ON A C^*-ALGEBRA FORMALISM FOR CONTINUOUS FIRST ORDER LOGIC

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Abstract.

All C^*-algebras are assumed to be unital and commutative.

A purely relational language $\mathcal{L}$ consists of a set $\mathcal{R}^{\mathcal{L}}$ whose members are called *predicate symbols*. Each predicate symbol $P \in \mathcal{R}^{\mathcal{L}}$ also carries an *arity* $n_P \in \mathbb{N}$ and a compact *spectrum* $\Sigma_P \subseteq \mathbb{C}$. An $\mathcal{L}$*-structure* $\mathcal{M}$ consists of a set M along with interpretations of the predicate symbols:

$$P^{\mathcal{M}} \colon M^{n_P} \to \Sigma_P.$$

Let $\mathcal{L}^{\mathcal{M}} = B(M^{\mathbb{N}}, \mathbb{C})$ denote the C^*-algebra of bounded functions $\varphi\colon M^{\mathbb{N}} \to \mathbb{C}$. If $\varphi \in \mathcal{L}^{\mathcal{M}}$ and $(a_i)_{i\in\mathbb{N}} \in M^{\mathbb{N}}$ we may write $\varphi(a_i)_{i\in\mathbb{N}}$ as a *substitution* $\varphi[{}^{a_i}/_{x_i}]_{i\in\mathbb{N}}$, the idea being that we substitute each a_i for the variable x_i (which serves as a place holder of sorts) and then evaluate φ. This C^*-algebra admits the following structure:

- If P is an n-ary predicate symbol then $P^{\mathcal{M}} \in \mathcal{L}^{\mathcal{M}}$ via the addition of dummy variables, i.e., $P^{\mathcal{M}}(a_i)_{i\in\mathbb{N}} = P^{\mathcal{M}}(a_0, \ldots, a_{n-1})$.
- If $\varphi \in \mathcal{L}^{\mathcal{M}}$ and $\xi\colon \mathbb{N} \to \mathbb{N}$ then $\varphi^\xi \in \mathcal{L}^{\mathcal{M}}$ as well, where $\varphi^\xi[{}^{a_i}/_{x_i}] = \varphi[{}^{a_{\xi(i)}}/_{x_i}]$. The function φ^ξ can also be written as a substitution $\varphi[{}^{x_{\xi(i)}}/_{x_i}]_{i\in\mathbb{N}}$. The expected composition rules for substitutions hold, namely:

$$\varphi[{}^{x_{\xi(i)}}/_{x_i}][{}^{a_j}/_{x_j}] = \varphi[{}^{a_{\xi(i)}}/_{x_i}], \qquad \big(= \varphi[{}^{x_{\xi(i)}}/_{x_i}][{}^{a_{\xi(i)}}/_{x_{\xi(i)}}]\big)$$
$$\varphi[{}^{x_{\xi(i)}}/_{x_i}][{}^{x_{\zeta(j)}}/_{x_j}] = \varphi[{}^{a_{\zeta\circ\xi(i)}}/_{x_i}]. \qquad \big(= \varphi[{}^{x_{\xi(i)}}/_{x_i}][{}^{x_{\zeta\circ\xi(i)}}/_{x_{\xi(i)}}]\big)$$

- For any self-adjoint (i.e., real-valued) $\varphi \in \mathcal{L}^{\mathcal{M}}_{sa}$ and variable x, define:

$$\big(\sup_x \varphi\big)[{}^{a_i}/_{x_i}] = \sup\big\{\varphi[{}^{b}/_{x}, {}^{a_i}/_{x_i}]_{x_i\neq x} \colon b \in M\big\}$$

  Then $\sup_x \varphi \in \mathcal{L}^{\mathcal{M}}_{sa}$ as well.

We may consider a more general case where $\mathcal{L}$ consists of $\mathcal{R}^{\mathcal{L}}$ as above along with a set $\mathcal{F}^{\mathcal{L}}$ of *function symbols*. A function symbol $f \in \mathcal{F}^{\mathcal{L}}$ only carries an arity $n_f \in \mathbb{N}$. An $\mathcal{L}$-pre-structure $\mathcal{M}$ then consists of a set M along with interpretations:

$$f^{\mathcal{M}} \colon M^{n_f} \to M, \qquad P^{\mathcal{M}} \colon M^{n_P} \to \Sigma_P.$$

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Function symbols can be composed formally, yielding the *term algebra* $\mathcal{T}^{\mathcal{L}}$ of $\mathcal{L}$. The term algebra is the set generated freely from the formal variables $\{x_i\}_{i\in\mathbb{N}}$ by the composition operations $(\tau_0,\ldots,\tau_{n_f-1}) \mapsto f(\tau_0,\ldots,\tau_{n_f-1})$. Substitution of terms in other terms is defined in the natural manner:

$$x_i[{}^{\sigma_j}/_{x_j}] = \sigma_i, \qquad \big(f(\tau_0,\ldots,\tau_{n_f-1})\big)[{}^{\sigma_j}/_{x_j}] = f\big(\tau_0[{}^{\sigma_j}/_{x_j}],\ldots,\tau_{n_f-1}[{}^{\sigma_j}/_{x_j}]\big).$$

It will be convenient to consider the evaluation of a term τ in $\mathcal{M}$ as a substitution:

$$x_i^{\mathcal{M}}[{}^{a_j}/_{x_j}] = a_i, \qquad \big(f(\tau_0,\ldots,\tau_{n_f-1})\big)^{\mathcal{M}}[{}^{a_j}/_{x_j}] = f^{\mathcal{M}}\big(\tau_0^{\mathcal{M}}[{}^{a_j}/_{x_j}],\ldots,\tau_{n_f-1}^{\mathcal{M}}[{}^{a_j}/_{x_j}]\big).$$

The same composition rules hold:

$$\big(\rho[{}^{\tau_i}/_{x_i}]\big)^{\mathcal{M}}[{}^{a_j}/_{x_j}] = \rho^{\mathcal{M}}\big[{}^{\tau_i^{\mathcal{M}}[a_j/x_j]}/_{x_i}\big], \qquad \rho[{}^{\tau_i}/_{x_i}][{}^{\sigma_j}/_{x_j}] = \rho\big[{}^{\tau_i[\sigma_j/x_j]}/_{x_i}\big].$$

Now the algebra $\mathcal{L}^{\mathcal{M}}$ carries one additional operation:

- For any $\varphi \in \mathcal{L}^{\mathcal{M}}$ and terms $(\tau_i)_{i\in\mathbb{N}}$, the composition $\varphi \circ (\tau_i^{\mathcal{M}})$ also belongs to $\mathcal{L}^{\mathcal{M}}$. It can be written as a *term substitution* $\varphi[{}^{\tau_i}/_{x_i}]$. As usual:

$$\varphi[{}^{\tau_i}/_{x_i}][{}^{a_j}/_{x_j}] = \varphi\big[{}^{\tau_i^{\mathcal{M}}[a_j/x_j]}/_{x_i}\big], \qquad \varphi[{}^{\tau_i}/_{x_i}][{}^{\sigma_j}/_{x_j}] = \varphi\big[{}^{\tau_i[\sigma_j/x_j]}/_{x_i}\big].$$

These operations make sense in every $\mathcal{L}$-structure $\mathcal{M}$, leading us to the following abstraction:

Definition 0.1. A $\mathcal{L}$-*algebra* with variables $\{x_n\}_{n\in\mathbb{N}}$ is a C^*-algebra $\mathscr{A}$ equipped with the following additional structure:

- *Atomic formulae:* A mapping $\mathcal{R}^{\mathcal{L}} \to \mathcal{A}$, associating to each predicate symbol P a member $P^{\mathcal{A}} \in \mathcal{A}$. It satisfies:

$$\sigma(P^{\mathcal{A}}) \subseteq \Sigma_P.$$

- *Term substitution:* For each sequence $(\tau_i)_{i\in\mathbb{N}}$, a endomorphism $[{}^{\tau_i}/_{x_i}]_{i\in\mathbb{N}}\colon \mathcal{A} \to \mathcal{A}$.
- *Quantification:* For each variable x a mapping $\sup_x\colon \mathcal{A}_{sa} \to \mathcal{A}_{sa}$.

They are required to satisfy the following properties:

(N)	$\Vert\varphi[{}^{\tau_i}/_{x_i}]\Vert \leq \Vert\varphi\Vert, \qquad \Vert\sup_x \varphi\Vert \leq \Vert\varphi\Vert$	
(S1)	$P^{\mathcal{A}}[{}^{\tau_i}/_{x_i}]_{i\neq 0,\ldots,n_P-1} = P^{\mathcal{A}}$	
(S2)	$\varphi[] = \varphi$	
(S3)	$\varphi[{}^{\tau_i}/_{x_i}][{}^{\sigma_j}/_{x_j}] = \varphi\big[{}^{\tau_i[\sigma_j/x_j]}/_{x_i}\big]$	
(S4)	$(\sup_x \varphi)[{}^{\tau_i}/_{x_i}] = \sup_x \varphi[{}^{\tau_i}/_{x_i}]_{x_i\neq x}$	if x does not appear in any τ_i
(Q1)	$\varphi \leq \psi \Longrightarrow \sup_x \varphi \leq \sup_x \psi$	i.e., $\sup_x$ is monotone
(Q2)	$\sup_x \varphi \geq \varphi[{}^{\tau}/_{x}]$	
(Q3)	$\sup_x \varphi[{}^{y}/_{x}] = \varphi[{}^{y}/_{x}]$	if $x \neq y$

Remark 0.2. A reader familiar with classical first order logic may wish to compare the two. We have chosen to avoid syntax altogether, whence our axiomatic, rather than syntactic, definition of substitutions. One advantage is that we avoid the issue of *correctness* of substitution (the restriction we impose in axiom S4 is reminiscent of it, though). The axioms for quantifiers Q1-3 correspond very closely to the quantifier axioms for formal deductions:

$$\forall x\,(\varphi \to \psi) \to \forall x\,\varphi \to \forall x\,\psi \tag{A1}$$

$$\forall x\,\varphi \to \varphi[\tau/x] \tag{A2}$$

$$\varphi \to \forall x\,\varphi \qquad \text{if } x \text{ is not free in } \varphi \tag{A3}$$

A morphism of $\mathcal{L}$-algebras is a morphism of C^*-algebras which respects the additional structure. While being a tedious construction, it is fairly standard to see that the category of $\mathcal{L}$-algebras admits an initial object, i.e., a *free* $\mathcal{L}$-algebra. We call this algebra $\mathcal{L}$. Members of $\mathcal{L}$ are called *formulae*. Strictly speaking, $\mathcal{L}$ depends on the set of variables we chose. When we wish to make this explicit we may use the notation $\mathcal{L}(x)_{x\in\mathcal{V}}$ or $\mathcal{L}_{\mathcal{V}}$ or something reasonable XXXXX.

Proposition 0.3 (Induction principle)**.** *Let $\mathcal{L}' \subseteq \mathcal{L}$ be a sub-algebra, containing all the atomic formulae and closed under term substitution and under quantification. Then $\mathcal{L}'$ is dense in $\mathcal{L}$. If $\mathcal{L}'$ is closed then $\mathcal{L}' = \mathcal{L}$.*

Proof. Since substitutions and quantifiers are all continuous, the closed algebra $\overline{\mathcal{L}'}$ is also closed under these operations. In this case $\overline{\mathcal{L}'}$ is itself an $\mathcal{L}$-algebra and admits a unique morphism $\mathcal{L} \to \overline{\mathcal{L}'}$. The composition of $\mathcal{L} \to \overline{\mathcal{L}'} \subseteq \mathcal{L}$ must be $\mathrm{id}_{\mathcal{L}}$, so $\overline{\mathcal{L}'} = \mathcal{L}$. $\blacksquare_{0.3}$

Let φ be a formula, X a set of variables. We say that X *determines* φ if $\varphi[{\tau_i}/{x_i}] = \varphi[{\sigma_i}/{x_i}]$ whenever $\tau_i = \sigma_i$ for all $x_i \in X$. We define $\mathcal{L}_f$ to denote the collection of formulae which are determined by a finite set of variables.

Lemma 0.4. *$\mathcal{L}_f$ is a dense sub-algebra of $\mathcal{L}$.*

Proof. We prove by induction:
– $\mathcal{L}_f$ is a sub-algebra of $\mathcal{L}$. Indeed, if X_φ determines φ and X_ψ determines ψ then $X_\varphi \cup X_\psi$ determines $\varphi + \psi$, $\varphi\psi$, and so on.
– $\mathcal{L}_f$ contains all the atomic formulae by axiom S1.
– $\mathcal{L}_f$ is closed under term substitution. Indeed, let $\psi = \varphi[{\rho_j}/{x_j}]$ where $\varphi \in \mathcal{L}_f$. Let X_φ be a finite set which determines φ and let X_ψ consist of all variables appearing in $\{\rho_j : x_j \in X_\varphi\}$. Then X_ψ is finite. Assume that $\tau_i = \sigma_i$ for $x_i \in X_\psi$. Then $\rho_j[{\tau_i}/{x_i}] = \rho_j[{\sigma_i}/{x_i}]$ for $j \in X_\varphi$, whereby:

$$\psi[{\tau_i}/{x_i}] = \varphi\left[{\rho_j[{\tau_i}/{x_i}]}//{x_j}\right] = \varphi\left[{\rho_j[{\sigma_i}/{x_i}]}//{x_j}\right] = \psi[{\sigma_i}/{x_i}].$$

– $\mathcal{L}_f$ is closed under quantification. Indeed, let $\varphi \in \mathcal{L}_f$ be determined by X_φ. Without loss of generality we restrict ourselves to $\sup_{x_0} \varphi$. Assume that $\tau_i = \sigma_i$ for $x_i \in X_\varphi$, and let $\tau_i' = \tau_i[{}^{x_{j+1}}/_{x_j}]$, $\sigma_i' = \sigma_i[{}^{x_{j+1}}/_{x_j}]$. Then:

$$(\sup_{x_0}\varphi)[{}^{\tau_i'}/_{x_i}] = \sup_{x_0}\varphi[{}^{\tau_i'}/_{x_i}]_{i\neq 0} = \sup_{x_0}\varphi[{}^{\sigma_i'}/_{x_i}]_{i\neq 0} = (\sup_{x_0}\varphi)[{}^{\sigma_i'}/_{x_i}],$$

whereby:

$$\begin{aligned}(\sup_{x_0}\varphi)[{}^{\tau_i}/_{x_i}] &= (\sup_{x_0}\varphi)[{}^{\tau_i'}/_{x_i}][{}^{x_j}/_{x_{j+1}}] = (\sup_{x_0}\varphi)[{}^{\sigma_i'}/_{x_i}][{}^{x_j}/_{x_{j+1}}]\\ &= (\sup_{x_0}\varphi)[{}^{\sigma_i}/_{x_i}]. \qquad \blacksquare_{0.4}\end{aligned}$$

Lemma 0.5. *For every formula φ there is a (unique) smallest countable set of variables* $\mathrm{DVar}(\varphi)$ *which determines φ. Moreover:*

$$\begin{aligned}\mathrm{DVar}(\varphi) &= \{x : \varphi \neq \varphi[{}^{\tau}/_{x}] \text{ for some term } \tau\}\\ &= \{x : \varphi \neq \varphi[{}^{y}/_{x}] \text{ for every variable } y \neq x\}.\end{aligned}$$

Proof. Let Ξ_φ be the collection of all sets of variables X which determine φ. It is easy to see that Ξ_φ is closed under finite intersections. Thus, if $\varphi \in \mathcal{L}_f$ then Ξ_φ contains a smallest (finite) member $\mathrm{DVar}(\varphi)$. For a general formula φ there is a sequence $\{\varphi_n\} \subseteq \mathcal{L}_f$, $\varphi_n \to \varphi$, and it is easy to check that $\bigcup_n \mathrm{DVar}\,\varphi_n$ determines φ. Thus Ξ_φ contains a countable member. In order to show that Ξ_φ contains a smallest member it will be enough to show that if $X_0 \supseteq X_1 \supseteq \ldots$ is a decreasing sequence in Ξ_φ then $X = \bigcap_n X_n \in \Xi_\varphi$.

Assume that $\tau_i = \sigma_i$ for all $i \in X$, and we need to show that $\varphi[{}^{\tau_i}/_{x_i}] = \varphi[{}^{\sigma_i}/_{x_i}]$. Fix $\varepsilon > 0$, and choose $\psi \in \mathcal{L}_f$, $\|\psi - \varphi\| < \varepsilon$. Since $\mathrm{DVar}(\psi)$ is finite, for n big enough we have $X \cap \mathrm{DVar}(\psi) = X_n \cap \mathrm{DVar}(\psi)$. Let $\rho_i = \tau_i$ for $i \in \mathrm{DVar}(\psi)$ and $\rho_i = \sigma_i$ otherwise, so $\|\varphi[{}^{\tau_i}/_{x_i}] - \varphi[{}^{\rho_i}/_{x_i}]\| < 2\varepsilon$. If $i \in X_n$ then either $i \notin \mathrm{DVar}(\psi)$, in which case we chose $\rho_i = \sigma_i$, or $i \in \mathrm{DVar}(\psi)$, in which case $i \in X$ and again $\rho_i = \tau_i = \sigma_i$. Since $X_n \in \Xi_\varphi$ we have $\varphi[{}^{\rho_i}/_{x_i}] = \varphi[{}^{\sigma_i}/_{x_i}]$. We have thus shown that $\|\varphi[{}^{\tau_i}/_{x_i}] - \varphi[{}^{\sigma_i}/_{x_i}]\| < 2\varepsilon$ for all $\varepsilon > 0$, so $\varphi[{}^{\tau_i}/_{x_i}] = \varphi[{}^{\sigma_i}/_{x_i}]$, and $X \in \Xi\varphi$ as desired. This concludes the proof that It follows that Ξ_φ contains a countable smallest member, namely $\mathrm{DVar}(\varphi) = \bigcap \Xi_\varphi$.

We now prove the moreover part. If $x \notin \mathrm{DVar}(\varphi)$ then $\varphi = \varphi[{}^{\tau}/_{x}]$ for every term τ. Conversely, assume that $\varphi = \varphi[{}^{y}/_{x}]$ for some variable $y \neq x$. Assume that $\tau_i = \sigma_i$ whenever $x_i \neq x$. In particular let $\rho = \tau_j = \sigma_j$ for $x_j = y$. Then:

$$\varphi[{}^{\tau_i}/_{x_i}] = \varphi[{}^{y}/_{x}][{}^{\tau_i}/_{x_i}] = \varphi[{}^{\rho}/_{x}, {}^{\tau_i}/_{x_i}]_{x_i\neq x} = \varphi[{}^{y}/_{x}][{}^{\sigma_i}/_{x_i}] = \varphi[{}^{\sigma_i}/_{x_i}].$$

Thus $\mathrm{DVar}(\varphi) \smallsetminus \{x\} \in \Xi_\varphi$ whereby $x \notin \mathrm{DVar}(\varphi)$. $\blacksquare_{0.5}$

Lemma 0.6. *Let $\varphi \in \mathcal{L}$ be self-adjoint, x a variable. Then:*

$$\begin{aligned}&\mathrm{DVar}(\sup_x\varphi) \subseteq \mathrm{DVar}(\varphi) \smallsetminus \{x\},\\ &x \notin \mathrm{DVar}(\varphi) \iff \varphi = \sup_x\varphi,\\ &\sup_x\varphi = \sup_y\varphi[{}^{y}/_{x}]. \qquad \textit{if } y \notin \mathrm{DVar}(\varphi)\end{aligned}$$

Proof. Let x, y, z be distinct, $z \notin \mathrm{DVar}(\varphi)$. Then

$$\begin{aligned}(\sup_x\varphi)[{}^{y}/_{x}] &= \sup_x\varphi &&\Longrightarrow x \notin \mathrm{DVar}(\sup_x\varphi),\\ (\sup_x\varphi)[{}^{y}/_{z}] &= \sup_x\varphi[{}^{y}/_{z}] = \sup_x\varphi &&\Longrightarrow z \notin \mathrm{DVar}(\sup_x\varphi).\end{aligned}$$

Thus the first assertion is proved, and right to left of the second assertion follows. For left to right, observe that if $x \notin \mathrm{DVar}(\varphi)$ then:

$$\sup_x \varphi = \sup_x \varphi[{}^y/_x] = \varphi[{}^y/_x] = \varphi.$$

For the last assertion observe that for any variable y we have $\sup_x \varphi \geq \varphi[{}^y/_x]$, whereby $\sup_x \varphi = \sup_y \sup_x \varphi \geq \sup_y \varphi[{}^y/_x]$. If in addition $y \notin \mathrm{DVar}(\varphi)$ then $\varphi[{}^y/_x][{}^x/_y] = \varphi[{}^x/_y] = \varphi$, so by the same argument $\sup_x \varphi \leq \sup_y \varphi[{}^y/_x]$. $\blacksquare_{0.6}$

The inclusion $\mathrm{DVar}(\sup_x \varphi) \subseteq \mathrm{DVar}(\varphi) \setminus \{x\}$ may be strict. Indeed, assume $\varphi = -|P(x) - P(y)|$. Then $\mathrm{DVar}(\varphi) = \{x, y\}$, but $\mathrm{DVar}(\sup_x \varphi) = \mathrm{DVar}(0) = \varnothing$.

Lemma 0.7. *Let $\varphi \in \mathcal{L}$ be self-adjoint, x a variable. We can characterise $\sup_x \varphi$ as:*

(i) *The least formula satisfying $\psi \geq \varphi$ and $x \notin \mathrm{DVar}(\psi)$.*
(ii) *The least upper bound in $\mathcal{L}$ for the sets $\{\varphi[{}^\tau/_x] : \tau \in \mathcal{T}_\mathcal{L}\}$ and $\{\varphi[{}^y/_x] : y \neq x\}$:*

$$\sup_x \varphi = \sup\{\varphi[{}^\tau/_x] : \tau \in \mathcal{T}_\mathcal{L}\} = \sup\{\varphi[{}^y/_x] : y \neq x\}.$$

Proof. For the first item, we know that $\psi = \sup_x \varphi$ does indeed verify $\psi \geq \varphi$ and $x \notin \mathrm{DVar}(\psi)$. If ψ is another formula with the same properties then $\sup_x \varphi \leq \sup_x \psi = \psi$.

For the second item, we know that $\sup_x \varphi$ is an upper bound for $\{\varphi[{}^\tau/_x] : \tau \in \mathcal{T}_\mathcal{L}\}$. We need to show that if ψ is an upper bound for $\{\varphi[{}^y/_x] : y \text{ variable}\}$ then $\psi \geq \sup_x \varphi$.

Let $\inf_x \psi = -\sup_x -\psi$, noticing that $x \notin \mathrm{DVar}(\inf_x \psi)$ and $\inf_x \psi \leq \psi$. Choose $y \notin \mathrm{DVar}(\varphi) \cup \mathrm{DVar}(\psi)$, so $y \notin \mathrm{DVar}(\inf_x \psi)$ as well. Monotonicity of $\sup_x$ implies monotonicity of $\inf_x$, so $\psi \geq \varphi[{}^y/_x]$ implies $\inf_x \psi \geq \inf_x \varphi[{}^y/_x] = \varphi[{}^y/_x]$. Thus:

$$\inf_x \psi = (\inf_x \psi)[{}^x/_y] \geq \varphi[{}^y/_x][{}^x/_y] = \varphi[{}^x/_y] = \varphi.$$

By the first item, $\psi \geq \inf_x \psi \geq \sup_x \varphi$, as desired. $\blacksquare_{0.7}$

Lemma 0.8. *Assume $x \notin \mathrm{DVar}(\varphi)$. Then $\sup_x(\varphi + \psi) = \varphi + \sup_x \psi$.*

Proof. Immediate from the characterisation above. $\blacksquare_{0.8}$

We do Henkin stuff:

$$\begin{aligned}
\mathcal{V}_0 &= \mathcal{V}, \\
\mathcal{V}_{n+1} &= \mathcal{V}_n \cup \{z_{\varphi,y}\}_{(\varphi,y) \in \mathcal{L}_{\mathcal{V}_n} \times \mathcal{V}_n}, \\
\mathcal{V}' &= \bigcup_{n \in \mathbb{N}} \mathcal{V}_n, \\
\Sigma_h &= \{\sup_y \varphi - \varphi[{}^{z_{\varphi,y}}/_y] : (\varphi, y) \in \mathcal{L}_{\mathcal{V}_n} \times \mathcal{V}_n \text{ for some } n\}.
\end{aligned}$$

Lemma 0.9. *Assume $\mathcal{V} \subseteq \mathcal{V}'$ are two sets of variables. Then $\mathcal{L}_\mathcal{V}$ admits a canonical embedding in $\mathcal{L}_{\mathcal{V}'}$, whose image consists of all $\varphi \in \mathcal{L}_{\mathcal{V}'}$ such that $\mathrm{DVar}(\varphi) \subseteq \mathcal{V}$.*

Proof. The $(\mathcal{L}, \mathcal{V}')$-algebra $\mathcal{L}_{\mathcal{V}'}$ can be viewed as an $(\mathcal{L}, \mathcal{V})$-algebra, whence a canonical morphism of $(\mathcal{L}, \mathcal{V})$-algebras $\theta_1 \colon \mathcal{L}_\mathcal{V} \to \mathcal{L}_{\mathcal{V}'}$.

Assume first that $|\mathcal{V}| = |\mathcal{V}'|$, so there exists a bijection $\xi\colon \mathcal{V}' \to \mathcal{V}$. This bijection induces an isomorphism $\tilde{\xi}\colon \mathcal{L}_{\mathcal{V}'} \to \mathcal{L}_{\mathcal{V}}$ verifying:

$$\begin{aligned}
&\tilde{\xi}(P) = P[\xi(x_i)/x_i]_{i=0,\ldots,n_P-1},\\
&\tilde{\xi}\big(\varphi[\tau_i/x_i]_{x_i\in\mathcal{V}'}\big) = \big(\tilde{\xi}(\varphi)\big)\left[\tau_i[\xi(x)/x]_{x\in\mathcal{V}'}/\xi(x_i)\right]_{x_i\in\mathcal{V}'},\\
&\tilde{\xi}\big(\sup_x \varphi\big) = \sup_{\xi(x)} \tilde{\xi}(\varphi).
\end{aligned}$$

Define also two mappings $\theta_2, \theta_3\colon \mathcal{L}_{\mathcal{V}} \to \mathcal{L}_{\mathcal{V}}$ by:

$$\theta_2(\varphi) = \varphi[\xi(x)/x]_{x\in\mathcal{V}}, \qquad \theta_3(\varphi) = \varphi[x/\xi(x)]_{x\in\mathcal{V}}.$$

It is straightforward to check by induction that $\tilde{\xi}\circ\theta_1 = \theta_2$, and clearly $\theta_3\circ\theta_2 = \mathrm{id}_{\mathcal{L}_{\mathcal{V}}}$. Therefore θ_2 is injective (and isometric) and so is $\theta_1 = \tilde{\xi}^{-1}\circ\theta_2$.

If $\varphi = \theta_2(\psi)$ then clearly $\mathrm{DVar}(\varphi) \subseteq \xi(\mathcal{V})$. Conversely, if $\mathrm{DVar}(\varphi) \subseteq \xi(\mathcal{V})$ Then $\varphi = \theta_2(\theta_3(\varphi))$. In other words, the image of θ_2 consists precisely of those φ verifying $\mathrm{DVar}(\varphi) \subseteq \xi(\mathcal{V})$. Therefore the image of $\theta_1 = \tilde{\xi}^{-1}\circ\theta_2$ consists of those $\varphi\in\mathcal{L}_{\mathcal{V}'}$ verifying $\mathrm{DVar}(\varphi)\subseteq\mathcal{V}$.

If $|\mathcal{V}| < |\mathcal{V}'|$, consider $S = \{\mathcal{V}''\colon \mathcal{V}\subseteq\mathcal{V}''\subseteq\mathcal{V}' \text{ and } |\mathcal{V}| = |\mathcal{V}'|\}$. This is a directed system, and by the case we have handled so is $\{\mathcal{L}_{\mathcal{V}''}\}_{\mathcal{V}''\in S}$. The limit is easily checked to be the free $(\mathcal{L},\mathcal{V}')$-algebra, i.e., $\mathcal{L}_{\mathcal{V}'}$. The statement now follows from this construction of $\mathcal{L}_{\mathcal{V}'}$. $\blacksquare_{0.9}$

Lemma 0.10. *The natural mapping $\mathcal{L}_{\mathcal{V}} \to \mathcal{L}_{\mathcal{V}'}/\overline{\langle\Sigma_h\rangle}$ is an embedding.*

Proof. Assume not. Then there is a non zero $\varphi\in\mathcal{L}_{\mathcal{V}}$ which belongs to the closed ideal $\overline{\langle\Sigma_h\rangle}$. By standard properties of C^*-algebras we may assume that $\varphi = |\varphi| \geq 0$ is positive and actually belongs to the ideal generated algebraically by Σ_h. Since all members of Σ_h are positive, and possibly replacing φ with something smaller, this boils down to saying that there is a finite family of distinct pairs (ψ_i, y_i), $i\leq k$, such that:

$$\varphi \leq \sum_{i\leq k}\big(\sup_{y_i}\psi_i - \psi_i[z_i/y_i]\big),$$

where $z_i = z_{\psi_i,y_i}$. We may assume that k is minimal.

For each $i\leq k$ there is n_i such that $z_i\in\mathcal{V}_{n_i+1}-\mathcal{V}_n$. We may assume that n_k is maximal, which means that $z_k\notin\mathrm{DVar}(\psi_i)$ for $i\leq k$ and $z_k\neq z_i$ for $i<k$. Thus

$z_k \notin \mathrm{DVar}(\sup_{y_i} \psi_i)$ for $i \leq k$ and $z_k \notin \mathrm{DVar}(\psi_i[{}^{z_i}/{}_{y_i}])$ for $i < k$.

$$\begin{aligned}\varphi &= \sup_{z_k} \varphi \\ &\leq \sup_{z_k} \sum_{i \leq k} \big(\sup_{y_i} \psi_i - \psi_i[{}^{z_i}/{}_{y_i}]\big) \\ &= \sum_{i<k} \big(\sup_{y_i} \psi_i - \psi_i[{}^{z_i}/{}_{y_i}]\big) + \sup_{y_k} \psi_k - \sup_{z_k} \psi_k[{}^{z_k}/{}_{y_k}] \\ &= \sum_{i<k} \big(\sup_{y_i} \psi_i - \psi_i[{}^{z_i}/{}_{y_i}]\big) + \sup_{y_k} \psi_k - \sup_{y_k} \psi_k \\ &= \sum_{i<k} \big(\sup_{y_i} \psi_i - \psi_i[{}^{z_i}/{}_{y_i}]\big).\end{aligned}$$

This contradicts the minimality of k. $\blacksquare_{0.10}$

Theorem 0.11 (Completeness Theorem)**.** *Let $I \subseteq \mathcal{L}$ be a proper ideal. Then there exists an $\mathcal{L}$-structure $\mathcal{M}$ such that the mapping $\mathcal{L} \to \mathcal{L}^{\mathcal{M}}$ factors via $\mathcal{L}/I$.*

Proof. We have $\mathcal{L} = \mathcal{L}_{\mathcal{V}}$, and let $\mathcal{L}' = \mathcal{L}_{\mathcal{V}'}$ and Σ_h be as above. Let $I' \subseteq \mathcal{L}'$ be the ideal generated by I and Σ_h. By Lemma 0.10 it is a proper ideal, so it is contained in a maximal ideal $\mathfrak{m}$, corresponding to a homomorphism $\lambda\colon \mathcal{L}' \to \mathbb{C}$.

Let $M = \mathcal{T}_{\mathcal{L},\mathcal{V}'}$. The function symbols of $\mathcal{L}$ admit a natural interpretation on M. For the predicate symbols we define:

$$P^{\mathcal{M}}(\tau_0, \ldots, \tau_{n-1}) = \lambda\big(P[{}^{\tau_0}/{}_{x_0}, \ldots, {}^{\tau_{n-1}}/{}_{x_{n-1}}]\big).$$

We claim that for every formula $\varphi \in \mathcal{L}$:

$$\varphi^{\mathcal{M}}[{}^{\tau_i}/{}_{x_i}] = \lambda\big(\varphi[{}^{\tau_i}/{}_{x_i}]\big).$$

Indeed, let $\mathcal{L}_0$ consist of all formulae with this property which in addition only depend on finitely many variables. Then $\mathcal{L}_0$ is a sub-algebra and contains all atomic formulae. If $\varphi \in \mathcal{L}_0$ then $\varphi[{}^{\sigma_j}/{}_{x_j}] \in \mathcal{L}_0$ as well:

$$\begin{aligned}\varphi[{}^{\sigma_j}/{}_{x_j}]^{\mathcal{M}}[{}^{\tau_i}/{}_{x_i}] &= \varphi^{\mathcal{M}}\big[{}^{\sigma_j^{\mathcal{M}}[\tau_i/x_i]}/{}_{x_j}\big] = \varphi^{\mathcal{M}}\big[{}^{\sigma_j[\tau_i/x_i]}/{}_{x_j}\big] \\ &= \lambda\big(\varphi\big[{}^{\sigma_j[\tau_i/x_i]}/{}_{x_j}\big]\big) = \lambda\big(\varphi[{}^{\sigma_j}/{}_{x_j}][{}^{\tau_i}/{}_{x_i}]\big).\end{aligned}$$

Let us now show that if $\varphi \in \mathcal{L}_0$ then $\sup_x \varphi \in \mathcal{L}_0$. We may choose y which does not appear in the finite set of terms $\{\tau_i \colon x_i \in \mathrm{DVar}(\varphi)\}$. Let $\psi = \varphi[{}^{y}/{}_{x}, {}^{\tau_i}/{}_{x_i}]_{x_i \neq x} = \varphi[{}^{y}/{}_{x}][{}^{\tau_i}/{}_{x_i}]_{x_i \neq x,y}$. Then $\psi \in \mathcal{L}_0$,

$$(\sup_x \varphi)[{}^{\tau_i}/{}_{x_i}] = \big(\sup_y \varphi[{}^{y}/{}_{x}]\big)[{}^{\tau_i}/{}_{x_i}]_{x_i \neq x} = \sup_y \varphi[{}^{y}/{}_{x}][{}^{\tau_i}/{}_{x_i}]_{x_i \neq x,y} = \sup_y \psi.$$

Thus:

$$\begin{aligned}(\sup_x \varphi)^{\mathcal{M}}[{}^{\tau_i}/{}_{x_i}] &= \sup\big\{\varphi^{\mathcal{M}}[{}^{\sigma}/{}_{x}, {}^{\tau_i}/{}_{x_i}]_{x_i \neq x} \colon \sigma \in \mathcal{T}_{\mathcal{L}}\big\} \\ &= \sup\big\{\lambda\big(\varphi[{}^{\sigma}/{}_{x}, {}^{\tau_i}/{}_{x_i}]_{x_i \neq x}\big) \colon \sigma \in \mathcal{T}_{\mathcal{L}}\big\} \\ &= \sup\big\{\lambda\big(\psi[{}^{\sigma}/{}_{y}]\big) \colon \sigma \in \mathcal{T}_{\mathcal{L}}\big\} \leq \lambda(\sup_y \psi).\end{aligned}$$

On the other hand, by our finiteness assumption there exists n such that $\{y\}\cup\mathrm{DVar}(\psi) \subseteq \mathcal{V}_n$. For $z = z_{\psi,y} \in \mathcal{V}_{n+1}$:

$$\lambda(\sup_y \psi) = \lambda\big(\psi[{}^{z}/{}_{y}]\big) \leq (\sup_x \varphi)^{\mathcal{M}}[{}^{\tau_i}/{}_{x_i}]$$

Putting together both inequalities, we have shown that:

$$(\sup_x \varphi)^{\mathcal{M}}[{}^{\tau_i}/{}_{x_i}] = \lambda(\sup_y \psi) = \lambda\big((\sup_x \varphi)[{}^{\tau_i}/{}_{x_i}]\big).$$

We have shown by induction that $\mathcal{L}_0$ is dense in $\mathcal{L}$. Thus for $\varphi \in \mathcal{L}$ there is a sequence $\varphi_n \to \varphi$ in $\mathcal{L}_0$. Then $\varphi_n^{\mathcal{M}} \to \varphi^{\mathcal{M}}$, whereby $\varphi^{\mathcal{M}}[{}^{\tau_i}/{}_{x_i}] = \lambda\big(\varphi[{}^{\tau_i}/{}_{x_i}]\big)$ as well.

Finally, $I \subseteq \ker\lambda$ so $\mathcal{L} \to \mathcal{L}^{\mathcal{M}}$ factors through $\mathcal{L}/I$. $\blacksquare_{0.11}$

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