FRAÏSSÉ LIMITS OF METRIC STRUCTURES

ITAÏ BEN YAACOV

Abstract. We define *Fraïssé classes* of metric structures and *Fraïssé limits* thereof, proving uniqueness, universality and approximate homogeneity, with subsequent generalisations to *weak Fraïssé classes*. We introduce and use the formalism of *(strictly) approximate isomorphisms*, which has some advantages over the more familiar formalism of finite isomorphisms which are allowed to be subsequently modified by no more than a given error term.

Introduction

The notions of Fraïssé classes and Fraïssé limits were originally introduced by Roland Fraïssé [Fra54], as a method to construct countable homogeneous (discrete) structures:

(i) Every Fraïssé class $\mathcal{K}$ has a Fraïssé limit, which is unique (up to isomorphism). The limit is countable and ultra-homogeneous (or, in more model-theoretic terminology, quantifier-free-homogeneous).
(ii) Conversely, every countable ultra-homogeneous structure is the limit of a Fraïssé class, namely, its *age*.

Moreover, the limit is universal for countable $\mathcal{K}$-structures, namely for countable structures whose age is contained in $\mathcal{K}$.

Similar results hold for metric structures as well. Indeed, some general theory of this form is discussed by Schoretsanitis [Sch07]. Independently of this, Kubiś and Solecki [KS] treated the special case of the class of finite dimensional Banach spaces, essentially showing that their Fraïssé limit is the Gurarij space, which is therefore unique and universal, without ever actually uttering the phrase "Fraïssé limit" (and in a fashion which is very specific to Banach spaces). This multitude of somewhat incompatible approaches, reinforced by considerable nagging from Todor Tsankov and some personal interest in the generalisation to weak Fraïssé classes (see below), convinced the author of the potential usefulness of the present paper. It contains, in addition to what we hope is a clean and comprehensive treatment, two main novelties.

The first novelty is the use of the formalism of *approximate isometries* (which is just a fancy term for bi-Katětov maps) and *strictly approximate isomorphisms*. Approximate isometries allow us to code in a single, hopefully natural, object, notions such as a partial isometry between metric spaces, or even a "partial isometry only known up to an error $\varepsilon > 0$". On a technical level, approximate isometries are easier to manipulate than, say, partial isometries, and can be freely composed and inverted without loss of information (see for example the remarks at the beginning of Section 1). In our context, the use of approximate isometries dispenses with the need for several limit constructions – indeed, the only limit construction is in the back-and-forth argument of Theorem 2.16, and its counterpart Proposition 5.10, which, even in the discrete case, is in essence a limit construction. Had we followed the (hitherto?) standard formalism, using exact (rather than approximate) partial maps, and taken the condition of Corollary 2.17(iii) as the *definition* of a Fraïssé limit, the construction of such exact maps at intermediary stages would have required many additional limit constructions (which do not appear in the treatment of discrete Fraïssé classes), of the form "choose a finite isomorphism, then change it by ε, then change it by $\varepsilon/2$ more, and so on" (compare with the proofs of Facts 1.4 and 1.5 of [BU07], which could be accordingly simplified using the formalism introduced here). At the same time, the definition of a Fraïssé limit via strictly approximate isomorphisms is formally weaker than (although provably equivalent to)

2010 *Mathematics Subject Classification.* 03C30,03C52.

Key words and phrases. Fraïssé class, Fraïssé limit, metric structure, homogeneous structure, perturbation, Gurarij space.

Author supported by the Institut Universitaire de France.

The author wishes to thank Julien Melleray for many useful discussions, and Todor Tsankov for pushing him to write this paper.

Revision 1342 *of* 11th April 2012.

the condition of Corollary 2.17(iii), making it easier to prove that a given structure is a Fraïssé (we have in mind the Gurarij space and the proof of Theorem 3.3).

The second novelty is the introduction of *weak Fraïssé classes*. These can be viewed as "perturbed" versions of ordinary Fraïssé classes, somewhat in the spirit [Ben08b]. Uniqueness and universality of the limits still hold, but possibly only up to arbitrarily small error. Essentially all the results we prove for Fraïssé classes follow as special cases of their weak Fraïssé classes counterparts, rendering the treatment of the former technically superfluous. However, since the definition of weak Fraïssé classes is somewhat more involved than that of ordinary Fraïssé classes, relying on a further abstraction of the approximate isomorphism formalism, it seems best to treat the more concrete situation of Fraïssé classes first.

The formalism of approximate isometries is introduced in Section 1. In Section 2 we define Fraïssé classes and Fraïssé limits of metric structures, proving existence, uniqueness and universality of the Fraïssé limit. Some examples are given in Section 3. In Section 5 we present the generalisation to weak Fraïssé classes, whose relations with [Ben08b] are briefly explored in Section 6.

1. Approximate isometries

Approximate isometries, or bi-Katětov maps, provide a convenient manner to code partial information regarding an isometric map between metric spaces. We contend that in a sense, these form a more accurate generalisation to the metric setting of partial maps between sets (discrete spaces) than, say, partial isometric maps, and in any case they are much more flexible. Just as an example, let $f, g\colon X \to Y$ be very close (say they are total, and $d(fx, gx) < \varepsilon$ for all $x \in X$), and let $h\colon Y \dashrightarrow Z$ be partial, with $\operatorname{dom} h \supseteq \operatorname{img} f$ but $\operatorname{dom} h \cap \operatorname{img} g = \varnothing$. We should want hf and hg to be very close as well, which, if we try to compose them as partial maps, is either meaningless or, if we force things, simply false (with $\operatorname{dom} hg = \varnothing$ while $\operatorname{dom} hf = X$). On the other hand, the composition as approximate isometries is always meaningful and keeps pertinent information, so in the situation described above hf and hg are at least as close as f and g are.

Definition 1.1 (see also Uspenskij [Usp08]). Let X, Y and Z denote metric spaces.

(i) Given any $\psi\colon X \times Y \to [0, \infty]$ and $\varphi\colon Y \times Z \to [0, \infty]$ we define a *composition* $\varphi\psi\colon X \times Z \to [0, \infty]$ and a *pseudo-inverse* $\psi^*\colon Y \times X \to [0, \infty]$ by

$$\varphi\psi(x, z) = \inf_{y \in Y} \psi(x, y) + \varphi(y, z), \qquad \psi^*(y, x) = \psi(x, y).$$

We observe that $\psi = \psi d_X$ (respectively, $\psi = d_Y\psi$) if and only if ψ is 1-Lipschitz in the first (respectively, second) argument, or if it identically equal to ∞. We say that ψ is *Katětov* in the first (respectively, second) argument if $\psi = \psi d_X$ and $d_X \leq \psi^*\psi$ (respectively, if $\psi = d_Y\psi$ and $d_Y \leq \psi\psi^*$).

(ii) We say that $\psi\colon X \times Y \to [0, \infty]$ is an *approximate isometry* from X to Y, and write $\psi\colon X \rightsquigarrow Y$, if it is bi-Katětov, i.e., Katětov in both arguments. The approximate isometry $\psi = \infty$ identically is called the *empty approximate isometry*, and we observe that it is a destructive element for composition.

(iii) We identify an ordinary isometry $f\colon X \to Y$ with $\psi_f(x, y) = d(fx, y)$. More generally, we identify a partial isometry $f\colon X \dashrightarrow Y$ with $\psi_{f'}\psi_i^*$ where $f'\colon \operatorname{dom} f \to Y$ is f viewed as a total map on its domain, and $i\colon \operatorname{dom} f \to X$ is the inclusion. In particular, id_X is identified with d_X, the neutral element for composition.

(iv) Let $\psi\colon X \rightsquigarrow X$ and $\theta\colon X \rightsquigarrow Y$ be approximate isometries. Then we define, as for ordinary maps, $\psi^\theta = \theta\psi\theta^*\colon Y \rightsquigarrow Y$.

(v) Let $\psi\colon X \rightsquigarrow Y$ be an approximate isometry and $i\colon X \subseteq X'$, $j\colon Y \subseteq Y'$ isometric inclusions, or embeddings. Then $j\psi i^*\colon X' \rightsquigarrow Y'$ is called the *trivial extension* of ψ to $X' \rightsquigarrow Y'$ (compare with the analogous notion for ordinary partial maps).

(vi) If $\psi, \varphi\colon X \rightsquigarrow Y$ are approximate isometries, we say that ψ *approximates* φ, or that φ *refines* ψ, if $\psi \geq \varphi$. We write $\psi > \varphi$ if (and only if) there are finite $X_0 \subseteq X$, $Y_0 \subseteq Y$, and $\chi\colon X_0 \rightsquigarrow Y_0$ such that $\psi \geq \chi \geq \varphi + \varepsilon$ for some $\varepsilon > 0$ (which is equivalent, by finiteness of X_0 and Y_0, to $\psi \geq \chi > \varphi$ point by point). We then say that ψ *strictly approximates* φ, which *strictly refines* ψ. Notice that the empty approximate isometry strictly approximates all approximate isometries, including itself.

(vii) We say that an approximate isometry $\psi\colon X \rightsquigarrow Y$ is *r-total* for some $r > 0$ if $\psi^*\psi \leq \operatorname{id}_X + 2r$, or equivalently, if for all $x \in X$ and $s > r$ there is $y \in Y$ such that $\psi(x, y) < s$. If $\psi\psi^* \leq \operatorname{id}_Y + 2r$ then we say that ψ is *r-surjective* and if it is both then it is *r-bijective*.

We leave the following to the reader:

Lemma 1.2. (i) *Let* $\psi\colon X \times Y \to [0,\infty)$ *be given and let* $Z = X \amalg Y$. *Define* d_Z *extending* d_X *and* d_Y *by* $d(x,y) = d(y,x) = \psi(x,y)$. *Then* ψ *is an approximate isometry if and only if* d *is a pseudo-distance on* Z.

(ii) $(\psi^*)^* = \psi$, $(\varphi\psi)^* = \psi^*\varphi^*$.

(iii) *(Pseudo-)inversion is compatible with the identification of partial isometries with approximate ones. Similarly for composition* $\psi_g\psi_f = \psi_{gf}$ *when* $\operatorname{dom} g \supseteq \operatorname{img} f$ *or* $\operatorname{dom} g \subseteq \operatorname{img} f$.

(iv) *If* $\psi' > \psi\colon X \rightsquigarrow Y$ *and* $\varphi' > \varphi\colon Y \rightsquigarrow Z$ *then* $\varphi'\psi' > \varphi\psi$. *Conversely, if* $\rho > \varphi\psi$ *then there are* $\varphi' > \varphi$ *and* $\psi' > \psi$ *such that* $\rho \geq \varphi'\psi'$.

(v) *If* $\varphi > \psi\colon Y \rightsquigarrow Z$, $\rho\colon X \rightsquigarrow Y$ *and* X *is finite then* $\varphi\rho > \psi\rho$.

(vi) *Assume that* $\psi\colon X \rightsquigarrow Y$ *is an approximate isometry, with both* $X, Y \subseteq Z$, *and let us identify* ψ *with its trivial extension to* $\psi\colon Z \rightsquigarrow Z$. *Then* ψ *either approximates all of* id_X, id_Y *and* id_Z *(all viewed as approximate isometries* $Z \rightsquigarrow Z$*) or none. We shall therefore allow ourselves to say in such situations that "*ψ *approximates* id*" without further precision. Similarly, if* X *is finite then* $\psi > \mathrm{id}_X$ *if and only if* $\psi > \mathrm{id}_Z$, *and similarly for* Y, *in which case we say that* $\psi > \mathrm{id}$ *without further qualification.*

Lemma 1.3. *Let* X *be a metric space. Then the following are equivalent:*

(i) *The space* X *is totally bounded (and if complete, then it is compact).*

(ii) *For every* $\varepsilon > 0$ *we have* $\mathrm{id}_X + \varepsilon > \mathrm{id}_X$.

(iii) *For every* $\varepsilon > 0$ *there exist* Y *and approximate isometries* $\psi, \varphi\colon X \rightsquigarrow Y$ *such that* $\psi > \varphi$ *and* ψ *is* ε*-total.*

Proof. (i) $\Longrightarrow$ (ii). Choose a finite subset $X_0 \subseteq X$ such that $X \subseteq B(X_0, \varepsilon/3)$. Then $\mathrm{id}_X + \varepsilon \geq \mathrm{id}_{X_0} + \varepsilon/3 > \mathrm{id}_{X_0} \geq \mathrm{id}_X$.

(ii) $\Longrightarrow$ (iii). Immediate.

(iii) $\Longrightarrow$ (i). Assume that X is not totally bounded. Then there exists $\varepsilon > 0$ such that for every finite $X_0 \subseteq X$ there is $x \in X \smallsetminus B(X_0, \varepsilon)$. However, if $\psi > \varphi$ then we may assume that φ extends trivially from $X_0 \times Y$ for some finite X_0, in which case ψ cannot be ε-total. $\blacksquare_{1.3}$

Definition 1.4. Now let X be a metric space and $\mathcal{A} \subseteq \mathrm{Apx}(X)$.

(i) For two metric spaces X and Y we define

$$\mathrm{Apx}(X,Y) = \{\psi\colon X \rightsquigarrow Y\} \subseteq [0,\infty]^{X \times Y}, \qquad \mathrm{Apx}(X) = \mathrm{Apx}(X,X).$$

We equip this space with the topology of point-wise convergence, namely with the induced topology as a subspace of $[0,\infty]^{X\times Y}$.

(ii) Given $\mathcal{A} \subseteq \mathrm{Apx}(X,Y)$, we define $\mathcal{A}^\uparrow = \{\psi \in \mathrm{Apx}(X,Y) \colon \exists \varphi \in \mathcal{A}, \psi \geq \varphi\}$ to be the closure of $\mathcal{A}$ under approximation. We observe that its topological closure $\overline{\mathcal{A}^\uparrow}$ is still closed under approximation.

(iii) Given $\mathcal{A} \subseteq \mathrm{Apx}(X)$, we define $\langle \mathcal{A} \rangle$ to consist of the closure of $\mathcal{A}$ under pseudo-inversion and composition. We observe that its topological/approximation closure $\overline{\langle \mathcal{A} \rangle^\uparrow}$ is still closed under pseudo-inversion and composition.

Lemma 1.5. *The space* $\mathrm{Apx}(X,Y)$ *is compact, and the composition map* $\mathrm{Apx}(X,Y) \times \mathrm{Apx}(Y,Z) \to \mathrm{Apx}(X,Z)$ *is separately continuous, although not necessarily jointly so (unless* Y *is compact). In particular,* $\mathrm{Apx}(X)$ *is a semi-topological semi-group. Moreover, the interpretation of actual isometries as approximate isometries yields a topological embedding* $\mathrm{Iso}(X) \subseteq \mathrm{Apx}(X)$ *of the isometry group in this semi-group. See also Uspenskij* [Usp08].

2. Metric Fraïssé limits via approximate maps

Definition 2.1. Let $\mathcal{L}$ be denote a collection of symbols, each being either a *predicate symbol* or a *function symbol* and each having an associated natural number called its *arity*. An $\mathcal{L}$*-structure* $\mathfrak{A}$ consists of a complete metric space A, together with,

- For each n-ary predicate symbol R, a continuous interpretation $R^{\mathfrak{A}}\colon A^n \to \mathbf{R}$. It will be convenient to consider the distance as a (distinguished) binary predicate symbol.
- For each n-ary function symbol f, a continuous interpretation $f^{\mathfrak{A}}\colon A^n \to A$. A zero-ary function is also called a *constant*.

If $\mathfrak{A}$ is a structure and $A_0 \subseteq A$, then the smallest substructure of $\mathfrak{A}$ containing A_0 is denoted $\langle A_0 \rangle$, the substructure *generated* by A_0. Its underlying set is just the metric closure of A_0 under the interpretations of function symbols.

An *embedding* of $\mathcal{L}$-structures $\varphi \colon \mathfrak{A} \to \mathfrak{B}$ is a map which commutes with the interpretation of the language: $R^{\mathfrak{B}}(\varphi\bar{a}) = R^{\mathfrak{A}}(\bar{a})$ and $f^{\mathfrak{B}}(\varphi\bar{a}) = \varphi f^{\mathfrak{A}}(\bar{a})$ (in particular, $d^{\mathfrak{B}}(\varphi a, \varphi b) = d^{\mathfrak{A}}(a, b)$, so an embedding is always isometric).

Remark 2.2. The definition given here is more relaxed than definitions given in more general treatments of continuous logic, such as [BU10, BBHU08] for the bounded case and [Ben08a] for the general (unbounded) case, in that we only require plain continuity (rather than uniform), and no kind of boundedness. Indeed, let us consider the following properties of a map $f \colon X \to Y$ between metric spaces, which imply one another from top to bottom:

(i) The map f is uniformly continuous.
(ii) The map f sends Cauchy sequences to Cauchy sequences (equivalently, f admits a continuous extension to the completions, $\hat{f} \colon \widehat{X} \to \widehat{Y}$). Not having found an explicit name for this in the literature, let us call this *Cauchy continuity*.
(iii) The map f is continuous.

If X is complete then the last two properties coincide, if X is totally bounded then the first two coincide, and if X is compact then all three do. Thus Cauchy continuity is intimately connected with completeness. Similarly, although possibly less clearly, uniform continuity is intimately related with compactness: on the one hand, compactness implies uniform continuity, while on the other hand, uniform continuity of the language is a crucial ingredient in the proof of compactness for first order continuous logic (similarly, in unbounded logic, compactness below every bound corresponds to uniform continuity on bounded sets).

In light of this, and since compactness will not intervene in any way in our treatment, plain continuity on complete spaces, and Cauchy continuity on incomplete ones, are what we need.

Convention 2.3. We equip products of metric spaces with the supremum distance, so for two n-tuples $\bar{a}$ and $\bar{b}$ we have $d(\bar{a}, \bar{b}) = \max_i d(a_i, b_i)$.

Definition 2.4. Let $\mathcal{K}$ be a class of finitely generated $\mathcal{L}$-structures. For $n \geq 0$, we let $\mathcal{K}_n$ denote the class of all pairs $(\bar{a}, \mathfrak{A})$, where $\mathfrak{A} \in \mathcal{K}$ and $\bar{a} \in A^n$ generates $\mathfrak{A}$. By an abuse of notation, we shall refer to $(\bar{a}, \mathfrak{A}) \in \mathcal{K}_n$ by $\bar{a}$ alone, and denote the generated structure $\mathfrak{A}$ by $\langle \bar{a} \rangle$.

Notice that we consider $\bar{a}$ as an ordered tuple, and that we allow repetitions, so in situations where we have several tuples, we may usually assume they are of the same length.

Definition 2.5. The *age* of an $\mathcal{L}$-structure $\mathfrak{M}$, denoted $\mathrm{Age}(\mathfrak{M})$, is the class of finitely generated structures which embed in $\mathfrak{M}$. If $\mathcal{K}$ is a class of finitely generated structures then by a *$\mathcal{K}$-structure* we mean an $\mathcal{L}$-structure whose age is contained in $\mathcal{K}$.

We now define notions of (strictly) approximate isomorphisms and distance on $\mathcal{K}_n$ which are intrinsic to a class $\mathcal{K}$ of finitely generated structures. Since in this section we shall only consider these intrinsic notions, we shall most of the time drop the qualifier "intrinsic". However, more general notions will be considered in Section 5.

Definition 2.6. Let $\mathcal{K}$ be a class of finitely generated structures, $\bar{a} \in \mathcal{K}_n$, $\bar{b} \in \mathcal{K}_m$. We say that an approximate isometry $\psi \colon \bar{a} \rightsquigarrow \bar{b}$ is a *($\mathcal{K}$-intrinsic) strictly approximate isomorphism* if there are $\mathfrak{C} \in \mathcal{K}$ and embeddings $i \colon \langle \bar{a} \rangle \to \mathfrak{C}$, $j \colon \langle \bar{b} \rangle \to \mathfrak{C}$ such that $j\psi i^* > \mathrm{id}$, or equivalently $\psi > j^*i$:

$$\begin{array}{ccc} \bar{a} & \xrightarrow{\psi > j^* i} & \bar{b} \\ \downarrow i & & \downarrow j \\ \mathfrak{C} & \overset{j\psi i^* > \mathrm{id}}{=\!=\!=} & \mathfrak{C} \end{array}$$

We let $\mathrm{Stx}(\bar{a}, \bar{b})$ denote the set of strictly approximate isomorphisms from $\bar{a}$ to $\bar{b}$. Given a $\mathcal{K}$-structure $\mathfrak{B}$, an approximate isometry $\psi \colon \bar{a} \rightsquigarrow B$ is said to be a strictly approximate isomorphism if it approximates some $\psi' \in \mathrm{Stx}(\bar{a}, \bar{b})$ where $\bar{b} \in B^m$, and we write $\psi \in \mathrm{Stx}(\bar{a}, \mathfrak{B})$. We define $\mathrm{Stx}(\mathfrak{A}, \bar{b})$ and $\mathrm{Stx}(\mathfrak{A}, \mathfrak{B})$ similarly.

Since the identity of a $\mathcal{K}$-structure is *not* a strictly approximate isomorphism, we also define the set $\mathrm{Apx}(\mathfrak{A}, \mathfrak{B})$ of *approximate isomorphisms* to consist of all approximate isometries $\psi \colon A \to B$ all of whose *strict* approximations are in $\mathrm{Stx}(\mathfrak{A}, \mathfrak{B})$.

If $\psi \in \mathrm{Stx}(\bar{a}, \bar{b})$ then there exists $\delta > 0$ such that $\psi - \delta \in \mathrm{Stx}(\bar{a}, \bar{b})$, and we define $\Gamma(\psi)$ to be the supremum of all such δ.

Finally, for $\psi \in \mathrm{Stx}(\mathfrak{M}, \mathfrak{N})$, we define

$$\mathrm{Stx}^{<\psi}(\mathfrak{M}, \mathfrak{N}) = \{\varphi \in \mathrm{Stx}(\mathfrak{M}, \mathfrak{N}) \colon \varphi < \psi\},$$

and similarly for obvious variations.

It is clear from the definition that if ψ is a (strictly) approximate isomorphism then so is ψ^* and that an approximate isometry between $\mathcal{K}$-structures is a strictly approximate isomorphism if and only if it is a strict approximation of an approximate isomorphism. Also, $\mathrm{id} \in \mathrm{Apx}(\mathfrak{A}, \mathfrak{A})$, and if $\mathcal{K}$ is a Fraïssé class then by the following, $\mathcal{K}$ together with approximate isomorphisms form a category – more precisely, they form an *approximate category* as we shall define in Section 5.

Definition 2.7. Let $\mathcal{K}$ be a class of finitely generated structures. We define a *($\mathcal{K}$-intrinsic) pseudo-distance* on $\mathcal{K}_n$ by

$$d^{\mathcal{K}}(\bar{a}, \bar{b}) = \inf_{\psi \in \mathrm{Stx}(\bar{a}, \bar{b})} d(\psi), \qquad \text{where } d(\psi) = \max_i \psi(a_i, b_i).$$

Equivalently, $d(\bar{a}, \bar{b})$ is the infimum of all possible $d(\bar{a}, \bar{b})$ under embeddings of $\langle \bar{a} \rangle$ and $\langle \bar{b} \rangle$ into some $\mathfrak{C} \in \mathcal{K}$. In fact this is only a pseudo-distance under some additional hypotheses, and in particular it is if strictly approximate isomorphisms compose.

Lemma 2.8. *The following are equivalent for a class $\mathcal{K}$ of finitely generated structures:*

(i) *Say $\mathfrak{A}, \mathfrak{B}_i \in \mathcal{K}$ and $\varphi_i \colon \mathfrak{A} \to \mathfrak{B}_i$ are embeddings, for $i < 2$. Then for every finite tuple $\bar{a} \in A^n$ and $\varepsilon > 0$ there are $\mathfrak{C} \in \mathcal{K}$ and embeddings $\psi_i \colon \mathfrak{B}_i \to \mathfrak{C}$ such that $d(\psi_0\varphi_0\bar{a}, \psi_1\varphi_1\bar{a}) < \varepsilon$.*
(ii) *The composition of any two strictly approximate isomorphisms in $\mathcal{K}$ is one as well.*
(iii) *The composition of any two approximate isomorphisms in $\mathcal{K}$ is one as well.*
(iv) *Every partial isomorphism, in the classical sense, between members of $\mathcal{K}$, is an approximate isomorphism.*

Under these equivalent conditions, $d^{\mathcal{K}}$ is a pseudo-distance on $\mathcal{K}_n$.

Proof. (i) $\Longrightarrow$ (ii). Easy.
(ii) $\Longrightarrow$ (iii). By Lemma 1.2(iv).
(iii) $\Longrightarrow$ (iv). Since an embedding is an approximate isomorphism.
(iv) $\Longrightarrow$ (i). Immediate. $\blacksquare_{2.8}$

Definition 2.9. A *Fraïssé class* (of $\mathcal{L}$-structures) is a class $\mathcal{K}$ of finitely generated $\mathcal{L}$-structures having the following properties:

- *HP (Hereditary Property)*: Every finitely generated structure which embeds in a member of $\mathcal{K}$ is in $\mathcal{K}$.
- *JEP (Joint Embedding Property)*: Every two members of $\mathcal{K}$ embed in a third one.
- *NAP (Near Amalgamation Property)*: Any of the equivalent conditions of Lemma 2.8.
- *PP (Polish Property)*: The pseudo-metric $d^{\mathcal{K}}$ is separable and complete on $\mathcal{K}_n$ for each n.
- *CP (Continuity Property)*: Each n-ary predicate symbol P (respectively, function symbol f) is continuous on $\mathcal{K}$, by which we mean that if $(\bar{a}_k, \bar{b}_k) \to (\bar{a}, \bar{b})$ in $(\mathcal{K}_{n+m}, d^{\mathcal{K}})$ then $P^{\langle \bar{a}_k, \bar{b}_k \rangle}(\bar{a}_k) \to P^{\langle \bar{a}, \bar{b} \rangle}(\bar{a})$ (respectively, $(\bar{a}_k, \bar{b}_k, f^{\langle \bar{a}_k, \bar{b}_k \rangle}(\bar{a}_k)) \to (\bar{a}, \bar{b}, f^{\langle \bar{a}, \bar{b} \rangle}(\bar{a}))$ in $(\mathcal{K}_{n+m+1}, d^{\mathcal{K}})$).

We say that it is an *incomplete Fraïssé class* if instead of PP & CP we have:

- *WPP (Weak Polish Property)*: The pseudo-metric $d^{\mathcal{K}}$ is separable on $\mathcal{K}_n$ for each n.
- *CCP (Cauchy Continuity Property)*: For each n-ary predicate symbol P (respectively, function symbol f) and Cauchy sequence $(\bar{a}_k, \bar{b}_k)$ in $(\mathcal{K}_{n+m}, d^{\mathcal{K}})$, the sequence $P^{\langle \bar{a}_k, \bar{b}_k \rangle}(\bar{a}_k)$ (respectively, $(\bar{a}_k, \bar{b}_k, f^{\langle \bar{a}_k, \bar{b}_k \rangle}(\bar{a}_k))$) is Cauchy as well.

Remark 2.10. We observe that:

(i) CP implies that the kernel of $d^{\mathcal{K}}$ on $\mathcal{K}_n$ is exactly the isomorphism relation, namely $d^{\mathcal{K}}(\bar{a}, \bar{b}) = 0$ implies that there exists a (necessarily unique) isomorphism $\varphi \colon \langle \bar{a} \rangle \to \langle \bar{b} \rangle$ sending $\bar{a} \mapsto \bar{b}$.
(ii) Together with PP this implies that a $\mathcal{K}$-structure generated by a set of cardinal κ has density character at most $\kappa + \aleph_0$ (even if the language contains more than κ symbols). In particular, every member of $\mathcal{K}$ is separable.

(iii) Every Fraïssé class is in particular an incomplete Fraïssé class, and conversely, every incomplete Fraïssé class $\mathcal{K}$ admits a unique *completion* $\widehat{\mathcal{K}}$, consisting of all limits of Cauchy sequences in $\mathcal{K}$ (that is, in $\mathcal{K}_n$, as n varies), which is a Fraïssé class.

(iv) JEP is equivalent to saying that the empty approximate isometry is always an approximate isomorphism. Modulo NAP, JEP is further equivalent to there being a unique $\varnothing$-generated (empty, if there are no constant symbols) structure in $\mathcal{K}$.

We give some examples for these definitions in Section 3

Definition 2.11. Let $\mathcal{K}$ be a Fraïssé class. By a *limit* of $\mathcal{K}$ we mean a separable $\mathcal{K}$-structure $\mathfrak{M}$, satisfying that for every $\bar{a} \in \mathcal{K}_n$, $\psi \in \mathrm{Stx}(\bar{a}, \mathfrak{M})$ and $\varepsilon > 0$ there exists an ε-total $\varphi \in \mathrm{Stx}^{<\psi}(\bar{a}, \mathfrak{M})$.

Lemma 2.12. *Let $\mathcal{K}$ be a Fraïssé class, and $\mathfrak{M}$ a $\mathcal{K}$-structure. Let $M_0 \subseteq M$ be dense, and for each n let $\mathcal{K}_{n,0} \subseteq \mathcal{K}_n$ be $d^{\mathcal{K}}$-dense. Assume furthermore that the property of a Fraïssé limit holds whenever $\bar{a} \in \mathcal{K}_{n,0}$, $\psi \in \mathrm{Stx}(\bar{a}, \bar{b})$ for some $\bar{b} \in M_0^n$, $\varepsilon \in \mathbf{Q}^{>0}$, and $\psi\restriction_{\bar{a}\times\bar{b}}$ only takes rational values. Then $\mathfrak{M}$ is a limit of $\mathcal{K}$, and moreover, φ as in the conclusion of Definition 2.11 can be taken to belong to* $\mathrm{Stx}^{<\psi}(\bar{a}, \bar{b}')$*, where $\bar{b}' \in M_0^\ell$ as well (for some ℓ), and to take rational values of $\bar{a} \times \bar{b}'$.*

Proof. Let $\bar{a} \in \mathcal{K}_n$, $\psi \in \mathrm{Stx}(\bar{a}, \mathfrak{M})$ and $\varepsilon > 0$, and we may assume that $\psi \in \mathrm{Stx}(\bar{a}, \bar{b})$ for some $\bar{b} \in M^m$. Possibly increasing n, and extending $\bar{a}$ and $\bar{b}$ arbitrarily, we may assume that $m = n$, and decreasing ε we may assume it is rational. Let $\delta = \frac{1}{4}\min \varepsilon, \Gamma(\psi)$. Choose $\bar{b}' \in M_0^n$ with $d(\bar{b}, \bar{b}') < \delta$, and let $\chi = d\restriction_{\bar{b}\times\bar{b}'} \in \mathrm{Apx}(\bar{b}, \bar{b}')$. Let also $\bar{a}' \in \mathcal{K}_{n,0}$ with $d^{\mathcal{K}}(\bar{a}, \bar{a}') < \delta$, and let $\rho \in \mathrm{Stx}(\bar{a}, \bar{a}')$ witness this, namely satisfy $d(\rho) < \delta$ as per Definition 2.7. Finally, let $\psi' = \chi\psi\rho - 3\delta \in \mathrm{Stx}(\bar{a}', \bar{b}')$, and then choose $\psi'' \in \mathrm{Stx}^{<\psi'}(\bar{a}', \bar{b}')$ which in addition only takes rational values. By assumption there exists an $(\varepsilon - \delta)$-total $\psi'' > \varphi' \in \mathrm{Stx}(\bar{a}', \mathfrak{M})$. Then $\varphi = \varphi'\rho^* \in \mathrm{Stx}(\bar{a}, \mathfrak{M})$ is ε-total and $\varphi < \chi\psi\rho\rho^* - 3\delta < \psi$, as desired.

We leave the moreover part to the reader. ■$_{2.12}$

Lemma 2.13. *Every Fraïssé class $\mathcal{K}$ admits a limit.*

Proof. We construct an increasing chain of $\mathfrak{A}_n \in \mathcal{K}$, starting with $\mathfrak{A}_0$ being the unique $\varnothing$-generated structure in $\mathcal{K}$. For each n we fix a countable $d^{\mathcal{K}}$-dense subset of $\mathcal{K}_n$, call it $\mathcal{K}_{n,0}$, and a countable dense subset $A_{n,0} \subseteq A_n$, such that $A_{n,0} \subseteq A_{n+1,0}$. For each $\bar{a} \in A_{n,0}^n$, $\bar{b} \in \mathcal{K}_{n,0}$ and rational-valued $\psi \in \mathrm{Stx}(\bar{b}, \bar{a})$ we make sure there is some m and an embedding $\psi > \varphi\colon \langle\bar{b}\rangle \to \mathfrak{A}_m$. By PP and CP, the chain $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \ldots$ admits a unique limit in the category of $\mathcal{K}$-structures, which we denote by $\mathfrak{M} = \bigcup \mathfrak{A}_n$. Now $M_0 = \bigcup A_{n,0} \subseteq M$ is a countable dense subset, and by Lemma 2.12, $\mathfrak{M}$ is a limit. ■$_{2.13}$

In fact, we can do better. For $\bar{a} \in \mathcal{K}_n$ let $[\bar{a}]$ denote the equivalence class $\bar{a}/\ker d^{\mathcal{K}}$, and let $\overline{\mathcal{K}}_n = \mathcal{K}_n/\ker d^{\mathcal{K}}$ denote the quotient space, equipped with the quotient metric (which is separable and complete, by PP). For each n we have a natural map $\overline{\mathcal{K}}_{n+1} \to \overline{\mathcal{K}}_n$, sending $[a_0, \ldots, a_n] \mapsto [a_0, \ldots, a_{n-1}]$, giving rise to an inverse system with a limit $\overline{\mathcal{K}}_\omega = \varprojlim \overline{\mathcal{K}}_n$, equipped with the topology induced from $\prod_n \overline{\mathcal{K}}_n$. A member of $\overline{\mathcal{K}}_\omega$ will be denoted by ξ, represented by a compatible sequence $(\xi_n)_{n\in\mathbf{N}}$. Considering limits of increasing chains as in the proof of Lemma 2.13, we see that for every $\xi \in \overline{\mathcal{K}}_\omega$ there exists a $\mathcal{K}$-structure $\mathfrak{M}^\xi$ along with a generating sequence $\bar{a}^\xi = (a_i^\xi)_{i\in\mathbf{N}} \subseteq M^\xi$, such that $\xi_n = [a_{<n}^\xi]$ for all n, and this pair $(\mathfrak{M}^\xi, \bar{a}^\xi)$ is determined by ξ up to a unique isomorphism. Conversely, any pair of a (separable) $\mathcal{K}$-structure $\mathfrak{M}$ and a generating $\mathbf{N}$-sequence is of this form.

Lemma 2.14. *Let $\bar{b} \in \mathcal{K}_n$ and $\psi\colon \bar{b} \times \ell \to \mathbf{R}^{>0}$. For $\xi \in \overline{\mathcal{K}}_\ell$ let $\bar{a}^\xi \in \mathcal{K}_\ell$ be any representative, and say that ψ^ξ is defined if $\psi(b_k, i) = \psi(b_k, j)$ whenever $i < j < \ell$ and $a_i^\xi = a_j^\xi$. If so, define $\psi^\xi(b_k, a_i^\xi) = \psi(b_k, i)$. Then the set of $\xi \in \overline{\mathcal{K}}_\ell$ such that ψ^ξ is defined and belongs to* $\mathrm{Stx}(\bar{b}, \bar{a}^\xi)$ *(a property which does not depend on the choice of $\bar{a}^\xi$) is open.*

Proof. Easy. ■$_{2.14}$

Proposition 2.15. *Let $\mathcal{K}$ be a Fraïssé class, and let $\overline{\mathcal{K}}_\omega$ be as above. Then $\overline{\mathcal{K}}_\omega$ is a Polish space, and the set of $\xi \in \overline{\mathcal{K}}_\omega$ for which $\mathfrak{M}^\xi$ is a limit of $\mathcal{K}$ is a dense G_δ set. Moreover, the set of $\xi \in \overline{\mathcal{K}}_\omega$ for which $\mathfrak{M}^\xi$ is a limit and every tail of the sequence (a_i^ξ) is dense in $\mathfrak{M}^\xi$ is a dense G_δ set.*

Proof. That this is a Polish space is clear. Now fix n, $\bar{b} \in \mathcal{K}_n$, $\varepsilon > 0$, and a map $\psi\colon \bar{b} \times n \to \mathbf{R}^{>0}$. For $\xi \in \overline{\mathcal{K}}_\omega$ let $(\mathfrak{M}^\xi, \bar{a}^\xi)$ be as above, and let $\psi^\xi = \psi^{\xi_n}\colon \bar{b} \times a_{<n}^\xi \to \mathbf{R}$, if defined, as in Lemma 2.14. Let $X \subseteq \overline{\mathcal{K}}_\omega$ consist of all ξ such that if ψ^ξ is defined, and belongs to $\mathrm{Stx}(\bar{b}, a_{<n}^\xi)$, then there exists an

ε-total $\varphi \in \mathrm{Stx}^{<\psi^\xi}(\bar{b}, \mathfrak{M}^\xi)$, which is moreover a trivial extension from $\bar{b} \times a^\xi_{<\ell}$ (for some ℓ). Then by Lemma 2.12, it will be enough to show that X is a dense G_δ set.

By Lemma 2.14 it is the union of a closed set (where ψ^ξ is not defined, or is not in $\mathrm{Stx}(\bar{b}, a^\xi_{<n})$) and an open set (where φ exists), so it is G_δ.

Finally, let $U \subseteq \overline{\mathcal{K}}_\omega$ be open and $\xi \in U$. If $\psi^\xi \notin \mathrm{Stx}(\bar{b}, a^\xi_{<n})$ then $\xi \in X \cap U$ and we are done. Otherwise, we may assume that U is the inverse image in $\overline{\mathcal{K}}_\omega$ of an open set $V \subseteq \overline{\mathcal{K}}_\ell$, with $\ell \geq n$ and $\xi_\ell \in V$. Since $\mathcal{K}$ is a Fraïssé class, there exists an extension $\langle a^\xi_{<\ell} \rangle \subseteq \mathfrak{C} \in \mathcal{K}$ and an embedding $\psi^\xi > \varphi \colon \langle \bar{b} \rangle \to \mathfrak{C}$, and we may assume that $\mathfrak{C} = \langle \bar{c} \rangle$ where $\bar{c} = a^\xi_{<\ell}, \varphi\bar{b}$, so $\bar{c} \in \mathcal{K}_{\ell+n}$. Let $\zeta \in \overline{\mathcal{K}}_\omega$ be any such that $\zeta_{\ell+n} = [\bar{c}]$. Then $\zeta \in U \cap X$, as desired.

The moreover part is proved using similar reasoning and is left to the reader. $\blacksquare_{2.15}$

Theorem 2.16. *Let $\mathcal{K}$ be a Fraïssé class, $\mathfrak{M}$ and $\mathfrak{N}$ separable $\mathcal{K}$-structures, and let $\psi \colon \mathfrak{M} \rightsquigarrow \mathfrak{N}$ be a strictly approximate isomorphism.*

(i) *If $\mathfrak{N}$ is a limit of $\mathcal{K}$ then ψ strictly approximates an embedding $\theta \colon \mathfrak{M} \to \mathfrak{N}$.*
(ii) *If both $\mathfrak{M}$ and $\mathfrak{N}$ are limits of $\mathcal{K}$ then ψ strictly approximates an isomorphism $\theta \colon \mathfrak{M} \cong \mathfrak{N}$.*

In particular (with ψ empty), the limit of $\mathcal{K}$ is unique up to isomorphism.

Proof. We only prove the second assertion, the first being similar and easier. Let $\{a_n\}$ and $\{b_n\}$ enumerate dense subsets of $\mathfrak{M}$ and $\mathfrak{N}$, respectively. We shall construct two increasing sequences of finite tuples $\bar{c}_n \in M^{m_n}$ and $\bar{d}_n \in N^{k_n}$, as well as a decreasing sequence of $\theta_n \in \mathrm{Stx}(\bar{c}_n, \bar{d}_n)$, such that $a_i \in \bar{c}_{i+1}$, $b_i \in \bar{d}_{i+1}$, and θ_n is 2^{-n}-total for odd n and 2^{-n}-surjective for even $n > 0$.

We start with $\bar{c}_0 \subseteq M$ and $\bar{d}_0 \subseteq N$ such that ψ approximates some $\theta_0 \in \mathrm{Stx}(\bar{c}_0, \bar{d}_0)$. Given θ_n, for n even, we let $\bar{c}_{n+1} = \bar{c}_n, a_n$. Then there exists a 2^{-n}-total $\theta_{n+1} \in \mathrm{Stx}^{<\theta_n}(\bar{c}_{n+1}, \bar{d}_{n+1})$ for some $\bar{d}_{n+1} \subseteq N$ which we may assume extends $\bar{d}_n, b_n$. The odd case is treated similarly.

Then $\theta = \lim \theta_n$ is the desired isomorphism. $\blacksquare_{2.16}$

The unique limit of $\mathcal{K}$ will be denoted by $\lim \mathcal{K}$. It can also be characterised in terms of actual maps.

Corollary 2.17. *Let $\mathcal{K}$ be a Fraïssé class and $\mathfrak{M}$ a separable $\mathcal{K}$-structure. Then the following are equivalent:*

(i) *The structure $\mathfrak{M}$ is a limit of $\mathcal{K}$.*
(ii) *For every $\mathfrak{A} = \langle \bar{a} \rangle \in \mathcal{K}$ (with $\bar{a}$ finite), $\mathcal{K}$-structure $\mathfrak{B}$, embedding $\psi \colon \mathfrak{A} \to \mathfrak{M}$ and $\varepsilon > 0$, there is an embedding $\varphi \colon \mathfrak{B} \to \mathfrak{M}$ such that $d(\varphi\bar{a}, \psi\bar{a}) < \varepsilon$.*
(iii) *Same, where $\mathfrak{B}$ is finitely generated (i.e., $\mathfrak{B} \in \mathcal{K}$).*

In addition, eve

Proof. (i) $\Longrightarrow$ (ii). By Theorem 2.16(i).

(ii) $\Longrightarrow$ (iii). Immediate.

(iii) $\Longrightarrow$ (i). Let $\bar{b} \in \mathcal{K}_n$ and $\psi \in \mathrm{Stx}(\bar{b}, \mathfrak{M})$, and we may extend $\bar{b}$ (and $\langle \bar{b} \rangle$) as we wish, as long as we keep it finite (and $\langle \bar{b} \rangle$ finitely generated). By definition of a strictly approximate isomorphism, possibly extending $\bar{b}$, there is a tuple $\bar{a} \in \langle \bar{b} \rangle$, an actual embedding $\psi' \colon \langle \bar{a} \rangle \to \mathfrak{M}$, and $\varepsilon > 0$, such that $\psi > \psi' + 2\varepsilon$. Applying the hypothesis to ψ' and ε, there is an embedding $\varphi' \colon \langle \bar{b} \rangle \to \mathfrak{M}$ such that $d(\varphi'\bar{a}, \psi'\bar{a}) < \varepsilon$, whereby $\psi > \varphi' + \varepsilon$. Let $\varphi = (\varphi' + \varepsilon){\restriction}_{\bar{b}}$. Then $\varphi \in \mathrm{Stx}(\bar{b}, \mathfrak{M})$, it is 2ε-total, and since ψ extends trivially from $\bar{b}$, we have $\psi > \varphi$. Since ε can be taken arbitrarily small, this is enough. $\blacksquare_{2.17}$

Definition 2.18. We say that a separable structure $\mathfrak{M}$ is *approximately ultra-homogeneous* if every isomorphism of finitely generated substructures of $\mathfrak{M}$ is arbitrarily close, on any finite set of generators, to the restriction of an automorphism of $\mathfrak{M}$. Equivalently, if every strict approximation of an isomorphism of finitely generated substructures of $\mathfrak{M}$ also strictly approximates an automorphism.

Theorem 2.19. *Let $\mathcal{K}$ be a class of finitely generated structures. Then the following are equivalent:*

(i) *The class $\mathcal{K}$ is a Fraïssé class.*
(ii) *The class $\mathcal{K}$ is the age of a separable approximately ultra-homogeneous structure $\mathfrak{M}$.*

Moreover, such a structure $\mathfrak{M}$ is necessarily a limit of $\mathcal{K}$, and thus unique up to isomorphism and universal for separable $\mathcal{K}$-structures.

Proof. The second item clearly implies the first, as well as the moreover part. Conversely, if $\mathcal{K}$ is a Fraïssé class then by Lemma 2.13 it has a limit $\mathfrak{M}$. By Theorem 2.16(i) we have $\mathrm{Age}(\mathfrak{M}) = \mathcal{K}$, continuity is by CP, and the last property follows from Theorem 2.16(ii). $\blacksquare_{2.19}$

Remark 2.20. Let $\mathcal{K}$ be a Fraïssé class, and let $\theta\colon [0,\infty] \to [0,1]$ be any increasing sub-additive map which is continuous and injective near zero. For example, plain truncation $x \mapsto x \wedge 1$ will do, or if one wants a homeomorphism, one may take $x \mapsto 1 - e^{-x}$ or $x \mapsto \frac{x}{x+1}$. The important point is that for any distance function d, θd is a bounded distance function, uniformly equivalent to d.

We define a new language $\mathcal{L}_{\mathcal{K}}$, consisting of one n-ary predicate symbol $P_{[\bar a]}$ for each equivalence class $[\bar a]$ in $\overline{\mathcal{K}}_n$ (or in a dense subset thereof). Then every $\mathcal{K}$-structure $\mathfrak{A}$ gives rise to an $\mathcal{L}_{\mathcal{K}}$-structure $\mathfrak{A}'$, with the same underlying set, where

$$d^{\mathfrak{A}'} = \theta d^{\mathfrak{A}}, \qquad P^{\mathfrak{A}}_{[\bar a]}(\bar b) = \theta d^{\mathcal{K}}(\bar a, \bar b).$$

Let $\mathcal{K}' = \bigcup_{\mathfrak{A}\in\mathcal{K}} \operatorname{Age}(\mathfrak{A}')$. Since $\mathcal{L}'$ is purely relational, all members of $\mathcal{K}'$ are necessarily finite, while members of $\mathcal{K}$ are merely finitely generated, and in general $\mathcal{K}' \neq \{\mathfrak{A}' \colon \mathfrak{A} \in \mathcal{K}\}$. However, for each n we do have canonical identification between $\mathcal{K}_n$ and $\mathcal{K}'_n$, with $d^{\mathcal{K}'} = \theta d^{\mathcal{K}}$. Then one checks that $\mathcal{K}'$ is a Fraïssé class, and that a $\mathcal{K}$-structure $\mathfrak{M}$ is a limit of $\mathcal{K}$ if and only if $\mathfrak{M}'$ is a limit of $\mathcal{K}'$.

We conclude that up to a change of language, any Fraïssé class or approximately ultra-homogeneous structure can be assume to be in a 1-Lipschitz, $[0,1]$-valued relational continuous language, and that our more relaxed definitions (see Remark 2.2), while convenient for some concrete examples, do not in truth add any more generality.

Another curious property of this construction is that $(\lim \mathcal{K})' = \lim \mathcal{K}'$ is always an atomic, and therefore prime, model of its continuous first order theory.

3. Examples of metric Fraïssé classes

3.1. **Standard examples.** Let $\mathcal{K}_M$ be the class of finite metric spaces; $\mathcal{K}_{M,1}$ the class of finite metric spaces of diameter at most one; $\mathcal{K}_H$ the class of finite dimensional Hilbert spaces; and $\mathcal{K}_P$ the class of finite probability algebra, each in the appropriate language. We leave it to the reader to check that these are all Fraïssé classes. We claim that the Urysohn space, the Urysohn sphere, ℓ^2, and the (probability algebra of the) Lebesgue space $([0,1], \lambda)$, are, respectively, limits of these classes. In fact, in each of these cases, the limits satisfy a strong version of Corollary 2.17(iii):

> For each extension $\mathfrak{A} \subseteq \mathfrak{B}$ of members of $\mathcal{K}$, every embedding $\mathfrak{A} \to \mathfrak{M}$ extends to an embedding $\mathfrak{B} \to \mathfrak{M}$.

A non example is the class of (real) L^p lattices, for some fixed $1 \le p < \infty$ (see [Mey91] for a formal definition and [BBH11] for a model-theoretic treatment) over finitely many atoms (this is in contrast with the class of finite probability algebras, which are all atomic). The culprit here is the completeness. Indeed, working inside $E = L^p[0,1]$, let $f(x) = 1$ and $g(x) = x$. Then on the one hand, $E = \langle f, g\rangle$ is non atomic, while on the other hand, approximating g by step functions, the pair (f, g) can be arbitrarily well approximated by pairs which do generate an atomic lattice. In fact, every separable L^p lattice is finitely generated, and the class of *all* separable L^p lattices is a Fraïssé class, whose limit is the unique separable atomless L^p lattice. Alternatively, one could add structure to atomic L^p lattices making embeddings preserve atoms. With this added structure, the class of L^p lattices over finitely many atoms is a Fraïssé class, with limit the unique atomic L^p with $\aleph_0$ atoms. The automorphism group of the latter is S_∞, the permutation group of $\mathbf{N}$.

3.2. **The Gurarij space.** We recall that

Definition 3.1. A *Gurarij space* is a separable Banach space $\mathbf{G}$ having the property that for any $\varepsilon > 0$, finite dimensional Banach space $E \subseteq F$, and isometric embedding $\psi\colon E \to \mathbf{G}$, there is a linear embedding $\varphi\colon F \to \mathbf{G}$ extending ψ such that in addition, for all $x \in F$, $(1-\varepsilon)\|x\| < \|\varphi x\| < (1+\varepsilon)\|x\|$.

Gurarij [Gur66] proved the existence and almost isometric uniqueness of such spaces, while actual (i.e., isometric) uniqueness of $\mathbf{G}$ was shown by Lusky [Lus76]. This uniqueness was more recently re-proved by Kubiś and Solecki [KS], in what essentially amounts to showing that it was the Fraïssé limit of the class of all finite dimensional Banach spaces, an observation we now have the tools to state and prove formally. From here on, $\mathcal{K} = \mathcal{K}_B$ is the class of finite dimensional Banach space. In order to show that this is a Fraïssé class, the only property which is not quite trivial is that $\mathcal{K}_n$ is separable. This follows, for example, from the fact that a separable universal Banach space (e.g., the Gurarij space) exists. For an alternative argument, let us state the following fact, due to Henson:

Fact 3.2. *Let $\bar{a}, \bar{b} \in \mathcal{K}_n$. Then*

$$d^{\mathcal{K}}(\bar{a}, \bar{b}) = \sup_{\sum |s_i|=1} \Big| \|\sum s_i a_i\| - \|\sum s_i b_i\| \Big|. \tag{1}$$

Moreover, there exists a canonical amalgam $E \supseteq \langle \bar{a} \rangle, \langle \bar{b} \rangle$ in which $d(a_i, b_i)$ is maximal possible for each i subject to the constraint that $\max d(a_i, b_i) = d^{\mathcal{K}}(\bar{a}, \bar{b})$.

Proof. See [BH]. $\blacksquare_{3.2}$

Let us fix n and some $M > 0$, and let us restrict our attention to $\mathcal{K}_n^{\leq M} = \{\bar{a} \in \mathcal{K}_n : \|a_i\| \leq M \text{ for all } i\}$. Then we can calculate $d^{\mathcal{K}}(\bar{a}, \bar{b})$ on $\mathcal{K}_n^{\leq M}$ up to an error smaller than any desired $\varepsilon > 0$ using in (1) only finitely many tuples $\bar{s}$ (whose choice depends on n, M and ε). It follows that $(\mathcal{K}_n^{\leq M}, d^{\mathcal{K}})$ is totally bounded, and therefore compact, and that $(\mathcal{K}_n, d^{\mathcal{K}})$ is separable. Thus $\mathcal{K}$ is indeed a Fraïssé class. Another consequence of Fact 3.2, somewhat informally stated, is that the neighbourhoods of $\bar{a} \in \mathcal{K}_n$ in the sense of $d^{\mathcal{K}}$ and of the Banach-Mazur distance are the same. This is used implicitly in the "only if" part of the following proof.

Theorem 3.3. *A Banach space G is a Gurarij space if and only if it is the Fraïssé limit of the class of all finite dimensional Banach space. In particular, the Gurarij space exists, is unique, and is universal for separable Banach spaces.*

Proof. Assume first that $G = \lim \mathcal{K}$. Let $E \subseteq F$ be two finite dimensional Banach spaces, with bases $\bar{a} \subseteq \bar{b}$, respectively, and let $\psi \colon E \to G$ be an isometric embedding. By Corollary 2.17 there exists an isometric $\varphi' \colon F \to G$ with $d(\bar{a}, \varphi\bar{a}) = \delta$ arbitrarily small. Define $\varphi \colon F \to G$ as ψ on $\bar{a}$ and φ' on $\bar{b} \setminus \bar{a}$. Taking δ sufficiently small, φ is injective, and both $\|\varphi\|$ and $\|\varphi^{-1}\|$ (with φ restricted to its image) arbitrarily close to one, so G is Gurarij.

Conversely, assume that G is Gurarij. Let $F = \langle \bar{b} \rangle \in \mathcal{K}$ and $\psi \in \mathrm{Stx}(\bar{b}, G)$. Then possibly extending F we may assume that there are $E \subseteq F$, $\varepsilon > 0$ (which may be chosen arbitrarily small) and an isometric embedding $\psi' \colon E \to G$ such that $\psi \geq \psi' + \varepsilon$. By assumption there exists a linear $\varphi \colon F \to G$ extending ψ', with $\|\varphi\|, \|\varphi^{-1}\|$ arbitrarily close to one, and by Fact 3.2 we can then have $d^{\mathcal{K}}(\bar{b}, \varphi\bar{b}) < \varepsilon$. Then there exists $\varphi' \in \mathrm{Stx}(\bar{b}, \varphi\bar{b}) \subseteq \mathrm{Stx}(\bar{b}, G)$ with $\varphi'(b_i, \varphi b_i) < \varepsilon$, so in particular φ' is ε-total on $\bar{b}$. Now for $a \in G$ we have $\psi(b_i, a) \geq d(\varphi b_i, a) + \varepsilon > \varphi'(b_i, a)$, so $\psi > \varphi'$, and G is a limit. $\blacksquare_{3.3}$

4. Approximate categories

5. Weak Fraïssé classes

Unlike partial isomorphisms in classical (discrete) logic, in our treatment of a metric Fraïssé classes the intrinsic (strict) approximate isomorphisms between structures depend on the class. This seems unavoidable (unless we are willing to impose additional constraints, such as a Lipschitz interpretation of the language), and in fact suggests we could go even further, and equip a class of structures with a family of "approximate isomorphisms" other than the intrinsic ones. This allows to consider Fraïssé classes and limits up to small perturbations, somewhat in the spirit of [Ben08b].

Definition 5.1. Let $\mathcal{MA}$ denote the category of complete metric spaces and approximate isometries. Let $\mathcal{C}$ be a category, and $F \colon \mathcal{C} \to \mathcal{MA}$ a faithful functor. For $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$, let A and B denote their respective images under F, and let $\mathrm{Apx}(\mathfrak{A}, \mathfrak{B}) \subseteq \mathrm{Hom}_{\mathcal{MA}}(A, B)$ denote the image of $\mathrm{Hom}_{\mathcal{C}}(\mathfrak{A}, \mathfrak{B})$ under F, which we call the set of *approximate isomorphisms* from $\mathfrak{A}$ to $\mathfrak{B}$.

By an *approximate category* we mean such a pair $(\mathcal{C}, F)$, such that in addition, for every $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$:

- (i) The set of approximate isomorphisms $\mathrm{Apx}(\mathfrak{A}, \mathfrak{B})$ is closed in the topology of point-wise convergence (on functions $A \times B \to [0, \infty]$).
- (ii) The set $\mathrm{Apx}(\mathfrak{A}, \mathfrak{B})$ is closed under approximation: if $\varphi \geq \psi \in \mathrm{Apx}(\mathfrak{A}, \mathfrak{B})$ (and $\varphi \colon A \rightsquigarrow B$ is an approximate isometry) then $\varphi \in \mathrm{Apx}(\mathfrak{B}, \mathfrak{A})$.
- (iii) If, in addition, $\mathrm{Apx}(\mathfrak{A}, \mathfrak{B})^* = \mathrm{Apx}(\mathfrak{B}, \mathfrak{A})$, we say that $(\mathcal{C}, F)$ is a *symmetric approximate category.*

When there is a natural choice for F, e.g., when it is a natural forgetful functor, we shall omit it.

Definition 5.2. Let $\mathcal{K}$ be a symmetric approximate category of finitely generated $\mathcal{L}$-structures. For $\mathcal{K}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ we define $\mathrm{Apx}(\mathfrak{A}, \mathfrak{B})$ as the closure of $\bigcup_{\bar{a} \in A^n, \bar{b} \in B^m} \mathrm{Apx}(\langle \bar{a} \rangle, \langle \bar{b} \rangle)$ in the topology of point-wise convergence. For $\bar{a} \in \mathcal{K}_n$ and a $\mathcal{K}$-structure $\mathfrak{B}$, define the set of *strictly approximate isomorphisms* from $\bar{a}$ to $\mathfrak{B}$ by

$$\mathrm{Stx}(\bar{a}, \mathfrak{B}) = \{\psi \colon \bar{a} \rightsquigarrow B \colon \mathrm{Apx}^{<\psi}(\langle \bar{a} \rangle, \mathfrak{B}) \neq \varnothing\},$$

and similarly $\mathrm{Stx}(\mathfrak{A}, \bar{b})$, $\mathrm{Stx}(\bar{a}, \bar{b})$. We then define $d^{\mathcal{K}}$ on $\mathcal{K}_n$ as in Definition 2.7.

We say that $\mathcal{K}$ is a *weak Fraïssé class* if it satisfies the following properties:

- HP, together with the requirement that every embedding $\mathfrak{A} \to \mathfrak{B}$ belongs to $\mathrm{Apx}(\mathfrak{A}, \mathfrak{B})$.
- *NJEP (Near Joint Embedding Property)*: For every two $\bar{a}, \bar{b} \in \mathcal{K}_n$ and $\varepsilon > 0$ there exists $\mathfrak{C} \in \mathcal{K}$ and ε-total $\psi \in \mathrm{Stx}(\bar{a}, \mathfrak{C})$, $\varphi \in \mathrm{Stx}(\bar{b}, \mathfrak{C})$.
- *DAP (Distant Amalgamation Property)*: For every $\varepsilon > 0$ there exists $\delta > 0$ satisfying that for every $\bar{a}, \bar{b} \in \mathcal{K}_n$, $\psi \in \mathrm{Stx}(\bar{a}, \bar{b})$ and $\eta > 0$, if $\inf \psi < \delta$ then there are $\mathfrak{C} \in \mathcal{K}$, an ε-total $\varphi \in \mathrm{Stx}(\bar{a}, \mathfrak{C})$ and an η-total $\rho \in \mathrm{Stx}(\bar{b}, \mathfrak{C})$ such that $\varphi < \rho\psi$.
- PP & CP

We say that it is an *incomplete weak Fraïssé class* if instead of PP & CP we have WPP & CCP.

Remark 5.3. The first three items of Remark 2.10 hold just as well for weak Fraïssé classes. For the third item (completion), one first defines $\mathrm{Stx}(\bar{a}, \bar{b})$ for $\bar{a} \in \widehat{\mathcal{K}}_n$, $\bar{b} \in \widehat{\mathcal{K}}_m$ as the set of all approximate isometries ψ such that for some $\bar{a}' \in \mathcal{K}_n$, $\bar{b}' \in \mathcal{K}_m$ and $\psi' \in \mathrm{Stx}(\bar{a}', \bar{b}')$ we have $\psi' + d^{\mathcal{K}}(\bar{a}, \bar{a}') + d^{\mathcal{K}}(\bar{b}, \bar{b}') < \psi$, and then defines $\mathrm{Apx}(\mathfrak{A}, \mathfrak{B})$, for $\mathfrak{A}, \mathfrak{B} \in \widehat{\mathcal{K}}$, as consisting of all approximate isometries all of whose strict approximations also approximate a member of $\mathrm{Stx}(\bar{a}, \bar{b})$ for some $\bar{a} \in A^n$, $\bar{b} \in B^m$.

Remark 5.4. In the conclusion of DAP, one may obtain the stronger property $\rho^*\varphi < \psi$. Indeed, we may assume that $\eta < \frac{1}{2}\Gamma(\psi)$, in which case there are φ and ρ as stated there such that $\varphi < \rho\psi - 2\eta$, in which case

$$\rho^*\varphi < \rho^*\rho\psi - 2\eta \leq \psi.$$

We leave it as an exercise to the reader to check that every Fraïssé class (with its intrinsic approximate isomorphisms) is a weak Fraïssé class, and that moreover $\delta = \infty$ suffices for every $\varepsilon > 0$ in DAP.

Definition 5.5. Let $\mathcal{K}$ be a weak Fraïssé class. By a *limit* of $\mathcal{K}$ we mean a separable non empty $\mathcal{K}$-structure $\mathfrak{M}$, satisfying that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\bar{a} \in \mathcal{K}_n$ and $\psi \in \mathrm{Stx}(\bar{a}, \mathfrak{M})$, if $\inf \psi < \delta$ then there exists an ε-total $\varphi \in \mathrm{Stx}^{<\psi}(\bar{a}, \mathfrak{M})$.

For a Fraïssé class $\mathcal{K}$ we therefore have two notions of limit, one as a Fraïssé class and one as a weak Fraïssé class. For the time being, let us agree that we only mean the latter notion, forgetting the former. It will follow easily from later work (Corollary 5.14) that the two notions always agree, and in fact a relatively simple direct proof can also be given. From here on, much of Section 2 generalises to weak Fraïssé classes. At first it will be convenient to consider limits with respect to moduli, just so that we can cleanly dispense with these later on.

Definition 5.6. By a *modulus* we mean a weakly increasing, lower semi-continuous function $\Delta \colon (0, \infty) \to (0, \infty]$. To every modulus we associate a *weak inverse* $\Delta^{-1} \colon (0, \infty) \to [0, \infty]$, $\Delta^{-1}(\delta) = \inf\{\varepsilon > 0 \colon \Delta(\varepsilon) > \delta\}$. The crucial property is that $\Delta(\varepsilon) > \delta$ if and only if $\varepsilon > \Delta^{-1}(\delta)$.

We say that a modulus Δ is an *amalgamation modulus* for a weak Fraïssé class $\mathcal{K}$ if DAP holds with $\delta = \Delta(\varepsilon)$. We observe below that $\mathcal{K}$ always admits an amalgamation modulus, and clearly the supremum of all amalgamation moduli for $\mathcal{K}$ is again such, so $\mathcal{K}$ admits a maximal amalgamation modulus which will be denoted $\Delta_{\mathcal{K}}$.

If Definition 5.5 holds with $\delta = \Delta(\varepsilon)$, we say that $\mathfrak{M}$ is a Δ*-limit*.

If $\mathcal{K}$ is a weak Fraïssé class then it admits an amalgamation modulus Δ, since any function $\varepsilon \mapsto \delta = \Delta_0(\varepsilon)$ can be replaced with

$$\Delta(\varepsilon) = \sup_{0 < \varepsilon' < \varepsilon} \Delta_0(\varepsilon').$$

By the same reasoning, a structure $\mathfrak{M}$ is a limit if and only if it is a Δ-limit for some modulus Δ.

Lemma 5.7. *Let Δ be a modulus. Then the criterion of Lemma 2.12 holds* mutatis mutandis *for Δ-limits of a weak Fraïssé class.*

Proof. Essentially identical. In order to get ε rational, we may replace it with any $\Delta^{-1}(\inf \psi) < \varepsilon' < \varepsilon$. $\blacksquare_{5.7}$

The definition/construction of $\overline{\mathcal{K}}_n$ and $\overline{\mathcal{K}}_\omega$ hold for a weak Fraïssé class just as well, and we prove

Lemma 5.8. *Let $\mathcal{K}$ be a weak Fraïssé class, and let Δ be an amalgamation modulus for $\mathcal{K}$. The statement of Proposition 2.15 holds for Δ-limits of weak Fraïssé classes, namely the set of points in $\overline{\mathcal{K}}_\omega$ which define a Δ-limit of $\mathcal{K}$ (along with a sequence of generators all whose tails are dense) is a dense G_δ.*

Conversely, if there exists any Δ-limit of $\mathcal{K}$, for a modulus Δ, then Δ is an amalgamation modulus for $\mathcal{K}$.

Proof. We observe that Lemma 2.14 holds for weak Fraïssé classes as well. Now the same proof holds with the following modifications.

First, we only consider ψ such that $\inf\psi < \Delta(\varepsilon)$.

Then, once we have V with $\xi_\ell \in V$, we have in fact $B(\xi_\ell, 2\eta) \subseteq V$ for some $\eta > 0$, and we may assume that $4\eta < \Gamma(\psi^\xi)$ and $\inf\psi - 4\eta < \inf\psi < \Delta(\varepsilon - \eta)$. By hypothesis, there exists a $\mathfrak{C} \in \mathcal{K}$, an ε-total $\varphi \in \mathrm{Stx}(\bar b, \mathfrak{C})$ and an η-total $\rho \in \mathrm{Stx}(a^\xi_{<\ell}, \mathfrak{C})$ such that $\rho\psi^\xi - 4\eta > \varphi$. There is a tuple $\bar a' \in C^\ell$ such that $\rho(a^\xi_i, a'_i) < 2\eta$, and we may assume that $\mathfrak{C} = \langle \bar a' \bar c\rangle$ (for some $\bar c$) and $\bar a' \bar c = a^\zeta_{<\ell+k}$ for some $\zeta \in \overline{\mathcal{K}}_\omega$. Then $[\bar a'] \in V$, so $\zeta \in U$, and if ψ^ζ is defined then

$$\psi^\zeta = \mathrm{id}_{\bar a^\zeta_{<n}} \psi^\zeta > \rho\rho^*\psi^\zeta - 4\eta > \rho\psi^\xi - 4\eta > \varphi \in \mathrm{Stx}(\bar b, \mathfrak{M}^\zeta).$$

Thus $\zeta \in X$.

The converse is left to the reader. $\blacksquare_{5.8}$

Next in line to generalise is Theorem 2.16. It is in fact most convenient to break it up into several separate results.

Definition 5.9. For an approximate isometry $\psi\colon X \rightsquigarrow Y$ and $n > 0$, let $\psi^{n*} = \ldots\psi\psi^*\psi$, of length n, and $\psi^{-n*} = (\psi^*)^{n*} = \ldots\psi^*\psi\psi^*$.

Let $\mathcal{K}$ be a weak Fraïssé class, $\mathfrak{M}$ and $\mathfrak{N}$ $\mathcal{K}$-structures. We say that $\theta \in \mathrm{Apx}(\mathfrak{M}, \mathfrak{N})$ is *minimal* if for every $n \in \mathbf{Z} \smallsetminus \{0\}$ and a, b in M or in N, as appropriate, if $\theta^{n*}(a, b) > r \in \mathbf{R}$ then there is $\psi > \theta$ such that $\varphi^{n*}(a, b) > r$ for all $\varphi \in \mathrm{Apx}^{<\psi}(\mathfrak{M}, \mathfrak{N})$. The set of minimal $\theta \in \mathrm{Apx}(\mathfrak{M}, \mathfrak{N})$ will be denoted $\mathrm{Apx}^{\min}(\mathfrak{M}, \mathfrak{N})$.

We observe that every $\theta \in \mathrm{Apx}^{\min}(\mathfrak{M}, \mathfrak{N})$ is minimal with respect to $\leq$, applying the definition with $n = 1$ (the converse need not hold).

Proposition 5.10. *Let $\mathcal{K}$ be a weak Fraïssé class, $\mathfrak{M}$ and $\mathfrak{N}$ separable $\mathcal{K}$-structures. Then for every $\psi \in \mathrm{Stx}(\mathfrak{M}, \mathfrak{N})$ there exists $\theta \in \mathrm{Apx}^{\min,<\psi}(\mathfrak{M}, \mathfrak{N})$.*

Proof. Let $M_0 \subseteq M$ and $N_0 \subseteq N$ be countable and dense, and let us enumerate $M_0 \times N_0 \times (\mathbf{Z} \smallsetminus \{0\})$ as $(a_k, b_k, n_k)_{k\in\mathbf{N}}$, repeating each triplet infinitely often. Now define $\psi_0 = \psi$, and given $\psi_k \in \mathrm{Stx}(\mathfrak{M}, \mathfrak{N})$, choose $\psi_{k+1} \in \mathrm{Stx}^{<\psi_k}(\mathfrak{M}, \mathfrak{N})$ such that $\psi^{n_k*}_{k+1}(a_k, b_k) < \inf\bigl\{\varphi^{n_k*}(a_k, b_k)\colon \varphi \in \mathrm{Stx}^{<\psi_k}(\mathfrak{M}, \mathfrak{N})\bigr\} + 2^{-k}$. Then $\theta = \lim\psi_k$ is minimal. $\blacksquare_{5.10}$

Definition 5.11. Let Δ be a modulus. We say that an approximate isometry $\psi\colon X \rightsquigarrow Y$ is Δ-*total* if it is $\Delta^{-1}(\inf\psi)$-total. Similarly, we say that ψ is Δ-*surjective* if it is $\Delta^{-1}(\inf\psi)$-surjective, and that it is Δ-*bijective* if it is both.

Lemma 5.12. *Let $\mathcal{K}$ be a weak Fraïssé class, $\mathfrak{M}$ and $\mathfrak{N}$ separable $\mathcal{K}$-structures, with $\mathfrak{N}$ a Δ-limit of $\mathcal{K}$ for some modulus Δ. Then*

(i) *Every $\theta \in \mathrm{Apx}^{\min}(\mathfrak{M}, \mathfrak{N})$ is Δ-total.*
(ii) *For every $\varepsilon > 0$ there exists $\psi \in \mathrm{Stx}(\mathfrak{M}, \mathfrak{N})$ with $\inf\psi < \varepsilon$.*

Proof. For the first item, $a \in M$, $r = \theta^*\theta(a, a)$, and $\varepsilon > 0$. Since θ is minimal, there is $\psi > \theta$ such that $\varphi^*\varphi(a, a) > r - \varepsilon$ for all $\varphi \in \mathrm{Apx}^{<\psi}(\mathfrak{M}, \mathfrak{N})$. We may assume that $\inf\psi < \Delta\bigl(\Delta^{-1}(\inf\theta) + \varepsilon\bigr)$ and that ψ is a trivial extension from two finite tuples $\bar a \subseteq M$, $\bar b \subseteq N$, with $a_0 = a$. By Remark 5.4 there exist $\mathfrak{C} \in \mathcal{K}$, a $\bigl(\Delta^{-1}(\inf\theta) + \varepsilon\bigr)$-total $\psi_0 \in \mathrm{Stx}(\bar a, \mathfrak{C})$ and an $\frac{1}{2}\Delta(\varepsilon)$-total $\psi_1 \in \mathrm{Stx}(\bar b, \mathfrak{C})$ such that $\psi_1^*\psi_0 < \psi$. In particular, $\inf\psi_1 < \Delta(\varepsilon)$, and since $\mathfrak{N}$ is a limit, there exists an ε-total $\psi_2 \in \mathrm{Stx}^{<\psi_1^*}(\mathfrak{C}, \mathfrak{M})$.

Putting everything together, we see that $\psi_2\psi_0$ is $\bigl(\Delta^{-1}(\inf\theta) + 2\varepsilon\bigr)$-total, so

$$r < (\psi_2\psi_0)^*(\psi_2\psi_0)(a, a) + \varepsilon \leq 2\Delta^{-1}(\inf\theta) + 5\varepsilon.$$

Since ε was arbitrary, $r = 2\Delta^{-1}(\inf\theta)$, so θ is Δ-total.

For the second item we use a similar argument together with NJEP. $\blacksquare_{5.12}$

Theorem 5.13. *Let $\mathcal{K}$ be a weak Fraïssé class, $\mathfrak{M}$ and $\mathfrak{N}$ separable $\mathcal{K}$-structures.*

(i) *If $\mathfrak{N}$ is a limit of $\mathcal{K}$ (respectively, if both are) then for every $\varepsilon > 0$ there exists an ε-total (respectively ε-bijective) $\theta \in \mathrm{Apx}^{\min}(\mathfrak{M}, \mathfrak{N}) \subseteq \mathrm{Apx}(\mathfrak{M}, \mathfrak{N})$.*

(ii) *The $\mathcal{K}$-structure $\mathfrak{M}$ is a limit of $\mathcal{K}$ if and only if it is a $\Delta_{\mathcal{K}}$-limit (and therefore a Δ-limit for any amalgamation modulus Δ of $\mathcal{K}$).*

(iii) *Let $\psi \in \mathrm{Stx}(\mathfrak{M}, \mathfrak{N})$. Then there exists $\theta \in \mathrm{Apx}^{\min,<\psi}(\mathfrak{M}, \mathfrak{N}) \subseteq \mathrm{Apx}^{<\psi}(\mathfrak{M}, \mathfrak{N})$, such that if $\mathfrak{N}$ is a limit of $\mathcal{K}$ (respectively, if both are), then θ is $\Delta_{\mathcal{K}}^{-1}(\inf \theta)$-total (respectively, $\Delta_{\mathcal{K}}^{-1}(\inf \theta)$-bijective) (observing that $\Delta_{\mathcal{K}}^{-1}(\inf \theta) \leq \Delta_{\mathcal{K}}^{-1}(\inf \psi)$, and that $\Delta_{\mathcal{K}}^{-1} \leq \Delta^{-1}$ for any amalgamation modulus Δ of $\mathcal{K}$).*

(iv) *We may drop Δ from Lemma 5.8: the set of points in $\overline{\mathcal{K}}_\omega$ which define a limit of $\mathcal{K}$ (along with a sequence of generators all whose tails are dense) is a dense G_δ.*

Proof. Item (i) follows from Lemma 5.12, Proposition 5.10 and the fact that any two limits have common modulus. Item (ii) follows from (i) using the existence of $\Delta_{\mathcal{K}}$-limits by Lemma 5.8. Items (iii) and (iv) follow. $\blacksquare_{5.13}$

Corollary 5.14. *Let $\mathcal{K}$ be a Fraïssé class. Then the limits of $\mathcal{K}$ as a weak Fraïssé class and as a Fraïssé class are the same. In particular, any two limits of $\mathcal{K}$ are isomorphic.*

Proof. It is immediate that any limit of $\mathcal{K}$ as a Fraïssé class is also a limit as a weak Fraïssé class. For the converse, we use Theorem 5.13, together with the fact that if $\mathcal{K}$ is a Fraïssé class then $\Delta_{\mathcal{K}} = \infty$ and $\Delta_{\mathcal{K}}^{-1} = 0$. The uniqueness of the limit (which we have already proved in Section 2) also follows from this. $\blacksquare_{5.14}$

Our next goal is to generalise Theorem 2.19. For this, we need an appropriate notion of an approximately ultra-homogeneous structure.

Definition 5.15. Let $\mathfrak{M}$ be a structure and $\mathcal{A}$ a family of approximate isometries $M \rightsquigarrow M$.

(i) We define $\langle \mathcal{A} \rangle$ to consist of the closure of $\mathcal{A} \cup \mathrm{Aut}(\mathfrak{M})$ under pseudo-inversion and composition.

(ii) We define $\overline{\mathcal{A}}$ to consist of the closure of $\mathcal{A}$ under approximation and point-wise limits, observing that if $\mathcal{A} = \langle \mathcal{A} \rangle$ then $\overline{\mathcal{A}} = \overline{\langle \mathcal{A} \rangle}$.

(iii) We say that a set $\mathcal{A} = \overline{\langle \mathcal{A} \rangle}$ is *Δ-generated*, for a modulus Δ, if there exists a subset $\mathcal{A}_0 \subseteq \mathcal{A}$, consisting of Δ-bijective approximate isometries, such that $\mathcal{A} = \overline{\mathcal{A}_0}$.

(iv) We say that a sequence of tuples $\bar{a}_k \in M^n$ is *Cauchy modulo $\mathcal{A}$* if for every $\varepsilon > 0$ there is m such that for all $k, \ell \geq m$ there is $\psi \in \langle \mathcal{A} \rangle$ (or equivalently, $\psi \in \overline{\langle \mathcal{A} \rangle}$) with $\max_{i<n} \psi(a_{k,i}, a_{\ell,i}) < \varepsilon$.

Definition 5.16. An *approximate structure* is a pair $(\mathfrak{M}, \mathcal{A})$ such that:

(i) The set $\mathcal{A}$ consists of approximate isometries $M \rightsquigarrow M$, and $\mathcal{A} = \overline{\langle \mathcal{A} \rangle}$.

(ii) For each n-ary predicate symbol P (respectively, function symbol f) and sequence $(\bar{a}_k, \bar{b}_k)$ in M^{n+m} which is Cauchy modulo $\mathcal{A}$, the sequence $P^{\langle \bar{a}_k, \bar{b}_k \rangle}(\bar{a}_k)$ (respectively, $\big(\bar{a}_k, \bar{b}_k, f^{\langle \bar{a}_k, \bar{b}_k \rangle}(\bar{a}_k)\big)$) is Cauchy (modulo $\mathcal{A}$) as well.

In addition,

(iii) We say that $(\mathfrak{M}, \mathcal{A})$ is Δ-generated if $\mathcal{A}$ is, and that $(\mathfrak{M}, \mathcal{A})$ is separable if $\mathfrak{M}$ is.

(iv) We say that $(\mathfrak{M}, \mathcal{A})$ is *approximately ultra-homogeneous* if every isomorphism between substructures of $\mathfrak{M}$ belongs to $\mathcal{A}$. We then define $\mathrm{Age}(\mathfrak{M}, \mathcal{A})$ as the approximate category whose class of objects is $\mathrm{Age}(\mathfrak{M})$, together with $\mathrm{Apx}(\mathfrak{A}, \mathfrak{B}) = \{\psi{\restriction}_{A \times B} : \psi \in \mathcal{A}\}$ for any embedding of $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathfrak{M}$ (and this does not depend on the choice of embedding).

We observe that for any structure $\mathfrak{M}$, the pair $\big(\mathfrak{M}, \overline{\mathrm{Aut}(\mathfrak{M})}\big)$ is an ∞-generated approximate structure, and that it is approximately ultra-homogeneous if and only if $\mathfrak{M}$ is as per Definition 2.18.

Lemma 5.17. *Let $(\mathfrak{M}, \mathcal{A})$ be an approximately ultra-homogeneous, Δ-generated, separable, approximate structure. Then $\mathrm{Age}(\mathfrak{M}, \mathcal{A})$ is an incomplete weak Fraïssé class, whose completion $\widehat{\mathrm{Age}}(\mathfrak{M}, \mathcal{A})$ is a weak Fraïssé class, for which Δ is an amalgamation modulus.*

Conversely, let $\mathcal{K}$ be a weak Fraïssé class, and $\mathfrak{M}$ a $\mathcal{K}$-structure. Then $\big(\mathfrak{M}, \mathrm{Apx}(\mathfrak{M}, \mathfrak{M})\big)$ is an approximate structure.

Proof. Left to the reader (see Remark 5.3). $\blacksquare_{5.17}$

Theorem 5.18. *Let $\mathcal{K}$ be an approximate category of finitely generated structures, and Δ a modulus. Then the following are equivalent:*

(i) *The approximate category $\mathcal{K}$ is a weak Fraïssé class and Δ is an amalgamation modulus for $\mathcal{K}$.*

(ii) *The approximate category $\mathcal{K}$ is the completed age of a separable, approximately ultra-homogeneous, Δ-generated, approximate structure* $(\mathfrak{M}, \mathcal{A})$.

Moreover, such an approximate structure $(\mathfrak{M}, \mathcal{A})$ *is necessarily a limit of* $\mathcal{K}$, *with* $\mathcal{A} = \operatorname{Apx}(\mathfrak{M}, \mathfrak{M})$, *and it is* weakly unique *and* weakly universal *among separable* $\mathcal{K}$*-structures, meaning that for every separable* $\mathcal{K}$*-structure* $\mathfrak{N}$*:*

(i) *For every* $\varepsilon > 0$ *there exists an* ε*-total* $\psi \in \operatorname{Apx}^{\min}(\mathfrak{N}, \mathfrak{M})$.

(ii) *The structure* $\mathfrak{N}$ *is also a limit of* $\mathcal{K}$ *if and only if it is a* $\mathcal{K}$*-structure as well, and for arbitrarily small* $\varepsilon > 0$ *there exist* ε*-bijective* $\psi \in \operatorname{Apx}(\mathfrak{N}, \mathfrak{M})$, *and even* $\psi \in \operatorname{Apx}^{\min}(\mathfrak{M}, \mathfrak{N})$.

Proof. The second item clearly implies the first, as well as the moreover part. Conversely, if $\mathcal{K}$ is a weak Fraisse class then by Lemma 5.8 it has a limit $\mathfrak{M}$. Then $\big(\mathfrak{M}, \operatorname{Apx}(\mathfrak{M}, \mathfrak{M})\big)$ is a separable approximate structure, which is approximately ultra-homogeneous and Δ-generated by Theorem 5.13, and clearly $\mathcal{K} = \widehat{\operatorname{Age}}\big(\mathfrak{M}, \operatorname{Apx}(\mathfrak{M}, \mathfrak{M})\big)$. $\blacksquare_{5.18}$

Corollary 5.19. *A weak Fraïssé class* $\mathcal{K}$ *is a Fraïssé class, along with its intrinsic approximate isomorphisms, if and only if* $\Delta_{\mathcal{K}} = \infty$.

Proof. One direction has already been observed. For the other, assume that $\Delta_{\mathcal{K}} = \infty$ and let $\mathfrak{M}$ be a limit. Then by Lemma 5.12 we have $\operatorname{Apx}^{\min}(\mathfrak{M}, \mathfrak{M}) = \operatorname{Aut}(\mathfrak{M})$, which in turn implies that $\mathcal{K} = \operatorname{Age}\big(\mathfrak{M}, \overline{\operatorname{Aut}(\mathfrak{M})}\big)$ is a Fraïssé class. $\blacksquare_{5.19}$

Question 5.20. One could imagine a stronger universality property for limits of a weak Fraïssé class, namely:

Every separable $\mathcal{K}$-structure embeds in a limit of $\mathcal{K}$.

Indeed, this together with weak uniqueness of the limit implies weak universality. Is this true? (By some coding tricks, it would be enough to prove that for every weak Fraïssé class $\mathcal{K}$, every member of $\mathcal{K}$ (namely, every finitely generated $\mathcal{K}$-structure) embeds in a limit.)

Remark 5.21. The construction and conclusion of Remark 2.20 hold just as well for weak Fraïssé classes, with the exception of the last assertion, regarding atomicity and primeness. In a sense, $\lim \mathcal{K}'$ is atomic *up to perturbation*, as per the next section, but it is not clear in what way, if at all, this would imply any variant of primeness.

In fact, the original language for $\mathcal{K}$ does not truly intervene in this construction, beyond the distinction between finitely and finitely generated structures. Having the notion of an approximate category at our disposal, we may now improve Remark 5.21 (and therefore Remark 2.20) as follows.

Definition 5.22. Let $\mathcal{K}$ be an approximate category of finite objects. Define $\mathcal{K}_n$ as usual as the class of enumerations of n-tuples in objects of $\mathcal{K}$, and $d^{\mathcal{K}}$ on $\mathcal{K}_n$ as per Definition 2.7. Say that $\mathcal{K}$ is a *Fraïssé category* if it has:

- *HP (Hereditary Property)*: For every $\mathfrak{A} \in \mathcal{K}$, finite metric space B, and isometric embedding $\theta \colon B \to A$, there exists a unique $\mathfrak{B} \in \mathcal{K}$ with underlying space B such that $\theta \in \operatorname{Apx}(\mathfrak{B}, \mathfrak{A})$.
- *JEP (Joint Embedding Property)*: Every two members of $\mathcal{K}$ embed in a third one.
- *NAP (Near Amalgamation Property)*: For any $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ and $\psi \in \operatorname{Stx}(\mathfrak{A}, \mathfrak{B})$ there exist $\mathfrak{C} \in \mathcal{K}$ and embeddings $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{C}$ under which $\psi > \mathrm{id}$.
- *PP (Polish Property)*: The pseudo-metric $d^{\mathcal{K}}$ is separable and complete on $\mathcal{K}_n$ for each n.

We say that it is a *weak Fraïssé category* if it has instead:

- HP.
- *NJEP (Near Joint Embedding Property)*: For every two $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ and $\varepsilon > 0$ there exists $\mathfrak{C} \in \mathcal{K}$ and ε-total $\psi \in \operatorname{Stx}(\mathfrak{A}, \mathfrak{C})$, $\varphi \in \operatorname{Stx}(\mathfrak{B}, \mathfrak{C})$.
- *DAP (Distant Amalgamation Property)*: For every $\varepsilon > 0$ there exists $\delta > 0$ satisfying that for every $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$, $\psi \in \operatorname{Stx}(\mathfrak{A}, \mathfrak{B})$ and $\eta > 0$, if $\inf \psi < \delta$ then there are $\mathfrak{C} \in \mathcal{K}$, an ε-total $\varphi \in \operatorname{Stx}(\mathfrak{A}, \mathfrak{C})$ and an η-total $\rho \in \operatorname{Stx}(\mathfrak{B}, \mathfrak{C})$ such that $\varphi < \rho\psi$.
- PP.

We say that it is an *incomplete (weak) Fraïssé class* if instead of PP it has:

- *WPP (Weak Polish Property)*: The pseudo-metric $d^{\mathcal{K}}$ is separable on $\mathcal{K}_n$ for each n.

6. Weak Fraïssé classes and perturbations

In this last section we point out the relations between weak Fraïssé classes and approximate isomorphisms on the one hand, and perturbations of metric structures as defined in [Ben08b] on the other hand. Let us start with a fairly "trivial" example of a weak Fraïssé class which is not a Fraïssé class, namely with approximate isomorphisms which are not necessarily intrinsic, and with a limit which is (weakly unique but) not unique.

Example 6.1. We go back to the Gurarij space, the Fraïssé limit of finite dimensional Banach spaces. This is a good example of a structure which is approximately ultra-homogeneous but not ultra-homogeneous in the exact sense: indeed, any two vectors of norm one generate isomorphic substructures, but there may be no automorphism of $\mathbf{G}$ sending one to the other, since it may happen that one is smooth while the other is not. Equivalently stated, one Gurarij space with a named vector of norm one need not be isomorphic to another.

Let $\mathcal{K}$ be the class of finite dimensional Banach spaces equipped with a named vector of norm one. Such structures will be denoted $E = (E, v^E)$, where v^E is the named vector. We say that $\psi \in \mathrm{Apx}_0(E, F)$ if $\psi - \psi(v^E, v^F)$ is an approximate isomorphism in the sense of plain Banach spaces, and then define $\mathrm{Apx}(E, F) = \overline{\mathrm{Apx}_0(E, F)}$ (as per Definition 5.15). We leave it to the reader to check that $(\mathcal{K}, \mathrm{Apx})$ is a weak Fraïssé class, that its limits are exactly the Gurarij spaces with a named vector of norm one, and that every $\psi \in \mathrm{Apx}^{\min}\big((\mathbf{G}, v), (\mathbf{G}, w)\big)$ is of the form $\theta + d(\theta v, w)$ for some $\theta \in \mathrm{Aut}(\mathbf{G})$. In this case, weak uniqueness of the limit just means that for every $v, w \in \mathbf{G}$ of norm one there are $\theta \in \mathrm{Aut}(\mathbf{G})$ which send v arbitrarily close to w, a fact which also follows directly from the approximate ultra-homogeneity of $\mathbf{G}$ as a limit of an ordinary Fraïssé class.

(The class $\mathcal{K}$ also satisfies the NAP and therefore also admits an *intrinsic* notion of approximate isomorphisms, which can be generated as above if, for $\psi \in \mathrm{Apx}_0(E, F)$, we add the requirement that $\psi(v^E, v^F) = 0$.)

This is merely a special case of an entire family of examples, obtained from any Fraïssé class $\mathcal{K}$ by fixing some $\bar{a} \in \mathcal{K}_n$ and considering all members of $\mathcal{K}$ which contains a named substructure isomorphic to $\langle \bar{a} \rangle$, with approximate isomorphisms defined as above (with $\psi - \max_i \psi(a_i^{\mathfrak{A}}, a_i^{\mathfrak{B}})$ in place of $\psi - \psi(v^E, v^F)$).

When the limit $\mathfrak{M}$ of a Fraïssé class $\mathcal{K}$ is an $\aleph_0$-categorical structure, as is the case of the Gurarij space, or even more generally, when it is approximately $\aleph_0$-saturated, the weak uniqueness of $\mathfrak{M}$ together with a named finite (or, with the correct definitions, even arbitrary) tuple $\bar{a}$, as in Example 6.1, can also be accounted for in terms of perturbations: the structure $(\mathfrak{M}, \bar{a})$ is separable and approximately $\aleph_0$-saturated up to perturbation of the constant symbols naming $\bar{a}$, and is therefore unique up to such perturbations.

Let us explore this relation a little further, being intentionally brief. Let T be a complete theory, this time in the sense of ordinary uniformly continuous and bounded continuous logic, which we may assume has quantifier elimination, and let $\mathfrak{p}$ be a perturbation system for T, as defined in [Ben08b]. We recall that $\mathfrak{p}$ can be coded as a family of lower semi-continuous $[0, \infty]$-valued distance functions $d_{\mathfrak{p},n}$ on $\mathrm{S}_n(T)$, satisfying:

(i) Invariance under permutation of the variables.
(ii) If $p(\bar{x}, y) \in \mathrm{S}_{n+1}(T)$, $q(\bar{x}) \in \mathrm{S}_n(T)$, then $d(p{\restriction}_{\bar{x}}, q) \leq r$ if and only if there exists $q'(\bar{x}, y) \in \mathrm{S}_{n+1}(T)$ such that $q = q'{\restriction}_{\bar{x}}$ and $d(p, q') \leq r$.
(iii) Respect of equality: if $d_{\mathfrak{p}}(p, q) < \infty$ and $p \vDash x_i = x_j$ then $q \vDash x_i = x_j$ as well.

From now on, we just write $d_{\mathfrak{p}}$, since n is always clear from the context. For $\mathfrak{M}, \mathfrak{N} \vDash T$, the set $\mathrm{BiPert}_{\mathfrak{p}(r)}(\mathfrak{M}, \mathfrak{N})$ consists of all bijections $\theta \colon M \to N$ (not necessarily isometric), such that for all $\bar{a} \in M^k$: $d_{\mathfrak{p}}\big(\mathrm{tp}(\bar{a}), \mathrm{tp}(\theta\bar{a})\big) \leq r$. The conditions above are equivalent to the property that whenever $p, q \in \mathrm{S}_n(T)$ and $d(p, q) \leq r$, there exist $\mathfrak{M}, \mathfrak{N} \vDash T$, $\theta \in \mathrm{BiPert}_{\mathfrak{p}(r)}(\mathfrak{M}, \mathfrak{N})$, and $\bar{a} \in M^n$, such that $\bar{a} \vDash p$, $\theta\bar{a} \vDash q$.

Since members of $\mathrm{BiPert}(\mathfrak{M}, \mathfrak{N})$ are not required to be isometric, we need some means of obtaining an approximate isometry from an arbitrary bijection $\theta \colon X \to Y$ between metric spaces. We define

$$[\theta](x, y) = \sup_{z \in X} |d(x, z) - d(y, \theta z)|.$$

Indeed, this is clearly 1-Lipschitz in each argument. To see that it is Katětov in, say, the second argument, it is enough to observe that $[\theta](x, y) \geq d(\theta x, y)$, and similarly for the first. If θ is isometric then $[\theta] = \theta$.

Fact 6.2. *By a compactness argument, it follows from the requirement that $d_\mathfrak{p}$ is lower semi-continuous on $\mathrm{S}_2(T)$, that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\theta \in \mathrm{BiPert}_{\mathfrak{p}(\delta)}(\mathfrak{M}, \mathfrak{N})$ then $|d(x,y) - d(\theta x, \theta y)| \leq \varepsilon$ for all $x, y \in M$, or equivalently, $[\theta](x, \theta x) < \varepsilon$ for all $x \in M$.*

Let $\mathcal{K}$ be the class of finitely generated substructures of models of T. For $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ and $\psi\colon A \rightsquigarrow B$, say that $\psi \in \mathrm{Apx}(\bar{a}, \bar{b})$ if there are models $\mathfrak{A} \subseteq \mathfrak{M} \models T$, $\mathfrak{B} \subseteq \mathfrak{N} \models T$, a real number $r \geq 0$, and an r-perturbation $\theta \in \mathrm{BiPert}_{\mathfrak{p}(r)}(\mathfrak{M}, \mathfrak{N})$, such that $\psi \geq r + [\theta]$. If this is the case and, say, $\mathfrak{M} \subseteq \mathfrak{M}' \models T$, then for an appropriate ultra-filter $\mathscr{U}$ we have $\mathfrak{M}' \subseteq \mathfrak{M}^{\mathscr{U}}$, and $\theta^{\mathscr{U}} \in \mathrm{BiPert}_{\mathfrak{p}(r)}(\mathfrak{M}^{\mathscr{U}}, \mathfrak{N}^{\mathscr{U}})$ also satisfies $\psi \geq r + [\theta^{\mathscr{U}}]$. It follows by an elementary chain argument that if $\psi \in \mathrm{Apx}(\mathfrak{A}, \mathfrak{B})$ and $\varphi \in \mathrm{Apx}(\mathfrak{B}, \mathfrak{C})$, then there are witnesses $\theta_0 \in \mathrm{BiPert}_{\mathfrak{p}(r_0)}(\mathfrak{M}_0, \mathfrak{M}_1)$ and $\theta_1 \in \mathrm{BiPert}_{\mathfrak{p}(r_1)}(\mathfrak{M}_1, \mathfrak{M}_2)$ which witness these, with the same $\mathfrak{M}_1$, in which case $\theta_1\theta_0 \in \mathrm{BiPert}_{\mathfrak{p}(r_0+r_1)}(\mathfrak{M}_0, \mathfrak{M}_2)$ witnesses that $\varphi\psi \in \mathrm{Apx}(\mathfrak{A}, \mathfrak{C})$. Thus $(\mathcal{K}, \mathrm{Apx})$ is a category. In addition, $\mathrm{Apx}(\mathfrak{A}, \mathfrak{B})$ is closed under approximation and pseudo-inversion, and given a sequence $(\psi_n) \subseteq \mathrm{Apx}(\mathfrak{A}, \mathfrak{B})$ which converges to some φ, any (non principal) ultra-product of the respective witnesses will yield a witness that $\varphi \in \mathrm{Apx}(\mathfrak{A}, \mathfrak{B})$, so $(\mathcal{K}, \mathrm{Apx})$, with the obvious forgetful functor, is an approximate category. The space $\overline{\mathcal{K}_n}$ can then be identified with types in $\mathrm{S}_n(T)$, on which $d^{\mathcal{K}}$ is uniformly equivalent to $\tilde{d}_\mathfrak{p}$ as defined in [Ben08b] (specifically, $\tilde{d}_\mathfrak{p} \leq d^{\mathcal{K}} \leq 2\tilde{d}_\mathfrak{p}$).

A quick review of the definition of a weak Fraïssé class reveals that the only property which $\mathcal{K}$ may fail is separability (in particular, DAP frollows from Fact 6.2). Moreover, each formula is actually uniformly continuous (rather than merely continuous) with respect to $d^{\mathcal{K}}$, a fact which we shall use below. Now, if T admits a separable $\mathfrak{p}$-approximately $\aleph_0$-saturated model, then $\mathcal{K}$ is separable (that is, $\mathcal{K}_n$ is separable for all n), and therefore a weak Fraïssé class. Conversely, if $\mathcal{K}$ it a weak Fraïssé class then more or less by definition, the limits of $\mathcal{K}$ are exactly the separable $\mathfrak{p}$-approximately $\aleph_0$-saturated models of T, which then necessarily exist. As for uniqueness, we have, for separable $\mathfrak{p}$-approximately $\aleph_0$-saturated models $\mathfrak{M}$ and $\mathfrak{N}$:

(i) By [Ben08b, Proposition 2.7], for arbitrarily small $\varepsilon > 0$ there are actual maps $\theta \in \mathrm{BiPert}_{\mathfrak{p}(\varepsilon)}(\mathfrak{M}, \mathfrak{N})$.
(ii) By Theorem 5.18, for arbitrarily small $\varepsilon > 0$ there are ε-bijective approximate isomorphisms $\psi \in \mathrm{Apx}^{\min}(\mathfrak{M}, \mathfrak{N})$.

The first result implies the second, since for every $\varepsilon > 0$ we may take $0 < \delta < \varepsilon$ as in Fact 6.2, and if $\theta \in \mathrm{BiPert}_{\mathfrak{p}(\delta)}(\mathfrak{M}, \mathfrak{N})$ then $[\theta] + \delta \in \mathrm{Apx}(\mathfrak{M}, \mathfrak{N})$ is uniformly 2ε-bijective. If $\mathfrak{p}$ is such that all perturbation maps are isometric, then we have a converse: every $\psi \in \mathrm{Apx}^{\min}(\mathfrak{M}, \mathfrak{N})$ is of the form $\theta + r$ $(= [\theta] + r)$ with $\theta \in \mathrm{BiPert}_{\mathfrak{p}(r)}(\mathfrak{M}, \mathfrak{N})$. On the other hand, it is not at all clear why anything of this kind should be true without the isometry assumption.

Rather than conclude that our treatment here is weaker, in the case of non isometric perturbations, than that of [Ben08b], we suggest that it is merely a different (incompatible?), and in some sense, much better, generalisation of the isometric case. To argue that it is better, let us make two observations.

- Morally, one should be able to consider a $\varnothing$-definable set D as a sort unto itself. However, a perturbation map $\theta \in \mathrm{BiPert}_{\mathfrak{p}(\varepsilon)}(\mathfrak{M}, \mathfrak{N})$ need not restrict to a bijection of $D^{\mathfrak{M}}$ with $D^{\mathfrak{N}}$. This is for example the case of the Banach-Mazur distance perturbation on Banach spaces (viewed as a bounded metric structure via the emboundment construction), in which the unit ball is a definable subset, and is in fact often used as "the" home sort in which the entire Banach space is coded. Indeed, on the entire Banach space the perturbation maps are linear isomorphisms θ such that $\|\theta\|$ and $\|\theta^{-1}\|$ are close to one, and these do not restrict to bijections between the respective unit balls (see [Ben08a]).
- Let $\varphi(x, y)$ be a formula, and S_φ the sort of canonical parameters for φ, as explained in [BU10, Section 5]. It may very well happen that $\varphi(x, a)$ and $\varphi(x, b)$ coincide in $\mathfrak{M}$, while $\varphi(x, \theta a)$ and $\varphi(x, \theta b)$ differ in $\mathfrak{N}$, which means that θ need not even induce a well defined map between $S_\varphi^{\mathfrak{M}}$ and $S_\varphi^{\mathfrak{N}}$.

Since abstract model theory is ordinarily expected to "pass" to definable sets and imaginary sorts, one cannot but conclude that the idea that the perturbation of one structure into another should be given by a *bijective map* between the two is flawed. These can be overcome by presenting perturbations as nearly bijective approximate isometries. We content ourselves here with pointing out the main ideas.

- A set D is definable if the distance to D is a definable predicate. Somewhat informally, if an ε-bijective approximate isometry $\psi\colon \mathfrak{M} \rightsquigarrow \mathfrak{N}$ codes a perturbation small enough not to change the distance to D by more than ε', then its restriction to $D^{\mathfrak{M}} \times D^{\mathfrak{N}}$ is $(\varepsilon + \varepsilon')$-bijective.

- Let us treat the case of canonical parameters for a formula $\varphi(x,y)$ more carefully. Let Δ_φ be a uniform continuity modulus for φ with respect to $d^{\mathcal{K}}$, that is to say that if $d^{\mathcal{K}}(ab,a'b') < \Delta_\varphi(\varepsilon)$ then $\big|\varphi(a,b) - \varphi(a',b')\big| < \varepsilon$, and for $a \in M$, $a' \in N$ define

$$[\psi]_\varphi\big([a]_\varphi,[a']_\varphi\big) = \inf_{b\in M, b'\in N} d\big([a]_\varphi,[b]_\varphi\big) + \Delta_\varphi^{-1}\Delta_{\mathcal{K}}^{-1}\psi(b,b') + d\big([a']_\varphi,[b']_\varphi\big).$$

This is clearly 1-Lipschitz, so in order to check that it is an approximate isometry it will suffice, by symmetry, to check that for $a,a' \in M$ and $b,b' \in N$,

$$d\big([a]_\varphi,[a']_\varphi\big) \le \Delta_\varphi^{-1}\Delta_{\mathcal{K}}^{-1}\psi(a,b) + d\big([b]_\varphi,[b']_\varphi\big) + \Delta_\varphi^{-1}\Delta_{\mathcal{K}}^{-1}\psi(a',b').$$

Assume not, namely that for some $c \in M$ and $s,t,r \in \mathbf{R}$:

$$|\varphi(c,a) - \varphi(c,a')| > s+t+r, \qquad s > \Delta_\varphi^{-1}\Delta_{\mathcal{K}}^{-1}\psi(a,b),\ t > d\big([b]_\varphi,[b']_\varphi\big),\ r > \Delta_\varphi^{-1}\Delta_{\mathcal{K}}^{-1}\psi(a',b').$$

In particular, $\inf\psi < \Delta_{\mathcal{K}}\Delta_\varphi(\min r,s)$. Let $\psi' > \psi$ be such that all the strict inequalities above hold with ψ' instead of ψ. Then there exists $\psi'' < \psi$ and c' in $\mathfrak{N}$ (or in an extension thereof) with $\psi''(c,c') < \Delta_\varphi(\min s,r)$, in which case

$$\begin{aligned} |\varphi(c,a) - \varphi(c,a')| &\le |\varphi(c,a) - \varphi(c',b)| + |\varphi(c',b) - \varphi(c',b')| + |\varphi(c',b') - \varphi(c,a')| \\ &\le s + d\big([b]_\varphi,[b']_\varphi\big) + r < s+t+r, \end{aligned}$$

a contradiction. Thus $[\psi]_\varphi$ is indeed an approximate isometry, and if ψ is ε-bijective then $[\psi]_\varphi$ is $\Delta_\varphi^{-1}\Delta_{\mathcal{K}}^{-1}(\varepsilon)$-bijective.

A proper treatment of the approximate isometry approach to perturbations exceeds the intended scope of the present paper, and may be the topic of a future one.

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Itaï Ben Yaacov, Université Claude Bernard – Lyon 1, Institut Camille Jordan, CNRS UMR 5208, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France

URL: `http://math.univ-lyon1.fr/~begnac/`