

# MODEL THEORETIC STABILITY AND DEFINABILITY OF TYPES, AFTER A. GROTHENDIECK

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ABSTRACT. We point out how the “Fundamental Theorem of Stability Theory”, namely the equivalence between the “non order property” and definability of types, proved by Shelah in the 1970s, is in fact an immediate consequence of Grothendieck’s “Critères de compacité” from 1952. The familiar forms for the defining formulae then follow using Mazur’s Lemma regarding weak convergence in Banach spaces.

In a meeting in Kolkata in January 2013, the author asked the audience who it was, and when, to have first defined the notion of a stable formula, and to the expected answer replied that, no, it had been Grothendieck, in the fifties. This was meant as a joke, of course – a more exact statement would be that in Théorème 6 and Proposition 7 of Grothendieck [Gro52] there appears a condition (see (1) and (2) below) which can be recognised as the “non order property” (NOP)<sup>1</sup>. It took (us) a while longer to realise that one could ask, quite seriously, who first proved the “Fundamental Theorem of Stability Theory”, namely, the equivalence between NOP and definability of types, and the answer would essentially be the same. (As a model theoretic result, this was first proved by Shelah [She90], probably in the seventies, generalising Morley’s result that in a totally transcendental theory all types are definable.)

In everything that follows, if  $X$  is a topological space then  $C_b(X)$  denotes the Banach space of bounded, complex-valued functions on  $X$ , equipped with the supremum norm. A subset  $A \subseteq C_b(X)$  is *relatively weakly compact* if it has compact closure in the weak topology on  $C_b(X)$ .

**Fact 1** (Grothendieck [Gro52, Proposition 7]). *Let  $G$  be a topological group (in fact, it suffices that the product be separately continuous). Then the following are equivalent for a function  $f \in C_b(G)$ :*

- (i) *The function  $f$  is weakly almost periodic, i.e., the orbit  $G \cdot f \subseteq C_b(G)$ , say under right translation, is relatively weakly compact.*
- (ii) *Whenever  $g_n, h_n \in G$  form two sequences we have*

$$\lim_n \lim_m f(g_n h_m) = \lim_m \lim_n f(g_n h_m), \tag{1}$$

*as soon as both limits exist.*

This has been first brought to the author’s attention by A. BERENSTEIN (see [BBF11]). The first reference to (1) as “stability” is probably the Krivine-Maurey stability [KM81], where  $G$  is the additive group of a Banach space and  $f(x) = \|x\|$  (or rather,  $f(x) = \min(\|x\|, M)$  for some large  $M$ , since  $f$  should be bounded – in any case, Krivine and Maurey make no reference to Grothendieck’s result). As it happens, Fact 1 is a mere corollary of the following:

**Fact 2** (Grothendieck [Gro52, Théorème 6]). *Let  $X$  be an arbitrary topological space,  $X_0 \subseteq X$  a dense subset. Then the following are equivalent for a subset  $A \subseteq C_b(X)$ :*

- (i) *The set  $A$  is relatively weakly compact in  $C_b(X)$ .*
- (ii) *The set  $A$  is bounded, and whenever  $f_n \in A$  and  $x_n \in X_0$  form two sequences we have*

$$\lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x_m), \tag{2}$$

*as soon as both limits exist.*

Our aim in this note is to point out how, modulo standard translations between syntactic and topological formulations, the Fundamental Theorem is an immediate corollary of Fact 2. In fact, we prove a version of the Fundamental Theorem relative to a single model, as in Krivine-Maurey stability, which in

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<sup>1</sup>Iovino [Iov99] points out that NOP also appears in a characterisation of reflexive Banach spaces due to James [Jam64]. For a direct connection between weak almost periodicity and reflexive Banach spaces see for example Megrelishvili [Meg03, Theorem 4.6].

turn implies the usual version. Our argument adapts a similar argument given in [BT] in the context of  $\aleph_0$ -categorical structures.

Let us first recall a few definitions and facts regarding local types in standard first order logic. We fix a formula  $\varphi(x, y)$ , where  $x$  and  $y$  are disjoint tuples of variables, say singletons, for simplicity. If  $\mathbf{M}$  is a structure and  $a \in \mathbf{M}' \succeq \mathbf{M}$ , we define the  $\varphi$ -type  $\text{tp}_\varphi(a/M)$  as the collection of all instances  $\varphi(x, b)$ ,  $b \in M$ , such that  $\varphi(a, b)$  holds, and let  $S_\varphi(M)$  denote the space of all  $\varphi$ -types (we shall only consider  $\varphi$ -types over models). We equip  $S_\varphi(M)$  with the obvious topology, rendering it a compact, totally disconnected space. The clopen subsets of  $S_\varphi(M)$  are exactly those defined by Boolean combinations of instances of  $\varphi$  over  $M$  – we call such a Boolean combination a  $\varphi$ -formula over  $M$ . We say that a formula  $\psi(y)$  over  $M$  defines a  $\varphi$ -type  $p \in S_\varphi(M)$  if for every  $b \in M$  we have  $\varphi(x, b) \in p(x)$  if and only if  $\models \psi(b)$ .

In the setting of continuous logic (see [BU10] or [BBHU08]), the situation is essentially identical, *mutatis mutandis*. In fact, identifying *True* with the value zero and *False* with one, we can, and will, view the classical case described above as a special case of the following. Recall first that a *definable predicate* over  $M$  is a continuous combination of (possibly infinitely many) formulae over  $M$  (formulae being, by definition, finite syntactic objects), or equivalently, a uniform limit of formulae over  $M$ , or yet equivalently, a continuous function on  $S_n(M)$  where  $n$  is the number of arguments. For all semantic intents and purposes definable predicates are indistinguishable from formulae, and every  $\{0, 1\}$ -valued definable predicate is in fact a formula. We define  $p = \text{tp}_\varphi(a/M)$  as the function which associates to each instance  $\varphi(x, b)$ ,  $b \in M$ , the value  $\varphi(a, b)$ , which will then be denoted by  $\varphi(p, b)$ , or  $\varphi^b(p)$ . We equip  $S_\varphi(M)$  with the least topology in which all functions  $\varphi^b$  (for  $b \in M$ ) are continuous. It is compact and Hausdorff, and every continuous function on  $S_\varphi(M)$  can be expressed as a continuous combination of (possibly infinitely many, but at most countably many) functions of the form  $\varphi^b$ , or equivalently, as a uniform limit of finite continuous combinations – such a definable predicate will be called a  $\varphi$ -predicate over  $M$ . A definable predicate  $\psi(y)$  over  $M$  defines  $p \in S_\varphi(M)$  if  $\varphi_p(b) = \psi(b)$  for all  $b \in M$ .

Finally, as per [BU10, Appendix B], we say that  $\varphi(x, y)$  is *stable in a structure*  $\mathbf{M}$  if whenever  $a_n, b_n \in M$  form two sequences we have

$$\lim_n \lim_m \varphi(a_n, b_m) = \lim_m \lim_n \varphi(a_n, b_m), \quad (3)$$

as soon as both limits exist. We say that  $\varphi$  is *stable in a theory*  $T$  if it is stable in every model of  $T$ . We leave it to the reader to check that this is merely a rephrasing of the familiar NOP.

We first prove the Fundamental Theorem for stability inside a model.

**Theorem 3.** *Let  $\varphi(x, y)$  be a formula stable in a structure  $\mathbf{M}$ . Then every  $p \in S_\varphi(M)$  is definable by a (unique)  $\tilde{\varphi}$ -predicate  $\psi(y)$  over  $M$ , where  $\tilde{\varphi}(y, x) = \varphi(x, y)$  (in the case of classical logic, a  $\tilde{\varphi}$ -formula).*

*Proof.* Let  $X = S_\varphi(M)$  and let  $X_0 \subseteq X$  be the collection of those types realised in  $M$ , which is dense in  $X$ . Since  $X$  is compact we have  $C_b(X) = C(X)$ . For  $a \in M$  let  $\varphi_a = \tilde{\varphi}^a$ , so  $A = \{\varphi_a : a \in M\} \subseteq C(X)$  is bounded (since every formula is). Thus, by Fact 2,  $\varphi$  is stable in  $\mathbf{M}$  if and only if  $A$  is relatively weakly compact.

Assume now that  $\varphi$  is indeed stable in  $\mathbf{M}$ , let  $p \in S_\varphi(M)$ , and let  $a_i \in M$  form a net such that  $\text{tp}_\varphi(a_i/M) \rightarrow p$ . By Fact 2 we may assume that  $\varphi_{a_i}$  converges weakly to some  $\psi \in C(X)$ . Then  $\psi$  is a  $\tilde{\varphi}$ -predicate over  $M$ , and for  $b \in M$  we have

$$\varphi(p, b) = \lim \varphi(a_i, b) = \psi(b),$$

as desired. The uniqueness of  $\psi$  is by density of the realised types. ■

With small variations, this appears in Pillay [Pil83, Corollary 2.3] for classical logic, or in [BU10, Theorem B.4] for continuous logic (the latter also asserts that the defining formula is an *increasing* continuous combination of instances of  $\tilde{\varphi}$ , which follows from Theorem 3 modulo Mazur's Lemma, see Corollary 7). The Fundamental Theorem follows:

**Corollary 4** (Fundamental Theorem of Stability). *Let  $\varphi(x, y)$  be a formula and  $T$  a theory. Then the following are equivalent.*

- (i) *The formula  $\varphi$  is stable in  $T$ .*
- (ii) *For every model  $\mathbf{M} \models T$ , every  $\varphi$ -type over  $M$  is definable by a  $\tilde{\varphi}$ -predicate over  $M$ .*
- (iii) *For every model  $\mathbf{M} \models T$ , every  $\varphi$ -type over  $M$  is definable over  $M$  (by some definable predicate).*
- (iv) *Let  $\mathbf{M} \models T$ , and let  $d$  denote the metric of uniform convergence on  $S_\varphi(M)$  (i.e.,  $d(p, q) = \|\varphi(p, \cdot) - \varphi(q, \cdot)\|_\infty$ ). Then the density character of  $(S_\varphi(M), d)$  is at most the density character of  $M$  plus  $|T|$  (in the classical settings all the distances are discrete and the density character is the same as the cardinal).*

- (v) *There exists a cardinal  $\kappa \geq |T|$  (in fact, any  $\kappa = (\kappa_0 + |T|)^{\aleph_0}$  will do, and in the classical setting, any  $\kappa \geq |T|$  will do) such that if  $\mathbf{M} \models T$ ,  $|M| \leq \kappa$  then  $|S_\varphi(M)| \leq \kappa$  as well.*

*Proof.* The chain of implications (ii)  $\implies$  (iii)  $\implies$  ...  $\implies$  (i) is straightforward and only requires elementary model theoretic methods and counting arguments. The most “involved” implication is (i)  $\implies$  (ii), which is by [Theorem 3](#).  $\blacksquare$

The Banach space formalism also allows us to obtain slight improvements quite easily. First, regarding the case of stability in a single structure, we can improve [Theorem 3](#) as follows.

**Theorem 5.** *Let  $\varphi(x, y)$  be a formula and  $\mathbf{M}$  a structure, and for  $a \in M$  let  $\varphi_a = \tilde{\varphi}^a: q \mapsto \varphi(a, q)$ . Then the following are equivalent.*

- (i) *The formula  $\varphi$  is stable in  $\mathbf{M}$ .*
- (ii) *Every  $p \in S_\varphi(M)$  is definable by a  $\tilde{\varphi}$ -predicate  $\psi_p$  over  $M$ , and the map  $p \mapsto \psi_p$  is a homeomorphic embedding of  $S_\varphi(M)$  in the weak topology on  $C(S_{\tilde{\varphi}}(M))$ .*
- (iii) *If  $p \in S_\varphi(M)$  is an accumulation point of a sequence  $\{\text{tp}_\varphi(a_n/M)\}$  then there exists a sub-sequence  $a_{n_k}$  such that  $\varphi_{a_{n_k}}$  converges point-wise on  $S_{\tilde{\varphi}}(M)$  to a definition of  $p$ .*

Moreover, in this case every  $p \in S_\varphi(M)$  is the limit of a sequence of realised types.

*Proof.* We continue with the notations of the proof of [Theorem 3](#).

(i)  $\iff$  (ii). If such a homeomorphism exists then  $A$  is relatively weakly compact, and  $\varphi$  is stable. For the converse, by the proof of [Theorem 3](#) the map sending  $p \mapsto \psi_p$  is a bijection with the weak closure of  $A$  (if  $\varphi_{a_i}$  form a weakly convergent net then, possibly passing to a sub-net, we may assume that  $\text{tp}_\varphi(a_i/M)$  converge), which is weakly compact. Since its inverse is clearly continuous, it is a homeomorphism.

(i)  $\implies$  (iii). Let  $p \in S_\varphi(M)$  be defined by  $\psi$ . Since we are only interested in a single formula, we may assume that the language is countable, and find a separable (or countable)  $\mathbf{M}_0 \preceq \mathbf{M}$  containing the sequence  $\{a_n\}$ . Let  $Y = S_{\tilde{\varphi}}(M_0)$ , so we have  $X \twoheadrightarrow Y$  and  $\psi \in C(Y) \subseteq C(X)$  also defines the restriction  $p_0 = p|_{M_0}$ . Since  $M_0$  is separable, there exists a sub-sequence  $a_{n_k}$  such that  $\text{tp}_\varphi(a_{n_k}/M_0) \rightarrow p_0$ . Since  $\varphi$  is stable in  $M_0$ ,  $\varphi_{a_{n_k}} \rightarrow \psi$  point-wise on  $Y$  and therefore on  $X$ .

(iii)  $\implies$  (i). By the Eberlein-Šmulian Theorem (see Whitley [[Whi67](#)]), and since point-wise convergence of a bounded sequence in  $C(X)$  implies weak convergence (since  $X$  is compact, by the Dominated Convergence Theorem),  $A$  is relatively weakly compact, so  $\varphi$  is stable in  $\mathbf{M}$ .

For the moreover part, just argue as above, taking  $\mathbf{M}_0$  to contain the (countably many) parameters needed for the definition  $\psi$ , and taking  $a_n$  to be any sequence in  $M_0$  such that  $\text{tp}_\varphi(a_n/M_0) \rightarrow p_0$ .  $\blacksquare$

The second point is with respect to the form of the defining  $\tilde{\varphi}$ -predicate, and in particular uniform definability when the formula is stable in the theory.

**Fact 6** (Mazur’s Lemma). *Let  $E$  be a Banach space, and let  $A \subseteq E$ . Then the weak closure of  $A$  is contained in the closure of the convex hull of  $A$ .*

*Proof.* Since a closed convex set is weakly closed (Hahn-Banach Theorem, see Brezis [[Bre83](#)]).  $\blacksquare$

**Corollary 7.** *Let  $\varphi(x, y)$  be a formula.*

- (i) *If  $\varphi$  is stable in a structure  $\mathbf{M}$  then the definition of a type  $p \in S_\varphi(M)$  can be written as a uniform limit of formulae of the form  $\frac{1}{n} \sum_{i < n} \varphi(a_i, y)$ , where  $a_i \in M$ .*
- (ii) *If  $\varphi$  is  $\{0, 1\}$ -valued, as in classical logic, the definition can be written as a single “majority rule” combination of instances  $\varphi(a_i, y)$ .*
- (iii) *If  $\varphi$  is stable in a theory  $T$ , this can be done uniformly for all  $\varphi$ -types over models (i.e., with the rate of uniform convergence, or number of instances of which a majority is required, fixed).*

*Proof.* The first item is by Mazur’s Lemma, and implies the second. For the third item, add a new unary predicate  $P$ . Then it is expressible in first order continuous logic that  $P$  is the distance to an elementary sub-structure, and a standard compactness argument yields that if  $\varphi$ -types realised in models of  $T$  over elementary sub-models were not uniformly definable in this fashion, one would not be definable at all, and we are done.  $\blacksquare$

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