0}$ its completion. By Lemma 2.6, \mathfrak{M} is a limit. $\blacksquare_{2.7}$

Lemma 2.8. *Let \mathcal{K} be a Fraïssé class, let \mathfrak{M} and \mathfrak{N} two limits of \mathcal{K} and let $\psi: \mathfrak{M} \rightsquigarrow \mathfrak{N}$ be a finite approximate isomorphism (i.e., there are finite $\mathfrak{A} \subseteq \mathfrak{M}$ and $\mathfrak{B} \subseteq \mathfrak{N}$ such that $\psi: \mathfrak{A} \rightsquigarrow \mathfrak{B}$ is an approximate embedding) then ψ approximates an isomorphism $\theta: \mathfrak{M} \rightarrow \mathfrak{N}$.*

Proof. Let $\{a_n\}$ and $\{b_n\}$ enumerate dense subsets of \mathfrak{M} and \mathfrak{N} , respectively. We shall construct an increasing sequence of finite approximate isomorphisms $\theta_n: \mathfrak{A}_n \rightsquigarrow \mathfrak{B}_n$, $\mathfrak{A}_n \subseteq \mathfrak{M}$, $\mathfrak{B}_n \subseteq \mathfrak{N}$, such that $\{a_i\}_{i < n} \subseteq A_n$, $\{b_i\}_{i < n} \subseteq B_n$, and $\theta_n^* \theta_n < \text{id}_{A_n} + 2^{-n}$ for odd n , $\theta_n \theta_n^* < \text{id}_{B_n} + 2^{-n}$ for even $n > 0$.

We start with $\theta_0 = \psi$, $\mathfrak{A}_0 = \mathfrak{A}$, $\mathfrak{B}_0 = \mathfrak{B}$. Given θ_n , for n even, we let $A_{n+1} = A_n \cup \{a_n\}$. Choose $0 < \delta < \min 2^{-n-1}, \Delta(\theta_n)$, and find $\theta_{n+1}: \mathfrak{A}_{n+1} \rightsquigarrow \mathfrak{B}_{n+1} \subseteq \mathfrak{N}$, with \mathfrak{B}_{n+1} finite, such that $\theta_{n+1} \theta_n^* - \delta > \text{id}_N$ and $\theta_{n+1}^* \theta_{n+1} < \text{id}_{A_{n+1}} + \delta$. We may increase \mathfrak{B}_{n+1} so that $B_n \cup \{b_n\} \subseteq B_{n+1}$. Then $\theta_{n+1} < \theta_n \theta_n^* \theta_{n+1} - \delta < \theta_n$, and the construction may proceed. The odd case is treated similarly.

Then $\theta = \lim \theta_n$ is the desired isomorphism. $\blacksquare_{2.8}$

Theorem 2.9. *Let \mathcal{K} be a class of finite structures. Then \mathcal{K} is a Fraïssé class if and only if it is the age of an approximately qf-homogeneous, separable structure \mathfrak{M} . Such a structure \mathfrak{M} is necessarily a limit of \mathcal{K} , and thus unique up to isomorphism.*

Proof. If \mathfrak{M} is an approximately qf-homogeneous separable structure then its age is clearly a Fraïssé class. Conversely, if \mathcal{K} is a Fraïssé class then by Lemma 2.7 it has a limit \mathfrak{M} , which is unique and approximately qf-homogeneous by Lemma 2.8. By Lemma 2.7 again and uniqueness, $\text{Age}(\mathfrak{M}) = \mathcal{K}$. $\blacksquare_{2.9}$

Corollary 2.10. *Let \mathcal{K} be a Fraïssé class, $\mathfrak{M} = \lim \mathcal{K}$. Then for every $\mathfrak{B} \in \mathcal{K}$ and strictly approximate isomorphism $\psi: \mathfrak{M} \rightsquigarrow \mathfrak{B}$, there exists an embedding $\varphi: \mathfrak{B} \rightarrow \mathfrak{M}$ such that $\varphi \psi > \text{id}$.*

Proof. Since $\mathfrak{B} \in \mathcal{K} = \text{Age}(\mathfrak{M})$, we can embed $\varphi': \mathfrak{B} \rightarrow \mathfrak{M}$. By homogeneity, there exists an isomorphism $h < \varphi' \psi$. Then $\varphi = h^{-1} \varphi'$ will do. $\blacksquare_{2.10}$

We then denote the limit of \mathcal{K} by $\lim \mathcal{K}$. We also remark that if \mathfrak{M} is an arbitrary separable approximately homogeneous structure then we can always name a dense family of realised types in the language.

This has no effect on the cardinality of the language, on $\text{Aut}(\mathfrak{M})$, or on the collection of types realised in \mathfrak{M} . It has, on the other hand, the effect of rendering \mathfrak{M} approximately qf-homogeneous and a prime model of its theory. Thus, for the purposes of the present paper, one may always assume, if convenient, that $\lim \mathcal{K}$ is a prime model.

3. GENERIC AUTOMORPHISMS IN METRIC HOMOGENEOUS STRUCTURES

We now fix a Fraïssé class \mathcal{K} and let $\mathfrak{M} = \lim \mathcal{K}$, $G = \text{Aut}(\mathfrak{M})$.

Definition 3.1. For $m \in \mathbf{N}$, define $\mathcal{K}_{a,m}$ to consist of all $(\mathfrak{A}, \bar{\psi})$, where $\mathfrak{A} \in \mathcal{K}$ and $\bar{\psi}$ is an m -tuple of strictly approximate isomorphisms $\psi_i: \mathfrak{A} \rightsquigarrow \mathfrak{A}$. We write \mathcal{K}_a for $\mathcal{K}_{a,1}$.

An embedding $(\mathfrak{A}, \bar{\psi}) \subseteq (\mathfrak{B}, \bar{\varphi}) \in \mathcal{K}_{a,m}$ is an embedding $\mathfrak{A} \subseteq \mathfrak{B}$ such that $\bar{\varphi} \leq \bar{\psi}$. When $\mathfrak{B} \in \mathcal{K}$, or possibly $\mathfrak{B} = \mathfrak{M}$, and \bar{f} are actual automorphisms of \mathfrak{B} , we shall rather use the notation $(\mathfrak{A}, \bar{\psi}) \subset (\mathfrak{B}, \bar{f})$ to say that $\mathfrak{A} \subseteq \mathfrak{B}$ and $\bar{f} < \bar{\psi}$.

Given $(\mathfrak{A}, \psi) \in \mathcal{K}_a$, $\mathfrak{A} \subseteq \mathfrak{M}$, we define $[\psi] = \{g \in G: g < \psi\}$, observing that this is a non empty open set, and that the collection of sets of this form is a basis of open sets for G .

The following generalises Kechris and Rosendal [KR07, Theorem 2.1].

Theorem 3.2. *Let \mathcal{K} be a Fraïssé class and $\mathfrak{M} = \lim \mathcal{K}$, $G = \text{Aut}(\mathfrak{M})$. Then the following are equivalent:*

- (i) *The action of G on G^m by conjugation admits a dense orbit.*
- (ii) *The class $\mathcal{K}_{a,m}$ has JEP.*

Proof. For one direction, let $\bar{f} \in G^m$ have a dense orbit. Let $(\mathfrak{A}, \bar{\psi}), (\mathfrak{B}, \bar{\varphi}) \in \mathcal{K}_{a,m}$. We may assume that $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{M}$, in which case $\bar{f} \in [\bar{\psi}]^g \cap [\bar{\varphi}]^h$ for some $g, h \in G$. Then there is $(\mathfrak{C}, \bar{\rho}) \in \mathcal{K}_{a,m}$ such that $\mathfrak{A} \cup \mathfrak{B} \subseteq \mathfrak{C} \subseteq \mathfrak{M}$ and $\bar{f} \in [\bar{\rho}] \subseteq [\bar{\psi}]^g \cap [\bar{\varphi}]^h$, so both $(\mathfrak{A}, \bar{\psi})$ and $(\mathfrak{B}, \bar{\varphi})$ embed in $(\mathfrak{C}, \bar{\rho})$.

For the other direction, let us first show that if $(\mathfrak{A}, \bar{\psi}), (\mathfrak{B}, \bar{\varphi}) \in \mathcal{K}_{a,m}$ with $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{M}$ then $[\bar{\psi}]^g \cap [\bar{\varphi}] \neq \emptyset$ for some $g \in G$. Indeed, let us embed $(\mathfrak{A}, \bar{\psi}), (\mathfrak{B}, \bar{\varphi})$ in some $(\mathfrak{C}, \bar{\rho}) \in \mathcal{K}_{a,m}$, and we may in fact assume that $\bar{\rho} + 2\varepsilon < \min \bar{\psi}, \bar{\varphi}$. Let \mathfrak{A}' and \mathfrak{B}' denote the images of \mathfrak{A} and \mathfrak{B} in \mathfrak{C} , respectively, and let us fix some embedding $\mathfrak{C} \subseteq \mathfrak{M}$. Then there are $g, h \in G$ such that $d(gA, A'), d(hB, B') < \varepsilon$, in which case $[\bar{\psi}]^g \cap [\bar{\varphi}]^h \supseteq [\bar{\rho}] \neq \emptyset$, which is enough. It follows that if $U \subseteq G^m$ is any non empty open set then U^G is (open and) dense, and taking the intersection over a countable basis we see that the set of $\bar{f} \in G^m$ of dense orbit is co-meagre. ■_{3.2}

Now let $\mathfrak{A} \in \mathcal{K}$ and $\psi, \varphi: \mathfrak{A} \rightsquigarrow \mathfrak{A}$ two strictly approximate isomorphisms which we wish to compare, in the sense of the uniform distance on G . For this to make any sense we need to embed $\mathfrak{A} \subseteq \mathfrak{M}$ and compare $[\psi]$ and $[\varphi]$ as subsets of G , making sure that the result does not depend on the embedding.

Definition 3.3. Let $\mathfrak{A} \in \mathcal{K}$, let ψ and φ be two strictly approximately isomorphisms on \mathfrak{A} and let $\varepsilon > 0$.

- (i) We say that $\psi <_\varepsilon \varphi$ if there exists an embedding $\mathfrak{A} \subseteq \mathfrak{M}$ such that $[\psi] \subseteq ([\varphi - \delta])_\varepsilon$ as subsets of G for some $\delta > 0$.
- (ii) We say that $\psi \sim_\varepsilon \varphi$ if there exists an embedding $\mathfrak{A} \subseteq \mathfrak{M}$ such that $[\psi] \cap ([\varphi])_\varepsilon \neq \emptyset$.

Lemma 3.4. (i) *The conditions of Definition 3.3 do not depend on the choice of embedding $\mathfrak{A} \subseteq \mathfrak{M}$, i.e., if they hold for one such embedding they hold for all.*

- (ii) *The relation \sim_ε is symmetric.*
- (iii) *If $\psi <_\varepsilon \varphi$ then $\psi \sim_\varepsilon \varphi$.*
- (iv) *If $\psi \sim_\varepsilon \varphi$ then there is $(\mathfrak{A}, \psi) \subseteq (\mathfrak{B}, \rho)$ such that $\rho <_\varepsilon \varphi$.*

Proof. Let $i, j: \mathfrak{A} \hookrightarrow \mathfrak{M}$ be two embeddings. Assume first that $[\psi^i] \subseteq ([\varphi^i - \delta])_\varepsilon$, and let $g \in [\psi^j]$. Then for some very small $\eta > 0$ we have $g \in [\psi^j - 2\eta]$, and we may assume that $\eta < \delta/3$. Let $f \in G$ be such that $ij^* + \eta > f$, or equivalently, $\text{id}_{iA} + \eta > fji^*$. Then $g^f < \psi^{fj} - 2\eta = \psi^{fji^*i} - 2\eta < \psi^i$, i.e.,

$g^f \in [\psi^i] \subseteq ([\varphi^i - \delta])_\varepsilon$. Let $h \in B(1, \varepsilon)$ be such that $g^f h \in [\varphi^i - \delta]$, and by the same reasoning as above we conclude that $gh^{f^*} \in [\varphi^j - \delta/3]$, whereby $g \in ([\varphi^j - \delta/3])_\varepsilon$.

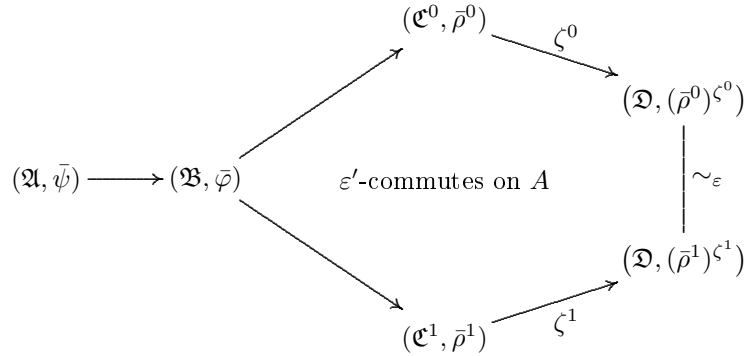
For \sim_ε , assume that $g \in [\psi^i] \cap ([\varphi^i])_\varepsilon$, so $gh \in [\varphi^i]$ for some $h \in B(1, \varepsilon)$. Then for small enough $\eta > 0$ we have $g \in [\psi^i - 2\eta]$ and $gh \in [\varphi^i - 2\eta]$, and we proceed as earlier.

Clearly \sim_ε is symmetric and weaker than $<_\varepsilon$.

Assume now that $\psi \sim_\varepsilon \varphi$, and let us fix some embedding $\mathfrak{A} \subseteq \mathfrak{M}$. Then there is $\delta > 0$ such that $U = [\psi] \cap ([\varphi - \delta])_\varepsilon \neq \emptyset$. As observed in [Ben], the set $([\varphi - \delta])_\varepsilon$ is open, so U is open. Therefore there is a strictly approximate isomorphism $\rho < \psi$ such that $[\rho] \subseteq U$, as desired. $\blacksquare_{3.4}$

We can now define an analogue of the existential isolation of Section 1. We shall in fact introduce two versions, one using embeddings, which is probably more “tangible”, and the other using approximate maps, with which it will be easier to work later on.

Definition 3.5. Let $(\mathfrak{B}, \bar{\varphi}) \in \mathcal{K}_{a,m}$, $A \subseteq B$, and $\varepsilon > \varepsilon' > 0$. Assume that whenever $(C^i, \bar{\rho}^i) \in \mathcal{K}_{a,m}$, $i < 2$, with embeddings $(\mathfrak{B}, \bar{\varphi}) \subseteq (\mathfrak{C}^i, \bar{\rho}^i)$, there exists $\mathfrak{D} \in \mathcal{K}$ and embeddings $\zeta^i: \mathfrak{C}^i \rightarrow \mathfrak{D}$ such that $d(\zeta^0 A, \zeta^1 A) < \varepsilon'$ and $(\bar{\rho}^0)^{\zeta^0} \sim_\varepsilon (\bar{\rho}^1)^{\zeta^1}$.



Then we say that $(\mathfrak{B}, \bar{\varphi})$ ε -isolates an existential type for A , or that it ε -isolates A for short.

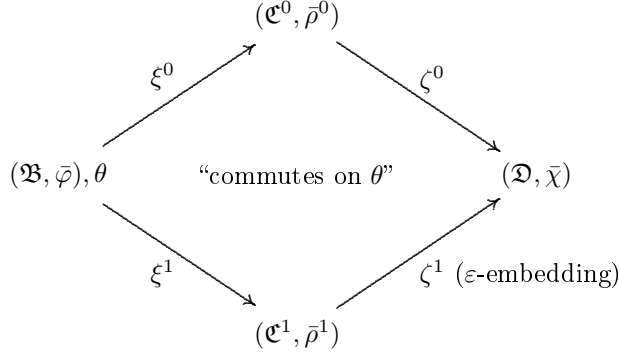
We say that $\mathcal{K}_{a,m}$ is *existentially atomic* if for every $(\mathfrak{A}, \bar{\psi}) \in \mathcal{K}_{a,m}$ and every $\varepsilon > 0$ there is $(\mathfrak{B}, \bar{\varphi}) \subseteq (\mathfrak{A}, \bar{\psi}) \in \mathcal{K}_{a,m}$ which ε -isolates A .

Definition 3.6. A *strictly approximate embedding* $\theta: (\mathfrak{A}, \bar{\psi}) \rightsquigarrow (\mathfrak{B}, \bar{\varphi})$ is a strictly approximate isomorphism $\theta: \mathfrak{A} \rightsquigarrow \mathfrak{B}$ which satisfies in addition $\bar{\psi} > \bar{\varphi}^{\theta^*}$.

A *strictly approximate ε -embedding* $\theta: (\mathfrak{A}, \bar{\psi}) \rightsquigarrow_\varepsilon (\mathfrak{B}, \bar{\varphi})$ is a strictly approximate embedding $\theta: (\mathfrak{A}, \bar{\psi}) \rightsquigarrow (\mathfrak{B}, \bar{\varphi}')$ for some $\bar{\varphi}' >_\varepsilon \bar{\varphi}$.

Definition 3.7. Let $(\mathfrak{B}, \bar{\varphi}, \theta) \in \mathcal{K}_{a,m+1}$, where $\theta > \text{id}$ and let $\varepsilon > 0$. Assume that whenever $(C^i, \bar{\rho}^i) \in \mathcal{K}_{a,m}$, $i < 2$, with strictly approximate embeddings and $\xi^i: (\mathfrak{B}, \bar{\varphi}) \rightsquigarrow (\mathfrak{C}^i, \bar{\psi}^i)$, there exists $(\mathfrak{D}, \bar{\chi}) \in \mathcal{K}_{a,m}$ and strictly approximate (ε) -embeddings $\zeta^0: (\mathfrak{C}^0, \bar{\rho}^0) \rightsquigarrow (\mathfrak{D}, \bar{\chi})$ and $\zeta^1: (\mathfrak{C}^1, \bar{\rho}^1) \rightsquigarrow_\varepsilon (\mathfrak{D}, \bar{\chi})$ such that

$$\theta > (\zeta^1 \xi^1)^* \zeta^0 \xi^0.$$



Then we say that $(\mathfrak{B}, \bar{\varphi})$ ε -isolates an existential type for θ , or that it ε -isolates θ for short.

Lemma 3.8. *Let $(\mathfrak{B}, \bar{\varphi}) \in \mathcal{K}_{a,m}$, $A \subseteq B$ and $\varepsilon > 0$. If $(\mathfrak{B}, \bar{\varphi})$ ε -isolates $\text{id}_A + \varepsilon$ (respectively, A) then there exists an embedding $(\mathfrak{B}, \bar{\varphi}) \subseteq (\mathfrak{B}', \bar{\varphi}') \in \mathcal{K}_{a,m}$ such that $(\mathfrak{B}', \bar{\varphi}')$ ε -isolates A (respectively, $\text{id}_A + \varepsilon$).*

Moreover, Definition 3.7 does not change if we restrict \mathfrak{C}^i and \mathfrak{D} to some $d^{\mathcal{K}}$ -dense sub-class of \mathcal{K} .

Proof. For the first direction, assume that $(\mathfrak{B}, \bar{\varphi})$ ε -isolates $\text{id}_A + \varepsilon$. Let $\delta = \Delta(\bar{\varphi})/4 = \min \Delta(\varphi_i)/4 > 0$, and let us show that $(\mathfrak{B}, \bar{\varphi} - 3\delta)$ ε -isolates A . Given any two embeddings $\xi^i: (\mathfrak{B}, \bar{\varphi} - 3\delta) \subseteq (\mathfrak{C}^i, \bar{\rho}^i)$, let $\tilde{\xi}^i = \xi^i + \delta$, so $\tilde{\xi}^i: (\mathfrak{B}, \bar{\varphi}) \rightsquigarrow (\mathfrak{C}^i, \bar{\rho}^i)$ are strictly approximate embeddings. Let $(\mathfrak{D}, \bar{\chi})$ and $\tilde{\zeta}^i: \mathfrak{C}^i \rightsquigarrow \mathfrak{D}$ be as in the conclusion of Definition 3.3. Then possibly increasing \mathfrak{D} , there are embeddings $\tilde{\zeta}^i > \zeta^i: \mathfrak{C}^i \rightarrow \mathfrak{D}$. On the one hand, $\bar{\rho}^0 > \bar{\chi}^{(\tilde{\zeta}^0)^*} > \bar{\chi}^{(\zeta^0)^*}$ implies $(\bar{\rho}^0)^{\zeta^0} > \bar{\chi}$, and similarly $(\bar{\rho}^1)^{\zeta^1} > \bar{\chi}' >_{\varepsilon} \bar{\chi}$, whence $(\bar{\rho}^0)^{\zeta^0} \sim_{\varepsilon} (\bar{\rho}^1)^{\zeta^1}$. On the other hand, we have $\text{id}_A + \varepsilon > (\tilde{\zeta}^1 \tilde{\xi}^1)^* \tilde{\zeta}^0 \tilde{\xi}^0 > (\zeta^1 \xi^1)^* \zeta^0 \xi^0 + 2\delta$, which gives the commutation up to $\varepsilon' = \varepsilon - 2\delta$ on A .

Conversely, assume that $(\mathfrak{B}, \bar{\varphi})$ ε -isolates A , as witnessed by $\varepsilon' < \varepsilon$, and let $\delta = (\varepsilon - \varepsilon')/4$. Then there exists an extension $(\mathfrak{B}, \bar{\varphi}) \subseteq (\mathfrak{B}', \bar{\varphi}')$ such that $\varphi_0^* \varphi_0' < \text{id}_A + \delta$, and let us show that $(\mathfrak{B}', \bar{\varphi}')$ ε -isolates $\text{id}_A + \varepsilon$. We may assume that $(\mathfrak{B}, \bar{\varphi}) = (\mathfrak{B}', \bar{\varphi}')$. We first make an observation. Assume that $\tilde{\xi}: (\mathfrak{B}, \bar{\varphi}) \rightsquigarrow (\mathfrak{C}, \bar{\rho})$ is a strictly approximate embedding. Then, possibly increasing \mathfrak{C} , there is an actual embedding $\tilde{\xi} > \xi: (\mathfrak{B}, \bar{\varphi}) \hookrightarrow (\mathfrak{C}, \bar{\rho})$. Since $\varphi_0 > \rho_0^{\tilde{\xi}^*}$,

$$\xi \text{id}_A + \delta > \xi \varphi_0^* \varphi_0 > \xi \tilde{\xi}^* \rho_0 \tilde{\xi} \tilde{\xi}^* \rho_0^* \tilde{\xi} > \text{id}_C \tilde{\xi} = \tilde{\xi}.$$

Now let $\tilde{\xi}^i: (\mathfrak{B}, \bar{\varphi}) \rightsquigarrow (\mathfrak{C}^i, \bar{\rho}^i)$ be strictly approximate embeddings, and possibly increasing \mathfrak{C}^i find actual embeddings $\xi^i < \tilde{\xi}^i < \xi^i \text{id}_A + \delta$ as above. Let $\zeta^i: \mathfrak{C}^i \rightarrow \mathfrak{D}$ be as in the conclusion of Definition 3.5. We may find $\bar{\chi}^i < (\bar{\rho}^i)^{\zeta^i}$ such that in addition $\bar{\chi}^0 <_{\varepsilon} \bar{\chi}^1$ and then there are strictly approximate embeddings $\tilde{\zeta}^i: (\mathfrak{C}^i, \bar{\rho}^i) \rightsquigarrow (\mathfrak{D}, \bar{\chi}^i)$ such that $\zeta^i < \tilde{\zeta}^i < \zeta^i + \delta$. Then $\tilde{\zeta}^1: (\mathfrak{C}^1, \bar{\rho}^1) \rightsquigarrow_{\varepsilon} (\mathfrak{D}, \bar{\chi}^0)$, and

$$(\tilde{\zeta}^1 \tilde{\xi}^1)^* \tilde{\zeta}^0 \tilde{\xi}^0 < (\zeta^1 \xi^1 \text{id}_A)^* \zeta^0 \xi^0 \text{id}_A + 4\delta < \text{id}_A + \varepsilon' + 4\delta = \text{id}_A + \varepsilon,$$

as desired.

The moreover part follows from the fact that the notion of a strictly approximate embedding in $\mathcal{K}_{a,m}$ allows for some small error. ■_{3.8}

Lemma 3.9. *The class $\mathcal{K}_{a,m}$ is existentially atomic if and only if for every $(\mathfrak{A}, \bar{\psi}) \in \mathcal{K}_{a,m}$ and $\varepsilon > 0$ there exists a strictly approximate embedding $\theta: (\mathfrak{A}, \bar{\psi}) \rightsquigarrow (\mathfrak{B}, \bar{\varphi}) \in \mathcal{K}_{a,m}$ such that $(\mathfrak{B}, \bar{\varphi})$ ε -isolates $\theta\theta^*$.*

Moreover, this remains true if we restrict \mathfrak{A} and \mathfrak{B} to some $d^{\mathcal{K}}$ -dense sub-class of \mathcal{K} .

Proof. By a similar reasoning. ■_{3.9}

Lemma 3.10. *The class $\mathcal{K}_{a,m}$ has JEP if and only if $(\emptyset, \bar{\emptyset})$ ε -isolates \emptyset for all $\varepsilon > 0$.*

Proof. One direction is immediate. For the other, we seek a joint embedding for $(\mathfrak{A}, \bar{\psi})$ and $(\mathfrak{B}, \bar{\varphi})$. Then $(\mathfrak{B}, \bar{\varphi} - \varepsilon) \in \mathcal{K}_{a,m}$ for some $\varepsilon > 0$, and embed $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{C}$ in a manner such that $\bar{\psi} \sim_\varepsilon (\bar{\varphi} - \varepsilon)$. If we embed $\mathfrak{C} \subseteq \mathfrak{M}$ then there exists $\bar{f} \in [\bar{\psi}] \cap ([\bar{\varphi} - \varepsilon])_\varepsilon \subseteq [\bar{\psi}] \cap [\bar{\varphi}]$, which is enough. $\blacksquare_{3.10}$

Lemma 3.11. *Assume that G^m contains a generic element, i.e., some \bar{f} such that $\overline{\bar{f}G^d}$ is co-meagre in G^m . Then $\mathcal{K}_{a,m}$ is existentially atomic.*

Proof. For convenience we prove for $m = 1$. We assume there is a generic automorphism f . Let $(\mathfrak{A}, \psi) \in \mathcal{K}_a$. We may assume that $\mathfrak{A} \subseteq \mathfrak{M}$ and $f \in [\psi]$. Let $G_{A, < s} = [\text{id}_A + s] \subseteq G$, and let $\Upsilon: G \rightarrow \mathbf{R}$ denote the (lower semi-continuous) function defined by

$$\{\Upsilon \leq s\} = \{g: \Upsilon(g) \leq s\} = \overline{(f)_s^{G_{A, < s}}}.$$

Clearly if gA and $g'A$ are very close then so are Υ^g and $\Upsilon^{g'}$, so $\Xi = \inf_G \Upsilon^g$ is equal to a countable infimum. By construction $\Xi|_{\overline{f}G^d} = 0$, so for every $s > 0$, the (closed) set $\{\Upsilon \leq s\}$ has non empty interior, or equivalently, for all s the open set $U_s = \{\Upsilon < s\}^\circ$ is non empty. In addition, by definition of Υ , the set $(f)_s^{G_{A, < s}}$ is dense in U_s , whereby $U_s \subseteq [\psi]$ for all s small enough, and in particular for some $s < \varepsilon/4$ which we now fix. For some small enough $\delta > 0$ we can find φ defined on some finite $\mathfrak{B} \subseteq \mathfrak{M}$ such that $\emptyset \neq [\varphi] \subseteq [\varphi + \delta] \subseteq U_s$.

Assume that $(\mathfrak{B}, \varphi) \subseteq (\mathfrak{C}^i, \rho^i)$. We embed \mathfrak{C}^i in \mathfrak{M} almost over \mathfrak{B} , in such a manner that still $\rho^i \leq \varphi + \delta$ and A is not moved by more than $\varepsilon/4$. Then there exist $h^i \in G_{A, < s}$ and $f^i \in (f)_s \cap [\rho^i]^{h^i}$, and letting $\mathfrak{D} = h^0 \mathfrak{C}^0 \cup h^1 \mathfrak{C}^1 \subseteq \mathfrak{M}$ we get the desired amalgamation. $\blacksquare_{3.11}$

Definition 3.12. Let Ξ consist of all $(\theta, \mathfrak{B}, \bar{\varphi})$, where $\theta: \mathfrak{M} \rightsquigarrow \mathfrak{B}$ and $(\mathfrak{B}, \bar{\varphi}) \in \mathcal{K}_{a,m}$, and $\Xi_\varepsilon \subseteq \Xi$ consist of those where $(\mathfrak{B}, \bar{\varphi})$ ε -isolates $\theta\theta^*$. An *embedding* $\zeta: (\theta, \mathfrak{B}, \bar{\varphi}) \rightsquigarrow (\xi, \mathfrak{C}, \bar{\rho})$ in Ξ is a strictly approximate embedding $\zeta: (\mathfrak{B}, \bar{\varphi}) \rightsquigarrow (\mathfrak{C}, \bar{\rho})$ such that in addition $\theta > \zeta^* \xi$, and similarly for a δ -embedding \rightsquigarrow_δ . Notice that embeddings compose, and that if $\mathcal{K}_{a,m}$ is existentially atomic then every member of Ξ embeds in a member of Ξ_ε , for any $\varepsilon > 0$. We say that $(\theta^i, \mathfrak{B}^i, \bar{\varphi}^i) \in \Xi$, $i < 2$, are ε -compatible if there exists $(\xi, \mathfrak{C}, \bar{\rho}) \in \Xi$ and (ε) -embeddings $(\theta^0, \mathfrak{B}^0, \bar{\varphi}^0) \rightsquigarrow (\xi, \mathfrak{C}, \bar{\rho})$ and $(\theta^1, \mathfrak{B}^1, \bar{\varphi}^1) \rightsquigarrow_\varepsilon (\xi, \mathfrak{C}, \bar{\rho})$, noticing that this is a symmetric relation. Say that $(\theta, \mathfrak{B}, \bar{\varphi}) \in \Xi$ (ε) -embeds in (\mathfrak{M}, f) if it (ε) -embeds in $(\text{id}, \mathfrak{M}, f)$ as above.

For $(\theta, \mathfrak{B}, \bar{\varphi}) \in \Xi$ and $r, \varepsilon > 0$, we define $U_{(\theta, \mathfrak{B}, \bar{\varphi}, r, \varepsilon)} \subseteq G^m$ to consist of all \bar{f} such that either $(\theta + \varepsilon, \mathfrak{B}, \bar{\varphi} + \varepsilon)$ $(r + 3\varepsilon)$ -embeds in (\mathfrak{M}, \bar{f}) , or some $(\xi, \mathfrak{C}, \bar{\rho}) \in \Xi_\varepsilon$ which is r -incompatible with $(\theta, \mathfrak{B}, \bar{\varphi})$ embeds in (\mathfrak{M}, \bar{f}) and in addition $\xi^* \xi < \theta^* \theta + 2\varepsilon$.

We define $X \subseteq G^m$ as the intersection of all $U_{(\theta, \mathfrak{B}, \bar{\varphi}, r, \varepsilon)}$ as defined above. If $\bar{f} \in X$ we say that (\mathfrak{M}, \bar{f}) is an *existentially atomic limit* of $\mathcal{K}_{a,m}$.

Lemma 3.13. *Let $(\theta, \mathfrak{B}, \bar{\varphi}) \in \Xi$ and $r, \varepsilon > 0$. Then*

- (i) *The set $U_{(\theta, \mathfrak{B}, \bar{\varphi}, r, \varepsilon)}$ is open, and if $\mathcal{K}_{a,m}$ is existentially atomic then it is also dense.*
- (ii) *Let $0 < \delta < \varepsilon$, $\theta \leq \theta' \leq \theta + \delta$, and $\bar{\varphi} \leq \bar{\varphi}' \leq \bar{\varphi} + \delta$ and $r + \delta \leq r' \leq r + 2\delta$. Then $U_{(\theta, \mathfrak{B}, \bar{\varphi}, r, \varepsilon)} \supseteq (U_{(\theta', \mathfrak{B}, \bar{\varphi}', r', \varepsilon - \delta)})_\delta$.*
- (iii) *Assume that $d^{\mathcal{K}}(\mathfrak{B}, \mathfrak{C}) < \varepsilon$, and let $\rho = [B \mapsto C] + \varepsilon: \mathfrak{B} \rightsquigarrow \mathfrak{C}$. Then $U_{(\theta, \mathfrak{B}, \bar{\varphi}, r, 5\varepsilon)} \supseteq U_{(\rho\theta, \mathfrak{C}, \bar{\varphi}^\rho, r, \varepsilon)}$.*

Proof. The first item is clear.

For the second, assume that $\bar{f} \in U_{(\theta', \mathfrak{B}, \bar{\varphi}', r', \varepsilon - \delta)} \cap (\bar{g})_\delta$. One possibility is that $(\theta' + \varepsilon - \delta, \mathfrak{B}, \bar{\varphi}' + \varepsilon - \delta)$ $(r' + 3\varepsilon - 3\delta)$ -embeds in (\mathfrak{M}, \bar{f}) , in which case $(\theta + \varepsilon, \mathfrak{B}, \bar{\varphi} + \varepsilon)$ $(r + 3\varepsilon)$ -embeds in (\mathfrak{M}, \bar{g}) (by the same approximate map). The other possibility is that there exists $(\xi, \mathfrak{C}, \bar{\rho}) \in \Xi_{\varepsilon - \delta}$ r' -incompatible with $(\theta', \mathfrak{B}, \bar{\varphi}')$, and *a fortiori* $(r + \delta)$ -incompatible with $(\theta, \mathfrak{B}, \bar{\varphi})$ which embeds in (\mathfrak{M}, \bar{f}) , such that in addition $\xi^* \xi < \theta^* \theta' + 2\varepsilon - 2\delta \leq \theta^* \theta + 2\varepsilon$. We may then assume that $(\mathfrak{C}, \bar{\rho}) \subset (\mathfrak{M}, \bar{f})$ and $\xi > \text{id}$.

Possibly increasing \mathfrak{C} , there are $\bar{\rho}'$ such that $(\xi, \mathfrak{C}, \bar{\rho}) \in \Xi_\varepsilon$, $(\mathfrak{C}, \bar{\rho}') \subset (\mathfrak{M}, \bar{g})$ and $\bar{\rho}' <_\delta \bar{\rho}$, so in particular $(\xi, \mathfrak{C}, \bar{\rho}')$ is r -incompatible with $(\theta, \mathfrak{B}, \bar{\varphi})$. Either way, $\bar{g} \in U_{(\theta, \mathfrak{B}, \bar{\varphi}, r, \varepsilon)}$, as desired.

For the third item, we have approximate embeddings $(\theta + 5\varepsilon, \mathfrak{B}, \bar{\varphi} + 5\varepsilon) \xrightarrow{\sim} (\rho\theta + \varepsilon, \mathfrak{C}, \bar{\varphi}^\rho + \varepsilon)$ and $(\rho\theta, \mathfrak{C}, \bar{\varphi}^\rho) \xrightarrow{\sim} (\theta, \mathfrak{B}, \bar{\varphi})$ (which are not strict, but this is still good enough for what follows). Then an $(r+3\varepsilon)$ -embedding of $(\rho\theta + \varepsilon, \mathfrak{C}, \bar{\varphi}^\rho + \varepsilon)$ in (\mathfrak{M}, \bar{f}) gives rise to an $(r+15\varepsilon)$ -embedding of $(\theta + 5\varepsilon, \mathfrak{B}, \bar{\varphi} + 5\varepsilon)$ there, and anything r -incompatible with $(\rho\theta, \mathfrak{C}, \bar{\varphi}^\rho)$ is r -incompatible with $(\theta, \mathfrak{B}, \bar{\varphi})$ as well. In addition $(\rho\theta)^*(\rho\theta) + \varepsilon < \theta^*\theta + 5\varepsilon$, yielding the desired inclusion. $\blacksquare_{3.13}$

Lemma 3.14. *Assume that $\mathcal{K}_{a,m}$ is existentially atomic. The set X is a dense, uniformly closed, G_δ set.*

Proof. That X is uniformly closed follows from Lemma 3.13(ii). By Lemma 3.13(iii), we may restrict to \mathfrak{B} in some fixed countable, $d^\mathcal{K}$ -dense family without chaining the intersection, and by Lemma 3.13(ii), we may restrict $\theta, \bar{\varphi}, r, \varepsilon$ to some countable family. Thus X is the intersection of countably many dense open sets, and is therefore itself a dense G_δ set. $\blacksquare_{3.14}$

Proposition 3.15. *Assume that $\mathcal{K}_{a,m}$ has JEP and is existentially atomic. Let $\bar{f} \in X$ as per Definition 3.12, and let $\bar{g} \in G^m$. Then the following are equivalent:*

- (i) $\bar{g} \in \overline{f^G}^d$.
- (ii) $\bar{g} \in X$.
- (iii) *The following two conditions hold:*
 - For every strictly approximate $\theta: \mathfrak{M} \rightsquigarrow \mathfrak{M}$ and $\varepsilon > 0$ there exists $(\mathfrak{B}, \bar{\varphi}) \subset (\mathfrak{M}, \bar{g})$ which in addition ε -isolates θ .
 - Assume that there are $\theta > \zeta: \mathfrak{M} \rightsquigarrow \mathfrak{B}$ such that $(\mathfrak{B}, \bar{g}^\zeta)$ ε -isolates $\theta\theta^*$. Then for every $\bar{\varphi} < \bar{g}^\zeta$ there exists $\chi: (\mathfrak{B}, \bar{\varphi}) \rightsquigarrow_{2\varepsilon} (\mathfrak{M}, \bar{g})$ such that in addition $\chi < \theta^*\theta^*$.

Proof. For (i) \implies (ii), we have $\overline{f^G}^d \subseteq X$, since X is invariant under conjugation and uniformly closed.

Let us prove (ii) \implies (iii). For the first point we may decrease ε as we wish. We may therefore assume that there is $\zeta: \mathfrak{M} \rightsquigarrow \mathfrak{M}$ such that $\zeta^*\zeta\zeta^*\zeta + 4\varepsilon < \theta$. Possibly decreasing ζ and ε even further, we can find $\mathfrak{B} \subseteq \mathfrak{M}$ and $\bar{\varphi}$ such that $(\zeta, \mathfrak{B}, \bar{\varphi}) \in \Xi$ and $(\zeta + \varepsilon, \mathfrak{B}, \bar{\varphi} + \varepsilon)$ does not 4ε -embed into (\mathfrak{M}, \bar{g}) . Since $\bar{g} \in U_{(\zeta, \mathfrak{B}, \bar{\varphi}, \varepsilon, \varepsilon)}$, there is $(\xi, \mathfrak{C}, \bar{\rho}) \in \Xi_\varepsilon$ which embeds in (\mathfrak{M}, \bar{g}) , such that $\xi^*\xi < \zeta^*\zeta + 2\varepsilon$. We may therefore assume that $(\mathfrak{C}, \bar{\rho}) \subset (\mathfrak{M}, \bar{g})$, and $\xi > \text{id}$. Now, $(\mathfrak{C}, \bar{\rho})$ ε -isolates $\xi\xi^*$, and a fortiori any of

$$\xi\xi^* < \xi^*\xi\xi^*\xi < \zeta^*\zeta\zeta^*\zeta + 4\varepsilon < \theta.$$

For the second point, let $0 < \delta < \Delta(\bar{\varphi}, \zeta)$. Assume that some $(\xi, \mathfrak{C}, \bar{\rho})$ embeds in (\mathfrak{M}, \bar{g}) , and we claim that in this case it is ε -compatible with $(\theta\theta^*\theta - \delta, \mathfrak{B}, \bar{\varphi} - \delta)$. For this purpose, we may, and shall, freely replace $(\xi, \mathfrak{C}, \bar{\rho})$ with anything in which it embeds. We may first assume that $(\mathfrak{B}, \bar{g}^\zeta), (\mathfrak{C}, \bar{\rho}) \subseteq \mathfrak{M}$ and $\zeta - \delta, \xi > \text{id}$, so in particular $\theta - \delta > \text{id}$, and we may further assume that $\mathfrak{B} \subseteq \mathfrak{C}$ and $\bar{\rho} < \bar{g}^\zeta$. Then by ε -isolation, we may further assume that there is $\sigma: (\mathfrak{B}, \bar{\varphi} - \delta) \rightsquigarrow_\varepsilon (\mathfrak{C}, \bar{\rho})$ such that $\sigma < \theta\theta^*$. Then $\sigma < (\theta^* - \delta)\theta\theta^* = \theta^*\theta\theta^* - \delta$, and replacing the other inclusions with close approximations thereof we see that $(\theta\theta^*\theta - \delta, \mathfrak{B}, \bar{\varphi} - \delta)$ and $(\xi, \mathfrak{C}, \bar{\rho})$ are indeed ε -compatible. Since $\bar{g} \in U_{(\theta\theta^*\theta, \mathfrak{B}, \bar{\varphi}, \varepsilon, \delta)}$, we conclude that $(\theta\theta^*\theta, \mathfrak{B}, \bar{\varphi})$ $(\varepsilon + 3\delta)$ -embeds in (\mathfrak{M}, \bar{g}) for arbitrarily small δ , and in particular that it 2ε -embeds there, as desired.

For (iii) \implies (i), given the previous implication, we assume that \bar{f} has all the properties of \bar{g} . We assume $m = 1$ for simplicity of notation. Given $\varepsilon > 0$, it will be enough to show that $g \in (f^G)_\varepsilon$. Let $\{a_n\}_{n \in \mathbb{N}}$ enumerate a dense subset of \mathfrak{M} and let $\delta_n = 4^{-n-1}\varepsilon$. We shall construct for each n finite $\mathfrak{A}_n, \mathfrak{B}_n, \mathfrak{C}_n, \mathfrak{D}_n \subseteq \mathfrak{M}$, strict approximate isomorphisms $\theta_n: \mathfrak{A}_n \rightsquigarrow \mathfrak{B}_n$ and $\zeta_n: \mathfrak{C}_n \rightsquigarrow \mathfrak{D}_n$, as well as $\delta_n > 0$, such that $\theta_n > \zeta_n \vee \theta_{n+1}$ XXXXXfor all n , and:

- (i) For all n , $\theta_n > \zeta_n$ and $\theta_{n+1}\theta_{n+1}^*\theta_{n+1}\theta_{n+1}^* < \theta_n\theta_n^*\theta_n\theta_n^*$.

- (ii) If n is even then $\{a_i\}_{i < n} \subseteq A_n$, $(\mathfrak{D}_n, f^{\zeta_n})$ δ_n -isolates $\theta_n \theta_n^*$, and $g \in ([f^{\zeta_n}]_{\varepsilon-3\delta_n})$.
- (iii) If n is odd then mirror image of the above.

For $n = 0$ we let all sets and approximate maps be empty, and the assumptions are satisfied by JEP.

Let n be even and θ_n, ζ_n be given.

- Let $B_{n+1} = D_n \cup a_{\leq n}$ and choose $\text{id} < \xi$ on B_{n+1} such that $\xi \xi^* \xi \xi^* \xi < \theta_n \theta_n^*$.
- Find $D_{n+1} \supseteq B_{n+1}$ and $\xi': \mathfrak{D}_{n+1} \rightsquigarrow \mathfrak{D}_{n+1}$ such that $\text{id} < \xi' < \xi$ and $(\mathfrak{D}_{n+1}, g^{\xi'})$ δ_{n+1} -isolates $\xi \xi^*$.
- Find a strictly approximate isomorphism $\varphi < f^{\zeta_n}$ on D_{n+1} such that $\varphi <_{\varepsilon-3\delta_n} g^{\xi'}$.
- There is $\chi: (\mathfrak{D}_{n+1}, \varphi) \rightsquigarrow_{2\delta_n} (\mathfrak{M}, f)$ such that in addition $\chi < \theta_n^* \theta_n \theta_n^*$.
- Let $h \in [\chi] \subseteq G$, so $f^{h^{-1}} \in ([\varphi]_{2\delta_n} \subseteq ([g^{\xi'}]_{\varepsilon-\delta_n}))$, and $f \in ([g^{h\xi'}]_{\varepsilon-3\delta_{n+1}})$.
- Let $A_{n+1} = h^{-1} B_{n+1}$, $C_{n+1} = h^{-1} D_{n+1}$, $\theta_{n+1} = (h\xi)^*$, $\zeta_{n+1} = (h\xi')^*$. In particular, $\zeta_{n+1} < \theta_{n+1}$, $f \in ([g^{\zeta_{n+1}}]_{\varepsilon-3\delta_{n+1}})$ and $(\mathfrak{C}_{n+1}, g^{\zeta_{n+1}})$ δ_{n+1} -isolates $\theta_{n+1}^* \theta_{n+1}$.
- All that is left to check is that $\theta_{n+1} \theta_{n+1}^* \theta_{n+1} \theta_{n+1}^* < \xi^* \xi \xi^* \xi \xi^* \chi^* < \theta_n \theta_n^* \theta_n \theta_n^*$.

The construction may therefore proceed, the opposite direction being treated similarly.

We may, in addition, make ξ at each stage as small as we wish, so the limit $h = \lim \theta_n \theta_n^* \theta_n \theta_n^*$ is an automorphism, and $g \in (f^h)_{\varepsilon}$, as desired. ■_{3.15}

We deduce a metric version of [KR07, Theorem 3.4]:

Theorem 3.16. *Let \mathcal{K} be a metric Fraïssé class, $\mathfrak{M} = \lim \mathcal{K}$, $G = \text{Aut}(\mathfrak{M})$. Then the following are equivalent:*

- (i) *The class $\mathcal{K}_{a,m}$ has JEP and is existentially atomic.*
- (ii) *The action of G on G^m by conjugation admits a co-meagre (dense G_{δ}) uniformly closed orbit.*
- (iii) *The class $\mathcal{K}_{a,m}$ has JEP and admits an existentially atomic limit.*

Proof. (i) \implies (ii). By Proposition 3.15 and Lemma 3.14.

(ii) \implies (i). By Lemma 3.11.

(i) \implies (iii). By Proposition 3.15.

(iii) \implies (i). By Proposition 3.15. ■_{3.16}

4. A FEW EXAMPLES OF GROUPS WITH AMPLE GENERICS

It was shown in [BBM] that the automorphism groups of ℓ^2 , of \mathbf{U}_1 (the Urysohn sphere) and of $([0, 1], \lambda)$ (the Lebesgue probability space) have topometric ample generics, i.e., that the second condition of Theorem 3.16 holds for all m , via approximation by discrete structures. We shall now prove the same by direct arguments.

4.1. Automorphisms of Hilbert spaces. In the case of $G = \text{Aut}(\ell^2)$, the uniform distance on G corresponds to uniform convergence on the unit ball of ℓ^2 , and not on the entire space. Here \mathcal{K} is the class of finite subsets of ℓ^2 together with the induced structure. Each member of \mathcal{K} generates a finite dimensional Euclidean space, and the two may, for most intents and purposes, be identified.

Lemma 4.1. *Let $(A, \bar{\psi}) \in \mathcal{K}_{a,m}$. Then there exists a finite dimensional Hilbert space B and $\bar{\varphi} \in \text{Aut}(B)^m$ such that $(A, \bar{\psi}) \subset (B, \bar{\varphi})$.*

Proof. Given $(A, \bar{\psi}) \in \mathcal{K}_{a,m}$, we may embed this first in $(B, \bar{\varphi}')$, where $\dim B < \infty$ and each φ'_j is an actual partial isomorphism whose domain is a subspace of B . We then extend φ'_j as we wish to $\varphi_j \in \text{Aut}(B)$. ■_{4.1}

Lemma 4.2. *For every n and $\varepsilon > 0$ there exists $\delta = \delta(n, \varepsilon) > 0$ such that for every two Hilbert spaces $B \subseteq C$, $\dim B = n$, linear isometry $\varphi: B \rightarrow C$ and $\rho \in \text{Aut}(C)$, if $\|\varphi - \rho \upharpoonright_B\| < \delta$ then there exists $\varphi' \in \text{Aut}(C)$ with $\|\varphi' - \rho\| < \varepsilon$.*

Proof. Let us restate the problem, replacing φ with $\rho^{-1}\varphi$. Then $\varphi: B \rightarrow C$ is an embedding satisfying $\|\varphi - \text{id}_B\| < \delta$, and we need to extend it to φ' such that $\|\varphi' - \text{id}_C\| < \varepsilon$. We now see that this is exactly what is proved in [BU07, Example 2.14]. $\blacksquare_{4.2}$

Lemma 4.3. *Let $\dim B < \infty$, $\bar{\varphi} \in \text{Aut}(B)^m$ and let $\varepsilon > 0$ be given. Then for some $\delta > 0$, $(B, \bar{\varphi} + \delta)$ ε -isolates B .*

Proof. Let $\delta = \delta(\dim B, \varepsilon/2) > 0$ as per Lemma 4.2. Let $(B \oplus C^i, \bar{\rho}^i) \in \mathcal{K}_{a,m}$ with $(B, \bar{\varphi} + \delta) \subset (B \oplus C^i, \bar{\rho}^i)$ be given for $i < 2$. By Lemma 4.1 we may assume that $\bar{\rho}^i > \bar{\rho}^i$ where $\bar{\rho}^i \in \text{Aut}(B \oplus C^i)^m$. For each $j < m$ we then have $\|\varphi_j - \rho_j^i|_B\| < \delta$. By Lemma 4.2 there exists $\theta_j^i \in \text{Aut}(C^i)$ such that $\|\varphi_j \oplus \theta_j^i - \rho_j^i\| < \varepsilon/2$.

Let $D = B \oplus C^0 \oplus C^1$. Then on D we have $\|\varphi_j \oplus \theta_j^0 \oplus \theta_j^1 - \rho_j^i \oplus \theta_j^{1-i}\| < \varepsilon/2$, where $\rho_j^1 \oplus \theta_j^0$ is understood “correctly”, whence $\|\rho_j^0 \oplus \theta_j^1 - \rho_j^1 \oplus \theta_j^0\| < \varepsilon$, concluding the proof. $\blacksquare_{4.3}$

Theorem 4.4. *The group $U(\ell^2)$ admits topometric ample generics.*

Proof. By the above and Theorem 3.16 (strictly speaking $\bar{\varphi} + \delta$ need not be finite since B is not, but there is always going to be a finite $\bar{\varphi}'$ such that $\bar{\varphi} + \delta > \bar{\varphi}' > \bar{\varphi}$, so this is not truly an obstacle). $\blacksquare_{4.4}$

4.2. Automorphisms of probability spaces. Rather than work with probability spaces, we work with the associated probability algebras (see Fremlin [Fre04], as well as [BU10, Example 3.4]). The (unique) separable atomless probability algebra is the Fraïssé limit of the class of finite probability algebras. Let \mathcal{U}_n denote the uniform probability algebra with n atoms, namely the power set of n equipped with the normalised counting measure. Notice that we have a natural embedding of $\mathcal{U}_n \subseteq \mathcal{U}_{n \times m} \cong \mathcal{U}_n \otimes \mathcal{U}_m$.

Lemma 4.5. *For every n and $(\mathcal{U}_n, \bar{\psi}) \in \mathcal{K}_{a,m}$ there is a finite k and automorphisms $\bar{\varphi} \in \text{Aut}(\mathcal{U}_{n \times k})^m$ such that $(\mathcal{U}_n, \bar{\psi}) \subset (\mathcal{U}_{n \times k}, \bar{\varphi})$.*

Proof. Let us first consider $m = 1$. We may embed $\mathcal{A} = \mathcal{U}_n$ in the separable atomless probability algebra \mathcal{B} and find $f \in \text{Aut}(\mathcal{B})$ such that $f < \psi$. It follows from Rokhlin’s Lemma that periodic automorphisms are uniformly dense, so we may assume that f is periodic. It follows that the algebra $\mathcal{C} = \langle \mathcal{A} \rangle_f = \langle f^\ell \mathcal{A} \rangle_{\ell \in \mathbb{Z}} \subseteq \mathcal{B}$ is finite, and again possibly changing f very little (so still $f < \psi$) we may further assume that the measures of the atoms of \mathcal{C} are rational. Then \mathcal{C} embeds over \mathcal{A} in some $\mathcal{D} = \mathcal{U}_{n \times k}$, and $f|_{\mathcal{C}}$ extends to \mathcal{D} , as desired.

For the general case, we treat each ψ_j separately, and then extend φ_j from $\mathcal{U}_{n \times k_j}$ to $\mathcal{U}_{n \times k_0 \times \dots \times k_{m-1}}$ as the identity on \mathcal{U}_{k_j} , for $j' \neq j$. $\blacksquare_{4.5}$

Lemma 4.6. *For all $n \in \mathbb{N}$ and $\varepsilon > 0$ be given there exists $\delta = \delta(n, \varepsilon) > 0$ such that for every embedding $\mathcal{B} = \mathcal{U}_n \subseteq \mathcal{C} = \mathcal{U}_{n \times k}$ and $\varphi \in \text{Aut}(\mathcal{B}) = \mathfrak{S}_n$, $\rho \in \text{Aut}(\mathcal{C}) = \mathfrak{S}_{n \times k}$, if $\|\varphi - \rho|_{\mathcal{B}}\| < \delta$ then there exists $\varphi' \in \text{Aut}(\mathcal{C})$ with $\|\varphi' - \rho\| < \varepsilon$.*

Proof. This is actually quite immediate. $\blacksquare_{4.6}$

Lemma 4.7. *Let $\mathcal{B} = \mathcal{U}_n$, $\bar{\varphi} \in \text{Aut}(\mathcal{B})^m$ and let $\varepsilon > 0$ be given. Then for some $\delta > 0$, $(\mathcal{B}, \bar{\varphi} + \delta)$ ε -isolates \mathcal{B} .*

Proof. Let $\delta = \delta(\dim B, \varepsilon/2) > 0$ as per Lemma 4.6. Consider first an embedding $(\mathcal{B}, \bar{\varphi} + \delta) \subset (\mathcal{C}, \bar{\rho}) \in \mathcal{K}_{a,m}$. One can always embed \mathcal{C} into something which is arbitrarily close to some $\mathcal{B} \otimes \mathcal{U}_k$, close enough that there exist strictly approximate isomorphisms $\bar{\rho}'$ on $\mathcal{B} \otimes \mathcal{U}_k$ such that $\bar{\rho}' > \bar{\rho}'$. This reduces our problem to embeddings of the form $(\mathcal{B}, \bar{\varphi} + \delta) \subset (\mathcal{B} \otimes \mathcal{U}_{k_i}, \bar{\rho}^i)$, which are treated as in the proof of Lemma 4.3. $\blacksquare_{4.7}$

Theorem 4.8. *The group $\text{Aut}([0, 1], \lambda)$ admits topometric ample generics.*

Proof. By the above, if $\mathcal{A} = \mathcal{U}_n$ then for every $(\mathcal{A}, \bar{\psi}) \in \mathcal{K}_{a,m}$ and $\varepsilon > 0$ we may embed $(\mathcal{A}, \bar{\psi}) \subset (\mathcal{B}, \bar{\varphi}) \in \mathcal{K}_{a,m}$ where $(\mathcal{B}, \bar{\varphi})$ ε -isolates \mathcal{B} , and in particular \mathcal{A} . The same holds for a finite \mathcal{A} in which all the atoms have rational measures, since it can be embedded in a \mathcal{U}_n . These being dense in \mathcal{K} , we conclude that $\mathcal{K}_{a,m}$ is approximately existentially atomic. $\blacksquare_{4.8}$

4.3. Isometries of the Urysohn space and sphere. The Urysohn space \mathcal{U} is the Fraïssé limit of the class of all finite metric spaces, while the Urysohn sphere \mathbf{U}_1 is the Fraïssé limit of finite metric spaces of diameter ≤ 1 . In what follows we shall only address the case of the Urysohn space. Considering spaces of diameter ≤ 1 and truncating everything below 1, the same arguments apply to the case of the Urysohn sphere.

Lemma 4.9. *Let $(A, \bar{\psi}) \in \mathcal{K}_{a,m}$. Then there exists a finite metric space B and $\bar{\varphi} \in \text{Iso}(B)^m$ such that $(A, \bar{\psi}) \subset (B, \bar{\varphi})$.*

Proof. Follows from Solecki [Sol05]. $\blacksquare_{4.9}$

Lemma 4.10. *Let $B \subseteq C^i$ be finite metric spaces for $i < 2$, and let D be the free metric amalgam of C^i over B , namely such that for $x \in C^0$ and $y \in C^1$ we have $d(x, y) = \min_{z \in B} d(x, z) + d(y, z)$. Let also $\varphi \in \text{Iso}(B)$, $\delta > 0$, and $(B, \varphi + \delta) \subset (C^i, \rho^i) \in \mathcal{K}_a$. Then there exist approximate isometries χ^i on D such that $(C^i, \rho^i) \subseteq (D, \chi^i)$ and (as functions on D^2) $\|\chi^0 - \chi^1\| < 2\delta$.*

Proof. First of all, we observe that if $x \in B$ then

$$\rho^i(x, y) + \delta > \rho^i(x, y) + \rho^i(x, \varphi x) \geq d(\varphi x, y),$$

and similarly, if $y \in B$ then $\rho^i(x, \varphi y) + \delta > d(x, y)$.

We claim that $\chi^i = \rho^i \wedge (\rho^{1-i} + 2\delta)$ will do. Indeed, all we have to show is that these are both approximate isometries, i.e., are bi-Katětov. Since each is 1-Lipschitz in each argument, as the minimum of two such functions, all we need to show is that $\chi^i(x, y) + \chi^i(x, y') \geq d(x, x')$ and similarly for the other argument. Since ρ^i and $\rho^i + 2\delta$ are all approximate isometries, we only need to check that

$$\rho^0(x, y) + \rho^1(x, y') + 2\delta \geq d(y, y')$$

(and similarly for the other argument). Without loss of generality we may assume that $x \in C^1$. Then

$$\begin{aligned} \rho^0(x, y) &= \min_{z \in C^0} \rho^0(z, y) + d(z, x) \\ &= \min_{z \in C^0, w \in B} \rho^0(z, y) + d(z, w) + d(w, x) \\ &= \min_{w \in B} \rho^0(w, y) + d(w, x) \\ &> \min_{w \in B} d(\varphi w, y) - \delta + \rho^1(x, \varphi w) - \delta \\ &\geq \rho^1(x, y) - 2\delta, \end{aligned}$$

and the desired inequality follows from ρ^1 being an approximate isometry. $\blacksquare_{4.10}$

Fact 4.11. *Let $A \subseteq \mathbf{U}$ be finite, and $f: A \rightarrow \mathbf{U}$ an isometry, such that $d(x, fx) \leq r$ for all $x \in A$. Then there exists $f' \in \text{Iso}(\mathbf{U})$ extending f such that $d(x, f'x) \leq r$ for all $x \in \mathbf{U}$.*

Proof. This is essentially [CV06, Lemma 3] (although it is stated for the rational Urysohn space, the same back-and-forth argument can be applied to a dense co unale subset of \mathbf{U}). $\blacksquare_{4.11}$

Lemma 4.12. *Let D be a finite metric space, and $\chi^i: D \rightsquigarrow D$ strictly approximate isometries of D , and let $\varepsilon > \|\chi^0 - \chi^1\|$. Then $\chi^0 \sim_\varepsilon \chi^1$.*

Proof. Let $\delta > 0$ be sufficiently small that $\chi^i - \delta$ are still approximate isometries, and let $\eta = \|\chi^0 - \chi^1\|$. Let D^i be isometric copies of D . We may define $E^i = D \cup D^i$, where $d(x, y^i) = \chi^i(x, y) - \delta$, observing that the assumption that $\chi^i - \delta$ is an approximate isometry is equivalent to this being a distance. We further define $E = E^0 \cup E^1 = D \cup D^0 \cup D^1$, with $d(x^0, y^1) = d(x, y) + \eta$ for all $x, y \in D$, observing that again the triangle inequality is respected. Embed $E \subseteq \mathbf{U}$. The partial isometry $D^0 \mapsto D^1$ has distance η from the identity, and by Fact 4.11 it extends to an isometry $f \in \text{Iso}(\mathbf{U})$ with $d(f, \text{id}) = \eta < \varepsilon$. Let $g \in \text{Iso}(\mathbf{U})$ be any isometry extending $D \mapsto D^0$. Then $fg \in [\chi^1]$ and $g \in [\chi^0] \cap ([\chi^1])_\varepsilon$, as desired. $\blacksquare_{4.12}$

Lemma 4.13. *Let B be a finite metric space, $\bar{\varphi} \in \text{Iso}(B)^m$ and let $\varepsilon > 0$ be given and $\delta = \varepsilon/2$. Then $(B, \bar{\varphi} + \delta)$ ε -isolates B .*

Proof. Immediate from the above. $\blacksquare_{4.13}$

Theorem 4.14. *The groups $\text{Iso}(\mathbf{U})$ and $\text{Iso}(\mathbf{U}_1)$ admits topometric ample generics.*

Proof. By the above and Theorem 3.16. $\blacksquare_{4.14}$

5. PERTURBED FRAÏSSÉ CLASSES

In earlier sections, and specifically in Definition 1.2 and Definition 3.5, we have defined when “an existential type is isolated” without having defined an existential type. This reflects the fact that the proofs as given there are merely the Quik-N-Eazy™ projection of a more conceptual argument, the “true” argument, in the author’s mind, which the terminology suggests, and which we shall now render explicit.

5.1. The discrete case. For the sake of simplicity we start with the discrete case, namely that of Section 1. The idea is that modulo $\mathcal{K}_{a,m}$ having JEP and being existentially atomic, what we seek to prove (the existence of a co-meagre, or even dense G_δ , orbit) essentially follows from general facts regarding Fraïssé limits, applied to a Fraïssé class appropriately derived from $\mathcal{K}_{a,m}$. In addition to folklore results, we shall require the following easy fact, which we have failed to find in the literature.

Proposition 5.1. *Let \mathcal{K} be a Fraïssé class in a language \mathcal{L} , such that the isomorphism class of each enumerated member of \mathcal{K} is isolated (in \mathcal{K}) by a quantifier-free \mathcal{L} -formula. Let X be the space of all \mathcal{L} -structures over \mathbf{N} whose age is contained in \mathcal{K} , equipped with the standard topology, and let $Y \subseteq X$ consist of those members which are limit of \mathcal{K} . Then X is Polish, and Y is a dense G_δ of X .*

We shall also require

Fact 5.2 ([Kec95, Theorem 3.11]). *Let X be Polish and $Y \subseteq X$. Then Y is Polish (in the induced topology) if and only if it is G_δ in X .*

For $n \in \mathbf{N}$ let us consider the family $\mathcal{S}_n^{\exists a, m}$ (or $\mathcal{S}_n^{\exists a, m}(\mathcal{K})$ if one wishes to be pedantic) of all $(\mathfrak{B}, \bar{\varphi}, \bar{b})$ where $(\mathfrak{B}, \bar{\varphi}) \in \mathcal{K}_{a, m}$, $\bar{b} \in B^n$, and in addition $(\mathfrak{B}, \bar{\varphi})$ isolates an existential type for $A = \{b_i\}_{i < n}$. We observe that the relation \equiv defined as “ $(\mathfrak{B}, \bar{\varphi}, \bar{b})$ and $(\mathfrak{C}, \bar{\rho}, \bar{c})$ amalgamate in $\mathcal{K}_{a, m}$ with \bar{b} and \bar{c} having the same image” is an equivalence relation on $\mathcal{S}_n^{\exists a}$. We let $S_n^{\exists a, m} = \mathcal{S}_n^{\exists a, m} / \equiv$, and call its members *atomic existential n -types (in $\mathcal{K}_{a, m}$)*. The class of $(\mathfrak{B}, \bar{\varphi}, \bar{b})$ is denoted $\text{tp}_{(\mathfrak{B}, \bar{\varphi})}^{\exists a}(\bar{b})$, the *existential type of \bar{b} in $(\mathfrak{B}, \bar{\varphi})$* . We define $\mathcal{L}^{\exists a, m}$ to consist of the original language \mathcal{L} along with a new n -ary predicate p for each $p \in S_n^{\exists a, m}$, observing that this is still countable.

Any $(\mathfrak{B}, \bar{\varphi}) \in \mathcal{K}_{a, m}$ which isolates an existential type for some $A \subseteq B$ determines an $\mathcal{L}^{\exists a, m}$ -structure $\tilde{\mathfrak{A}}$ on A , by taking first the induced \mathcal{L} -structure \mathfrak{A} , and for each n and $\bar{b} \in A^n$, saying that $p(\bar{b})$ is true where $p = \text{tp}_{(\mathfrak{B}, \bar{\varphi})}^{\exists a}(\bar{b})$ and $q(\bar{b})$ is false for all other $q \in S_n^{\exists a, m}$. We define $\mathcal{K}_{\exists a, m}$ to consist of all finite $\mathcal{L}^{\exists a, m}$ -structures $\tilde{\mathfrak{A}}$ obtained in this fashion. For $\tilde{\mathfrak{A}} \in \mathcal{K}_{\exists a, m}$ and $\bar{a} \in A^n$, we define $\text{tp}_{\tilde{\mathfrak{A}}}^{\exists a}(\bar{a})$ to be the

unique $p \in S_n^{\exists a, m}$ such that $p(\bar{a})$ holds in $\tilde{\mathfrak{A}}$. It is not difficult to check that if \bar{a} enumerates all of A then $\text{tp}^{\exists a}(\bar{a})$ determines $\tilde{\mathfrak{A}}$ (including the interpretation of \mathcal{L} thereon).

Each $\tilde{\mathfrak{A}} \in \mathcal{K}_{\exists a, m}$ determines an m -tuple of partial isomorphisms $\bar{\psi}: \mathfrak{A} \dashrightarrow \mathfrak{A}$, where $\psi_i(a) = b$ if there exists $(\mathfrak{B}, \bar{\varphi})$ inducing $\tilde{\mathfrak{A}}$ as above such that $\varphi_i(a) = b$. With some abuse of terminology, we shall call $(\mathfrak{A}, \bar{\psi})$ the $\mathcal{K}_{a, m}$ -reduct of $\tilde{\mathfrak{A}}$.

Now, $\mathcal{K}_{\exists a, m}$ has HP and is countable up to isomorphism, by definition. The class $\mathcal{K}_{a, m}$ is existentially atomic if and only if every $(\mathfrak{A}, \bar{\psi})$ embeds into the $\mathcal{K}_{a, m}$ -reduct of some $\tilde{\mathfrak{A}} \in \mathcal{K}_{\exists a, m}$. In this case $\mathcal{K}_{\exists a, m}$ also has AP, and it has JEP if and only if $\mathcal{K}_{a, m}$ does. In conclusion, if $\mathcal{K}_{a, m}$ has JEP and is existentially atomic then $\mathcal{K}_{\exists a, m}$ is a Fraïssé class.

Let $\tilde{\mathfrak{M}} = \lim \mathcal{K}_{\exists a, m}$ be its limit. Defining the reduct as above, we see that it is of the form (\mathfrak{M}, \bar{f}) , where $\mathfrak{M} = \lim \mathcal{K}$ and \bar{f} are necessarily total, surjective isomorphisms, i.e., $\bar{f} \in G^m = \text{Aut}(\mathfrak{M})^m$. We observe that for each $\tilde{\mathfrak{A}} \in \mathcal{K}_{\exists a, m}$ there exists $\mathfrak{A} \subseteq \mathfrak{B} \in \mathcal{K}_{\exists a, m}$ such that the reduct $(\mathfrak{B}, \bar{\varphi})$ determines $\tilde{\mathfrak{A}}$. If $\mathfrak{A} \subseteq \mathfrak{M}$ then we may assume that $\mathfrak{B} \subseteq \mathfrak{M}$, in which case \bar{f} determines $\bar{\varphi}$. Thus (\mathfrak{M}, \bar{f}) is not only determined by $\tilde{\mathfrak{M}}$, but also determines it.

Fix \mathfrak{M} , say with domain \mathbf{N} , and let $Z \subseteq G^m$ consist of all \bar{f} such that (\mathfrak{M}, \bar{f}) is a limit. Clearly Z is dense in G^m , and by uniqueness of the limit it is an orbit under conjugation, so all we need to show is that it is a G_δ set, or equivalently, that it is Polish. Indeed, let X be the space of all structures with domain \mathbf{N} and age in $\mathcal{K}_{\exists a, m}$, and $Y \subseteq X$ consist of those which are limit, as in Proposition 5.1. Let $Y' \subseteq X$ consist of those structures whose \mathcal{L} -reduct is \mathfrak{M} . The argument above shows that Z stands in a natural bijection with $Y \cap Y'$, and it is not hard to see that this is in fact a homeomorphism. Now, Y is G_δ and Y' is closed, so $Y \cap Y'$ is G_δ as well, and therefore Polish. Thus Z is Polish and in particular G_δ in G^m , and the proof of the “difficult” direction of Theorem 1.7 is complete.

5.2. The metric case.

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