GENERIC ORBITS AND TYPE ISOLATION IN THE GURARIJ SPACE

ITAÏ BEN YAACOV AND C. WARD HENSON

Abstract. We study the question of when the space of embeddings of a separable Banach space E in the separable Gurarij space $\mathbf{G}$ admits a generic orbit under the action of linear isometry group of $\mathbf{G}$. The question is recast in model-theoretic terms, namely type isolation and the existence of prime models, albeit without use of formal logic. We show that if the set of isolated types over E is dense then a dense G_δ orbit exists, and otherwise all orbits are meagre. We then study some (families of) examples with respect to this dichotomy. We also point out that the class of Gurarij spaces is the class of models of an $\aleph_0$-categorical theory with quantifier elimination, and calculate the density character of the space of types over E, answering a question of Avilés et al.

Contents

Introduction

In 1966, Gurarij [Gur66] defined what came to be known as the (separable) Gurarij space, and proved that it is almost isometrically unique. The isometric uniqueness of the Gurarij space was proved in 1976 by Lusky [Lus76]. In the same paper, Lusky points out that the arguments could be modified to prove also the isometric uniqueness of the separable Gurarij space equipped with a distinguished smooth unit vector. In other words, if $\mathbf{G}$ denotes the separable Gurarij space, then the set of smooth unit vectors in $\mathbf{G}$ forms an orbit under the action of the linear isometry group $\mathrm{Aut}(\mathbf{G})$. By Mazur [Maz33], this orbit is moreover a dense G_δ subset of the unit sphere.

These facts are strongly reminiscent of model theoretic phenomena, and indeed turn out to be special cases of such. It was observed some time ago by the second author that the uniqueness of the Gurarij space can be accounted for as it being the unique separable model of an $\aleph_0$-categorical theory, which moreover eliminates quantifiers. Similarly, the Gurarij space is atomic over a vector if and only if the latter is smooth, so Lusky's second uniqueness result is a special case of the uniqueness of the prime model.

These observations serve as a starting point for the present paper, whose goals are threefold:

- Our primary goal is to make the observations above precise, and generalise them to uniqueness results over a subset other than the empty set or a singleton – in other words, we study uniqueness and primeness of the Gurarij space over a subspace E of dimension possibly greater than one. We prove that if a separable space E admits a copy in $\mathbf{G}$ over which $\mathbf{G}$ is atomic then the set of such embeddings forms a dense G_δ orbit among all linear isometric embeddings of E in $\mathbf{G}$, and otherwise all orbits are meagre. We give some sufficient conditions for such an atomic

2010 *Mathematics Subject Classification.* 46B04 ; 03C30 ; 03C50 ; 03C98.
Key words and phrases. Gurarij space ; Banach space ; isolated type ; atomic model ; prime model ; group action ; generic orbit.
Research supported by the Institut Universitaire de France, by ANR project GruPoLoCo (ANR-11-JS01-008) and by a grant from the Simons Foundation (#202251 to the second author).
Revision 1989 *of* 27th November 2014.

embedding to exist for finite-dimensional spaces E (e.g., smooth, polyhedral, or of dimension ≤ 3 – see Theorem 6.4). We also give some examples for which an atomic embedding does not exist, leaving open the question of a general necessary and sufficient condition on a separable Banach space E to admit an atomic embedding in $\mathbf{G}$.
- A secondary goal is to present some tools and techniques of model theory in a manner accessible to non-logicians. Starting with a definition of types and type spaces which does not make any use of formal logic, we discuss general topics such as type isolation, the Tarski-Vaught Criterion, the Omitting Types Theorem, and the primeness and uniqueness of atomic models. While we do this in a fairly specific context, we present arguments that would be valid in the general case. Sometimes these are followed by separate results which improve the general ones in the specific context of the Gurarij space. There are few results which make explicit use of formal logic (essentially, Proposition 1.19 and Theorem 2.3); these serve mostly as parenthetical remarks required for the sake of completeness, and are not used in any way in other parts of the paper.
- A tertiary goal is to present to model theorists, who are familiar with the tools mentioned in the previous item in the context of classical logic, how these tools adapt to the metric setting.

In Section 1 we recall the definition of (quantifier-free) types and type spaces over a Banach space E, and study their properties. The topometric structure of the type space, a fundamental notion of metric model theory, is defined there, as well as (topometrically) isolated types, which are one of the main objects of study of this paper.

In Section 2 we start studying Gurarij spaces. At the technical level, we define and study Gurarij (and other) spaces which are atomic over a fixed separable parameter space E, and prove the Omitting Types Theorem (Theorem 2.12). We prove appropriate generalisations of the homogeneity and universality properties of the Gurarij space to homogeneity and universality over E. In particular, we show that the prime Gurarij spaces over E (see Corollary 2.13) are those Gurarij space which are separable and atomic over E, and that they are all isometrically isomorphic over E, denoted $\mathbf{G}[E]$. We also give the standard model theoretic criterion for the existence of $\mathbf{G}[E]$, namely that the isolated types over E are dense.

At this point we move on to the questions of when $\mathbf{G}[E]$ exists and how to characterise isolated types in a fashion suitable to the Banach space context. In Section 3 we consider the particularly easy case where $\dim E = 1$, giving a model-theoretic account of Lusky's result. Before considering the general case, we introduce an essential tool in Section 4, namely the presentation of 1-types as convex Katětov functions (as per [Ben14]), and the Legendre-Fenchel transformation of these. In Section 5 we characterise atomic 1-types in terms of their Legendre-Fenchel conjugate, which allows us to give in Section 6 some (families of) examples of spaces E such that $\mathbf{G}[E]$ is known to exist or not to exist.

We conclude in Section 7 with a "counting types" result, showing that the space of types over E is metrically separable if and only if E is finite-dimensional and polyhedral. This allows us to answer a question of Avilés et al. [ACC+11].

Throughout, E, F and so on denote normed spaces over the real numbers (with the exception of some parts of Section 4, where general locally convex topological vector spaces are considered). An embedding (or isomorphism, automorphism) of normed spaces is always isometric.

The topological dual of a normed space E will be denoted E^*. We shall often use the notation $B(E)$ for the closed unit ball of E, $\partial B(E)$ for the unit sphere (which, regardless of topology, is the boundary of $B(E)$ in the sense of convex geometry), and similarly for the dual space.

1. Quantifier-free types in Banach spaces

Before we start, let us state the following basic amalgamation result which we shall use many times, quite often implicitly.

Fact 1.1. *For any three Banach spaces E, F_0 and F_1, and isometric embeddings $f_i\colon E \to F_i$, there is a third Banach space G and isometric embeddings $g_i\colon F_i \to G$ such that $g_0 f_0 = g_1 f_1$.*

Proof. Equip the direct sum $F_0 \oplus F_1$ with the semi-norm $\|v + u\| = \inf_{w \in E} \|v + w\| + \|u - w\|$, divide by the kernel and complete. ■

We can now define the fundamental objects of study of this section, and, to a large extent, the entire paper.

Definition 1.2. Let E be a Banach space and X a sequence of symbols which we call *variables*. We let $E(X) = E \oplus \bigoplus_{x \in X} \mathbf{R}x$, and define $\mathrm{S}_X(E)$ to consist of all semi-norms on $E(X)$ which extend the norm on E, calling it the *space of types in X over E*. We shall denote members of $\mathrm{S}_X(E)$ by ξ, ζ and so on, and the corresponding semi-norms by $\|\cdot\|^\xi$, $\|\cdot\|^\zeta$ and so on (model-theoretic tradition would have us denote types by p, q and so on, but an expression such as $\|\cdot\|^p$ may be disastrously suggestive of a meaning other than the intended one).

Quite often X will be of the form $\{x_i\}_{i\in I}$ for some index set I, in which case we write $E(I) = E \oplus \bigoplus_{i\in I} \mathbf{R}x_i$ instead of $E(X)$, and similarly $\mathrm{S}_I(E)$, whose members are called I-types.

Definition 1.3. Given a Banach space extension $E \subseteq F$ and an I-sequence $\bar{a} = \{a_i\}_{i\in I} \subseteq F$, we define its *type over* E, in symbols $\xi = \mathrm{tp}(\bar{a}/E) \in \mathrm{S}_I(E)$, to be the semi-norm $\|b + \sum \lambda_i x_i\|^\xi = \|b + \sum \lambda_i a_i\|$, and say that $\bar{a}$ *realises* ξ. When a sequence $\bar{b}$ generates E, we may also write $\mathrm{tp}(\bar{a}/\bar{b})$ for $\mathrm{tp}(\bar{a}/E)$.

Conversely, given a type $\xi \in \mathrm{S}_I(E)$, we define the Banach space *generated* by ξ, in symbols $E[\xi]$, as the space obtained from $\big(E(I), \|\cdot\|^\xi\big)$ by dividing by the kernel and completing, together with the distinguished generators $\{x_i\}_{i\in I} \subseteq E[\xi]$.

Definition 1.4. We equip $\mathrm{S}_I(E)$ with a topological structure as well as with a metric structure *which may be distinct*. The *topology* on $\mathrm{S}_I(E)$ is the least one in which, for every member $x \in E(I)$, the map $\hat{x} \colon \xi \mapsto \|x\|^\xi$ is continuous. Given $\xi, \zeta \in \mathrm{S}_I(E)$, we define the *distance* $d(\xi, \zeta)$ to be the infimum, over all F extending E and over all realisations $\bar{a}$ and $\bar{b}$ of ξ and ζ, respectively, of $\sup_i \|a_i - b_i\|$.

Remark 1.5. A model-theorist will recognise types as we define them here as *quantifier-free* types, which do not, in general, capture "all the pertinent information". However, by Fact 1.1, they do capture a maximal existential type. Moreover, it follows from Lemma 1.14 below (and more specifically, from the assertion that $\pi_{\bar{x}} \colon \mathrm{S}_{\bar{x},y}(0) \to \mathrm{S}_{\bar{x}}(0)$ is open) that being an existentially closed Banach space is an elementary property, so the theory of Banach spaces admits a model companion. Then Fact 1.1 can be understood to say that the model companion eliminates quantifiers, so quantifier-free types and types are in practice the same. As we shall see later, the model companion is separably categorical, and its unique separable model is $\mathbf{G}$, the separable Gurarij space.

It is fairly clear that the distance refines the topology, and we shall see that unless the parameter space E is trivial and I is finite, they are in fact distinct. In a sense, the distance as defined on $\mathrm{S}_I(E)$ is "incorrect" when I is infinite (for more reasons than the mere fact that this distance can be infinite), and we should never have defined it for such I if not for Proposition 1.7 below holding for infinite I as well.

Lemma 1.6. *Let E, F be Banach spaces, I an index set, and consider tuples $\bar{a} = (a_i)_{i\in I} \in E^I$, $\bar{b} \in F^I$ and $\bar{\varepsilon} \in \mathbf{R}^I$. Also let $\mathbf{R}^{(I)}$ denote the set of all I-tuples in which all but finitely many positions are zero. The following conditions are equivalent.*

(i) *There exists a semi-norm $\|\cdot\|$ on $E \oplus F$ extending the respective norms of E and F, such that for each $i \in I$ one has $\|a_i - b_i\| \leq \varepsilon_i$.*
(ii) *For all $\bar{r} \in \mathbf{R}^{(I)}$, one has*

$$\left| \left\| \sum r_i a_i \right\| - \left\| \sum r_i b_i \right\| \right| \leq \sum |r_i| \varepsilon_i.$$

Proof. One direction being trivial, we prove the other. For $c + d \in E \oplus F$ define

$$\|c + d\|' = \inf_{\bar{r} \in \mathbf{R}^{(I)}} \left\| c - \sum r_i a_i \right\| + \left\| d + \sum r_i b_i \right\| + \sum |r_i| \varepsilon_i.$$

This is easily checked to be a semi-norm, with $\|c\|' \leq \|c\|$ for $c \in E$. Now, for $c \in E$ and $\bar{r} \in \mathbf{R}^{(I)}$ we have

$$\left\| c - \sum r_i a_i \right\| + \left\| \sum r_i b_i \right\| + \sum |r_i| \varepsilon_i \geq \left\| c - \sum r_i a_i \right\| + \left\| \sum r_i a_i \right\| \geq \|c\|.$$

Therefore $\|c\|' = \|c\|$, and similarly $\|d\|' = \|d\|$ for $d \in F$, concluding the proof. ■

Proposition 1.7. *Let $\xi, \zeta \in \mathrm{S}_I(E)$, and let $E(I)_1$ consist of all $a + \sum \lambda_i x_i \in E(I)$ (so $a \in E$ and all but finitely many of the λ_i vanish) such that $\sum |\lambda_i| = 1$. Then*

$$d(\xi, \zeta) = \sup_{x \in E(I)_1} \left| \|x\|^\xi - \|x\|^\zeta \right|.$$

Moreover, the infimum in the definition of distance between types is attained.

Proof. Immediate from Lemma 1.6. ■

Convention 1.8. When referring to the topological or metric structure of $\mathrm{S}_I(E)$, we shall follow the convention that unqualified terms taken from the vocabulary of general topology (open, compact and so on) apply to the topological structure, while terms specific to metric spaces (bounded, complete and so on) refer to the metric structure.

Excluded from this convention is the notion of isolation which will be defined in a manner which takes into account both the topology and the metric.

While this convention may seem confusing at first, it is quite convenient, as in the following.

Lemma 1.9. (i) *The space $\mathrm{S}_I(E)$ is Hausdorff, and every closed and bounded set thereof is compact.*

(ii) *The distance on $\mathrm{S}_I(E)$ is lower semi-continuous. In particular, the closure of a bounded set is bounded.*

(iii) *Assume that I is finite, say $I = n = \{0, 1, \ldots, n-1\} \in \mathbf{N}$. Then every bounded set is contained in an open bounded set. It follows that the space $\mathrm{S}_n(E)$ is locally compact, and that a compact subset of $\mathrm{S}_n(E)$ is necessarily (closed and) bounded.*

(iv) *A subset $X \subseteq \mathrm{S}_n(E)$ is closed if and only if its intersection with every compact set is compact.*

(v) *Let $m \leq n$, and let $\pi\colon \mathrm{S}_n(E) \to \mathrm{S}_m(E)$ denote the obvious variable restriction map. Then for every $\xi \in \mathrm{S}_n(E)$ and $\zeta \in \mathrm{S}_m(E)$ we have $d(\pi\xi, \zeta) = d(\xi, \pi^{-1}\zeta)$. Moreover there exists $\rho \in \pi^{-1}\zeta$ such that $d(\pi\xi, \zeta) = d(\xi, \rho)$ and $\|x_i\|^\rho = \|x_i\|^\xi$ for all $m \leq i < n$.*

In particular, the map π is metrically open.

Proof. For the first item, clearly $\mathrm{S}_I(E)$ is Hausdorff. If $X \subseteq \mathrm{S}_I(E)$ is bounded, then for every $x \in E(I)$ there exists M_x such that $\|x\|^\xi \leq M_x$ for all $\xi \in X$. We can therefore identify X with a subset of $Y = \prod_x [0, M_x]$, and if X is closed in $\mathrm{S}_I(E)$ then it is closed in Y and therefore compact.

The second item follows from Proposition 1.7, and the third is immediate.

For the fourth item, assume that $X \subseteq \mathrm{S}_n(E)$ is not closed, let $\xi \in \overline{X} \smallsetminus X$ and let U be a bounded neighbourhood of ξ, in which case $\overline{U} \cap X$ is not compact.

For (v), the inequality $\leq$ is immediate. For the opposite inequality, there exists an extension $F \supseteq E$ and realisations $\bar{a}$ of ξ and $\bar{b}$ of ζ in F such that $\|a_i - b_i\| < r$ for $i < m$. Letting $c_i = b_i$ for $i < m$, $c_i = a_i$ for $m \leq i < n$, we see that $\rho = \mathrm{tp}(\bar{c}/E)$ is as desired for both the main assertion and the moreover part. It follows that $\pi B(\xi, r) \supseteq B(\pi\xi, r)$, so π is metrically open. ■

This double structure makes $\mathrm{S}_I(E)$ a *topometric space*, in the sense of [Ben08b].

Definition 1.10. We say that a type $\xi \in \mathrm{S}_n(E)$ is *isolated* if the distance and the topology agree at ξ, i.e., if every metric neighbourhood of ξ is also a topological one.

This is the definition of isolation in a topometric space, taking into account both the metric and the topological structure. Ordinary topological spaces can be viewed as topometric spaces by equipping them with the discrete $0/1$ distance, in which case the notion of isolation as defined here coincides with the usual one.

Many results regarding ordinary topological spaces still hold, when translated correctly, with the topometric definitions. For example, the fact that a dense set must contain all isolated points becomes the following. Notice that in Lemma 1.16 below we prove that the set of isolated types it itself metrically closed.

Lemma 1.11. *Let E be a Banach space, $D \subseteq \mathrm{S}_n(E)$ a dense, metrically closed set. Then D contains all isolated types.*

Proof. If ξ is isolated then all metric neighbourhoods of ξ, which are also topological neighbourhoods, must intersect D. ■

One of our aims in this paper is to characterise isolated types. We start with the easiest situation.

Proposition 1.12. *Let 0 denote the trivial Banach space. Then every type in $\mathrm{S}_n(0)$ is isolated. In other words, the distance on $\mathrm{S}_n(0)$ is compatible with the topology.*

Proof. Given $N \in \mathbf{N}$, let $X_N \subseteq 0(n)_1$ be the finite set consisting of all $\sum \lambda_i x_i$ where $\sum |\lambda_i| = 1$ and each λ_i is of the form $\frac{k}{N}$. For $\xi \in \mathrm{S}_n(0)$, let $U_{\xi,N}$ be its neighbourhood consisting of all ζ such that

$$\forall x \in X_N \qquad \|x\|^\xi - 1/N < \|x\|^\zeta < \|x\|^\xi + 1/N.$$

This means in particular that $\|x_i\|^\zeta < \|x_i\|^\xi + 1$ for all $i < n$, and now an easy calculation together with Proposition 1.7 yields that there exists a constant $C(\xi)$ such that for all N, $U_{\xi,N}$ is contained in the ball of radius $C(\xi)/N$ around ξ, which is what we had to show. ■

This already allows us to construct the following useful tool of variable change in a type.

Definition 1.13. Given a linear map $\varphi\colon E(\bar y) \to E(\bar x)$ extending id_E, we define a pull-back map $\varphi^*\colon \mathrm{S}_{\bar x}(E) \to \mathrm{S}_{\bar y}(E)$ by $\|z\|^{\varphi^*\xi} = \|\varphi z\|^\xi$ (for $z \in E(\bar y)$). For $A \subseteq \mathrm{S}_{\bar y}(E)$, we define $\varphi_* A = (\varphi^*)^{-1}(A) \subseteq \mathrm{S}_{\bar x}(E)$ (this will be particularly convenient in the proof of Lemma 2.11).

Of course, φ is entirely determined by the image $\varphi\bar y$. Thus, when the variables $\bar y$ are known from the context, we may write $\xi{\restriction}_{\varphi\bar y}$ for $\varphi^*\xi$, so

$$\Big\|a + \sum \lambda_i y_i\Big\|^{\xi\restriction_{\bar z}} = \Big\|a + \sum \lambda_i z_i\Big\|^{\xi}.$$

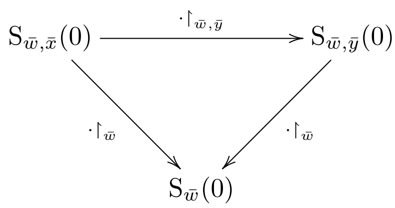
In fact, we shall often use this latter notation with $\bar z = \bar y$.

Lemma 1.14. *For a fixed tuple $\bar y \in E(\bar x)^m$, the map $\xi \mapsto \xi{\restriction}_{\bar y}$ is continuous and Lipschitz. If $\bar y$ are linearly independent over E then this map is also topologically and metrically open. Moreover, the metric openness is "Lipschitz" as well, in the sense that there exists a constant $C = C(\bar y)$ such that for all ξ and all $r > 0$ we have*

$$B(\xi, r){\restriction}_{\bar y} \supseteq B(\xi{\restriction}_{\bar y}, Cr).$$

Proof. Continuity and the Lipschitz condition are easy. We therefore assume that $\bar y$ are linearly independent over E, and we first prove the moreover part. In the special case where $\bar y$ generate $E(\bar x)$ over E, this is since $(\cdot{\restriction}_{\bar y})^{-1} = \cdot{\restriction}_{\bar x}\colon \mathrm{S}_{\bar y}(E) \to \mathrm{S}_{\bar x}(E)$ is Lipschitz. In the general case, we may complete $\bar y$ into a basis for $E(\bar x)$ over E, and using the special case above we reduce to the case where $y_i = x_i$ for $i < m$, which is just Lemma 1.9(v).

For topological openness, we proceed as follows. In the case where $E = 0$, this follows from metric openness and Proposition 1.12. Let us consider now the case where E is finite-dimensional. We fix a basis $\bar b$ for E and a corresponding tuple of variables $\bar w$. We may then identify $E(\bar x)$ with $0(\bar w, \bar x)$, and thus $\bar y$ with its image in $0(\bar w, \bar x)$. We already know that $\cdot{\restriction}_{\bar w,\bar y}\colon \mathrm{S}_{\bar w,\bar x}(0) \to \mathrm{S}_{\bar w,\bar y}(0)$ is open. In addition, we have a commutative diagram



and the map $\cdot{\restriction}_{\bar y}\colon \mathrm{S}_{\bar x}(E) \to \mathrm{S}_{\bar y}(E)$ is homeomorphic to the fibre of the horizontal arrow over $\mathrm{tp}(\bar b) \in \mathrm{S}_{\bar w}(0)$, so it is open as well. The infinite-dimensional case follows from the finite-dimensional one, since any basic open set in $\mathrm{S}_{\bar x}(E)$ can be defined using finitely many parameters in E. ∎

We leave it to the reader to check that if $\bar y$ are not linearly independent over E then $\cdot{\restriction}_{\bar y}$ is not metrically open, and *a fortiori* not topologically so (consider for example $\cdot{\restriction}_{x,x}\colon \mathrm{S}_1(0) \to \mathrm{S}_2(0)$).

Lemma 1.15. *Let $U \subseteq \mathrm{S}_n(E)$ be open and $r > 0$. Then $B(U, r)$ is open as well.*

Proof. Let $\bar x$ and $\bar y$ be two n-tuples of variables. Let us identify $\mathrm{S}_n(E)$ with $\mathrm{S}_{\bar x}(E)$, and let $W \subseteq \mathrm{S}_{\bar x,\bar y}(E)$ consist of all ξ such that $\|x_i - y_i\|^\xi < r$ for $i < n$. Then W is open, and by Lemma 1.14 so is $V = \big(W \cap (\cdot{\restriction}_{\bar x})^{-1}(U)\big){\restriction}_{\bar y} \subseteq \mathrm{S}_{\bar y}(E)$. Identifying $\mathrm{S}_{\bar y}(E)$ with $\mathrm{S}_n(E)$ as well, $V = B(U, r)$. ∎

Lemma 1.16. *Let E be a Banach space.*

(i) *A type in $\mathrm{S}_n(E)$ is isolated if and only if all its metric neighbourhoods have non empty interior.*
(ii) *The set of isolated types in $\mathrm{S}_n(E)$ is metrically closed.*

Proof. The first assertion follows easily from Lemma 1.15, and the second from the first. ∎

Another basic operation one can consider on types is the *restriction of parameters* $\mathrm{S}_n(F) \to \mathrm{S}_n(E)$ when $E \subseteq F$.

Lemma 1.17. *Let $E \subseteq F$ be an isometric inclusion of Banach spaces. Then the natural type restriction map $\pi\colon \mathrm{S}_n(F) \to \mathrm{S}_n(E)$ is continuous, closed, and satisfies $\pi B(\xi, r) = B(\pi\xi, r)$.*

In particular, π is both topologically and metrically a quotient map.

Proof. It is clear that π is continuous. To see that it is closed we use Lemma 1.9. Indeed, since closed sets are exactly those which intersect compact sets on compact sets, it will be enough to show that if $K \subseteq \mathrm{S}_n(E)$ is compact then so is $\pi^{-1}K$, which follows from the characterisation of compact sets as closed and bounded. Finally, it is clear that $d(\xi,\zeta) \geq d(\pi\xi,\pi\zeta)$ for $\xi,\zeta \in \mathrm{S}_n(F)$. Conversely, if $\zeta_0 \in \mathrm{S}_n(E)$ then using Fact 1.1, there exists $\zeta \in \pi^{-1}\zeta_0$ with $d(\xi,\zeta) \geq d(\pi\xi,\zeta_0)$, which proves that $\pi B(\xi,r) = B(\pi\xi,r)$. ■

We also obtain the following result, which is somewhat of an aside with respect to the rest of this paper. We shall therefore allow ourselves to be brief, and assume that the reader is familiar with continuous first order logic (see [BU10, BBHU08]), and, for the part regarding Banach spaces as unbounded metric structures, also with unbounded continuous logic (see [Ben08a]).

Lemma 1.18. *Let T be an inductive theory, and for $n \in \mathbf{N}$ let $\mathrm{S}^{\mathrm{qf}}_n(T)$ denote the space of quantifier-free types consistent with T, equipped with the natural logic topology. Assume that, first, every two models of T amalgamate over a common substructure, and, second, for every n, the variable restriction map $\mathrm{S}^{\mathrm{qf}}_{n+1}(T) \to \mathrm{S}^{\mathrm{qf}}_n(T)$ is open. Then T admits a model completion, namely a companion which eliminates quantifiers.*

(In fact, an approximate amalgamation property for models of T over a common finitely generated substructure suffices.)

Proof. Let $\varphi(\bar x,y)$ be a quantifier-free formula, inducing a continuous function $\hat\varphi\colon \mathrm{S}^{\mathrm{qf}}_{n+1}(T) \to \mathbf{R}$ (which has compact range, by compactness of $\mathrm{S}^{\mathrm{qf}}_{n+1}(T)$). Let $\pi\colon \mathrm{S}^{\mathrm{qf}}_{n+1}(T) \to \mathrm{S}^{\mathrm{qf}}_n(T)$ denote the variable restriction map, and define $\rho\colon \mathrm{S}^{\mathrm{qf}}_n(T) \to \mathbf{R}$ as the infimum over the fibre:

$$\rho(q) = \inf\{\hat\varphi(p) \colon \pi p = q\}.$$

Since π is continuous (automatically) and open (by hypothesis), ρ is continuous as well, and can therefore be expressed as a uniform limit of $\hat\psi_n\colon \mathrm{S}^{\mathrm{qf}}_n(T) \to \mathbf{R}$, where $\psi_n(\bar x)$ are quantifier-free formulae, say $\|\rho - \hat\psi_n\| \leq 2^{-n}$. One can now express that $\sup_{\bar x} |\psi_n(\bar x) - \inf_y \varphi(\bar x,y)| \leq 2^{-n}$ for all n by a set of sentences.

Let T^* consist of T together with all sentences constructed as above, for all possible quantifier-free formulae $\varphi(\bar x,y)$. Then, first, every existentially closed model of T is easily checked to be a model of T^* (using our amalgamation hypothesis), so T and T^* are companions. Moreover, by induction on quantifiers, every formula is equivalent modulo T^* to a uniform limit of quantifier-free formulae, so T^* eliminates quantifiers. ■

Proposition 1.19. *Consider Banach spaces either as metric structures in unbounded continuous logic, or as bounded metric structures via their closed unit balls, as explained, say, in* [Ben09]. *Then (in either approach) the theory of the class of Banach spaces is inductive, and admits a model completion T^* which is moreover complete and $\aleph_0$-categorical.*

When the entire Banach space is viewed as a structure then the types over a subspace are as per Definition 1.2 and Definition 1.3, and if one only considers the unit ball then the space of I-types over $B(E)$ is $\mathrm{S}^{\leq 1}_I(E) = \{\xi \in \mathrm{S}_I(E) \colon \|x_i\|^\xi \leq 1$ for all $i \in I\}$.

Proof. Let us consider the theory T of unit balls of Banach spaces. It is clearly inductive, and it is fairly easy to check that the space of quantifier-free I-types over a unit ball $B(E)$ is the space $\mathrm{S}^{\leq 1}_I(E)$ defined in the statement. By the moreover part of Lemma 1.9(v), variable restriction $\mathrm{S}^{\leq 1}_{n+1}(E) \to \mathrm{S}^{\leq 1}_n(E)$ is metrically open. For $E = 0$ this implies in particular that $\mathrm{S}^{\leq 1}_{n+1}(0) \to \mathrm{S}^{\leq 1}_n(0)$ is topologically open, but this latter is just $\mathrm{S}^{\mathrm{qf}}_{n+1}(T) \to \mathrm{S}^{\mathrm{qf}}_n(T)$. This, together with Fact 1.1, fulfils the hypotheses of Lemma 1.18. By quantifier elimination, $\mathrm{S}_n(T^*) = \mathrm{S}^{\mathrm{qf}}_n(T) = \mathrm{S}^{\leq 1}_n(0)$, so in particular, $\mathrm{S}_0(T^*)$ is a singleton, whereby T^* is complete. Finally, T^* is $\aleph_0$-categorical by the Ryll-Nardzewski Theorem (see [BU07]).

The case of Banach spaces as unbounded structures follows via the bi-interpretability of the whole Banach space with its unit ball. ■

2. The Gurarij space

Definition 2.1. We recall from, say, Lusky [Lus76] that a *Gurarij space* is a Banach space $\mathbf{G}$ having the property that for any $\varepsilon > 0$, finite-dimensional Banach space $E \subseteq F$, and isometric embedding $\varphi\colon E \to \mathbf{G}$, there is a linear map $\psi\colon F \to \mathbf{G}$ extending φ such that in addition, for all $x \in F$, $(1-\varepsilon)\|x\| \leq \|\psi x\| \leq (1+\varepsilon)\|x\|$.

Some authors add the requirement that a Gurarij space be separable, but from our point of view it seems more elegant to consider separability as a separate property.

Lemma 2.2. *Let F be a Banach space. Then the following are equivalent:*

(i) *The space F is a Gurarij space.*
(ii) *For every n, the set of realised types $\operatorname{tp}(\bar a/F)$, as $\bar a$ varies over F^n, is dense in $\mathrm{S}_n(F)$.*
(iii) *Same for $n = 1$.*

Proof. (i) $\Longrightarrow$ (iii). Let $U \subseteq \mathrm{S}_1(F)$ be open and $\xi \in U$. We may assume that U is defined by a finite set of conditions of the form $\big|\|a_i+r_ix\|-1\big| < \varepsilon$, where $\|a_i+r_ix\|^\xi = 1$. Let $E \subseteq F$ be the subspace generated by the a_i, and let $E' = E+\mathbf{R}x$ be the extension of E generated by the restriction of ξ to E. By hypothesis, there is a linear embedding $\psi\colon E' \to F$ extending the identity such that $(1-\varepsilon)\|y\| < \|\psi y\| < (1+\varepsilon)\|y\|$ for all $y \in E'$, and in particular for $y = a_i + r_ix$, so $\operatorname{tp}(\psi x/F) \in U$.

(iii) $\Longrightarrow$ (ii). We prove by induction on n, the case $n = 0$ being tautologically true. For the induction step, let $\varnothing \neq U \subseteq \mathrm{S}_{\bar x,y}(F)$ be open, and let $V = U{\restriction}_{\bar x} \subseteq \mathrm{S}_{\bar x}(F)$. By Lemma 1.14, V is open, and by the induction hypothesis there are $\bar b \in F^n$ such that $\operatorname{tp}(\bar b/F) \in V$. Now, consider the map $\theta\colon \mathrm{S}_y(F) \to \mathrm{S}_{\bar x,y}(F)$, sending $\operatorname{tp}(a/F) \mapsto \operatorname{tp}(\bar b, a/F)$. It is continuous (in fact, it is a topological embedding), so $\varnothing \neq \theta^{-1}U \subseteq \mathrm{S}_1(F)$ is open. By hypothesis, there is $c \in F$ such that $\operatorname{tp}(c/F) \in \theta^{-1}U$, i.e., such that $\operatorname{tp}(\bar b, c/F) \in U$, as desired.

(ii) $\Longrightarrow$ (i). Let $E \subseteq E'$ be finite-dimensional, with $E \subseteq F$, and let $\varepsilon > 0$. Let $\bar a$ be a basis for E, and let $\bar a, \bar b$ be a basis for E', say $|\bar a| = n$ and $|\bar b| = m$. For $N \in \mathbf{N}$, let $U_N \subseteq \mathrm{S}_m(F)$ be defined by the (finitely many) conditions of the form $\|\sum s_ia_i + \sum r_jx_j\| \in (1-\varepsilon, 1+\varepsilon)$, where s_i and r_j are of the form $\frac{k}{N}$ and $\|\sum s_ia_i + \sum r_jb_j\| \in (1-\varepsilon, 1+\varepsilon)$. By hypothesis there is a tuple $\bar c \in F^m$ such that $\operatorname{tp}(\bar c/F) \in U_N$, and we may define $\psi\colon E' \to F$ being the identity on E and sending $\bar b \mapsto \bar c$. For N big enough, it follows from the construction that if $y \in E'$, $\|y\| = 1$ then $\big|\|\psi y\| - 1\big| < 2\varepsilon$, which is good enough. ∎

Model theorists may find the second and third conditions of Lemma 2.2 reminiscent of a topological formulation of the Tarski-Vaught Criterion: a metrically closed subset A of a structure is an elementary substructure if and only if the set of types over A realised in A is dense. Indeed,

Theorem 2.3. *Let T^* be the model completion of the theory of Banach spaces, as per Proposition 1.19. Then its models are exactly the Gurarij spaces. In particular, since T^* is $\aleph_0$-categorical, there exists a unique separable Gurarij space (up to isometric isomorphism).*

Proof. Let E be a Banach space, and embed it in a model $F \vDash T^*$. Then, first, by quantifier elimination, E is a model of T^* if and only if $E \preceq F$. Second, by the topological Tarski-Vaught Criterion evoked above, $E \preceq F$ if and only if the set of types over E, in the sense of $\operatorname{Th}(F) = T^*$, realised in E, is dense.

By Proposition 1.19 the space of types over E (in the sense of $T^* = \operatorname{Th}(E)$) is $\mathrm{S}_n^{\leq 1}(E)$ as defined there. By a dilation argument, the set of types realised in E is dense in $\mathrm{S}_1(E)$ if and only if the set of types realised in $B(E)$ is dense in $\mathrm{S}_1^{\leq 1}(E)$, and we conclude by Lemma 2.2 (or, if one works with the whole space as an unbounded structure, the same holds without the dilation argument). ∎

As mentioned in the introduction, the isometric uniqueness of the separable Gurarij space was originally proved by Lusky [Lus76] using the Lazar-Lindenstrauss matrix representation of L^1 pre-duals. The same was recently re-proved by Kubiś and Solecki [KS13] using more elementary methods. Upon careful reading, their argument essentially consists of showing that the separable Gurarij space is the Fraïssé limit of the class of finite-dimensional Banach spaces, as is pointed out, alongside a general development of Fraïssé theory for metric structures (yielding yet another proof of the same result) by the first author [Ben]. From this point onward we shall leave continuous logic aside, and work entirely within the formalism of type spaces as introduced in Section 1. As we shall see, the uniqueness and existence also follow as easy corollaries from later results which do not depend explicitly on any form of formal logic (Corollary 2.7 and Lemma 2.11).

Definition 2.4. Let E be a Banach space. We say that a Banach space F is *atomic* over E if $E \subseteq F$ and the type over E of every finite tuple in F is isolated.

By Proposition 1.12, every Banach space is atomic over 0.

Theorem 2.5. *Let $E \subseteq F_0 \subseteq F_1$ be Banach spaces with $\dim F_0/E$ finite and F_1 separable and atomic over E, let $\mathbf{G} \supseteq E$ be a Gurarij space, and let $\varphi\colon F_0 \to \mathbf{G}$ be an isometric embedding extending id_E. Then there exist isometric embeddings $\psi\colon F_1 \to \mathbf{G}$ extending id_E with $\|\psi{\restriction}_{F_0} - \varphi\|$ arbitrarily small.*

In particular, any separable Banach space atomic over E embeds isometrically over E in any Gurarij space containing E.

Proof. It is enough to prove this in the case where $\dim F_1/F_0 = 1$. We may then choose a basis $\bar a \in F_1^{n+1}$ for F_1 over E, such that in addition $a_0, \ldots, a_{n-1}$ generate F_0. By hypothesis, $\xi = \operatorname{tp}(\bar a/E) \in \mathrm{S}_{n+1}(E)$ is isolated. Let $\rho\colon \mathrm{S}_{n+1}(\mathbf{G}) \to \mathrm{S}_{n+1}(E)$ be the parameter restriction map, and let $K = \rho^{-1}(\xi)$, observing that for any $\varepsilon > 0$, $B(K,\varepsilon) = \rho^{-1}B(\xi,\varepsilon)$ is a neighbourhood of K. We construct a sequence of tuples $\bar{c}_k \in \mathbf{G}^{n+1}$, each of which realising a type in $B(\xi, 2^{-k}r)$, as follows.

For $k = 0$, we let $V \subseteq \mathrm{S}_{n+1}(\mathbf{G})$ be the set defined by $\|x_i - \varphi a_i\| < r$ for $i < n$, which is open and intersects K. Then $V \cap B(K,r)^\circ \neq \varnothing$ (where $\cdot^\circ$ denotes topological interior), and we choose $\bar{c}_0$ to realise some type there. Given $\bar{c}_k$, we let $U_k \subseteq \mathrm{S}_{n+1}(\mathbf{G})$ be the set defined by $\|x_i - c_{k,i}\| < 2^{-k}r$ for $i \leq n$, which is again open intersecting K, and we choose $\bar{c}_{k+1}$ to realise a type in $U_k \cap B(K, 2^{-n-1}r)^\circ$.

We obtain a Cauchy sequence $(\bar{c}_k)$ converging to some $\bar{c} \in \mathbf{G}^{n+1}$, whose type $\mathrm{tp}(\bar{c}/E)$, being the metric limit of $\mathrm{tp}(\bar{c}_k/E)$, must be ξ. Then the linear map $\psi\colon F_1 \to \mathbf{G}$ which extends id_E by $a_i \mapsto c_i$ is an isometric embedding.

Finally, reading through our construction, we have $\|\varphi a_i - c_i\| < 3r$ for all $i < n$, and choosing r small enough, $\|\psi{\restriction}_{F_0} - \varphi\|$ is as small as desired. ■

In particular, any two separable Gurarij spaces atomic over E embed in one another, but we can do better.

Theorem 2.6. *Let $\mathbf{G}_i$ be separable Gurarij spaces atomic over E for $i = 0,1$, let $E \subseteq F \subseteq \mathbf{G}_0$ with $\dim F/E$ finite, and let $\varphi\colon F \to \mathbf{G}_1$ be an isometric embedding extending id_E. Then there exist isometric isomorphisms $\psi\colon \mathbf{G}_0 \cong \mathbf{G}_1$ extending id_E with $\|\psi{\restriction}_F - \varphi\|$ arbitrarily small.*

In particular, any two separable Gurarij spaces atomic over E are isometrically isomorphic over E.

Proof. Follows from Theorem 2.5 by a back-and-forth argument. ■

Since every Banach space is atomic over 0, we obtain the uniqueness and universality of the separable Gurarij space.

Corollary 2.7. *Every two separable Gurarij spaces are isometrically isomorphic, and every separable Banach space embeds isometrically in any Gurarij space (separable or not).*

Similarly, the Gurarij space is *approximately homogeneous*:

Corollary 2.8. *Let $\mathbf{G}$ be a separable Gurarij space, let $F \subseteq \mathbf{G}$ be finite-dimensional, and let $\varphi\colon F \to \mathbf{G}$ is an isometric embedding. Then there exist isometric automorphisms $\psi \in \mathrm{Aut}(\mathbf{G})$ such that $\|\psi{\restriction}_F - \varphi\|$ is arbitrarily small.*

Moreover, if $E \subseteq F$ is such that $\mathbf{G}$ is atomic over E, and $\varphi{\restriction}_E = \mathrm{id}$, then we may require that $\psi{\restriction}_E = \mathrm{id}$ as well.

Notation 2.9. We shall denote by $\mathbf{G}$ the unique separable Gurarij space. Similarly, for a separable Banach space E, we let $\mathbf{G}[E]$ denote the unique atomic separable Gurarij space over E, if such exists, observing that since all types over 0 are isolated, $\mathbf{G} = \mathbf{G}[0]$.

Corollary 2.10. *Let E be a separable Banach space, and let $H = \mathrm{Aut}(\mathbf{G})$ act by composition on the space of linear isometric embeddings $X = \mathrm{Emb}(E,\mathbf{G})$, where both are equipped with the topology of point-wise convergence (the strong operator topology).*

(i) *The space X is Polish, the action $H \curvearrowright X$ is continuous and all its orbits are dense.*
(ii) *If $\mathbf{G}[E]$ exists, then the set of $\varphi \in X$ such that $\mathbf{G}$ is atomic over φE (call these* atomic embeddings*) is a dense G_δ orbit under this action.*
(iii) *If $\mathbf{G}[E]$ does not exist then there are no atomic embeddings and all orbits are meagre.*

Proof. The first item is easy and left to the reader (density is by Corollary 2.8).

It follows from Theorem 2.6 that the set $Z \subseteq \mathrm{Emb}(E,\mathbf{G})$ of atomic embeddings forms a single orbit under $\mathrm{Aut}(\mathbf{G})$. By definition, $Z \neq \varnothing$ if and only if $\mathbf{G}[E]$ exists. Let $\mathscr{I}_n \subseteq \mathrm{S}_n(E)$ denote the set of isolated types. For $r > 0$, we know that $B(\mathscr{I}_n, r)$ is a neighbourhood of $\mathscr{I}_n$, so there exists an open set $U_{n,r}$ such that $\mathscr{I}_n \subseteq U_{n,r} \subseteq B(\mathscr{I}_n, r)$ (in fact one can show that $B(\mathscr{I}_n, r)$ is open, but we shall not require this). For each $\bar{b} \in \mathbf{G}^n$, we define $V_{\bar{b},r} \subseteq \mathrm{Emb}(E,\mathbf{G})$ to consist of all φ such that $\mathrm{tp}(\bar{b}/\varphi E) \in \varphi U_{n,r}$. It is easy to see that since $U_{n,r}$ is open, so is $V_{\bar{b},r}$. Since the set of isolated types is metrically closed, we have

$$Z = \bigcap_{n,\bar{b}\in\mathbf{G}^n,r>0} V_{\bar{b},r} = \bigcap_{n,\bar{b}\in\mathbf{G}_0^n,k} V_{\bar{b},2^{-k}},$$

where $\mathbf{G}_0 \subseteq \mathbf{G}$ is any countable dense subset. Thus, if $Z \neq \varnothing$ it is a dense G_δ orbit.

Assume now that $\mathbf{G}[E]$ does not exist, namely, that isolated types are not dense, and let $\psi \in X$. Then $\mathbf{G}$ necessarily realises some type $\psi\xi \in \mathrm{S}_n(\psi E)$ where $\xi \in \mathrm{S}_n(E)$ is non isolated. By Lemma 1.16, for $r > 0$ small enough, the closed metric ball $\overline{B}(\xi, r)$ is (topologically) closed of empty interior. For

$\bar{b} \in \mathbf{G}^n$, let $V_{\bar{b}} \subseteq \mathrm{Emb}(E, \mathbf{G})$ consist of all φ such that $\mathrm{tp}(\bar{b}/\varphi E) \notin \overline{B}(\varphi\xi, r)$. Reasoning as above, each $V_{\bar{b}}$ is a dense open set, and the set of $\varphi \in X$ such that $\mathbf{G}$ omits $\varphi\xi$ is co-meagre. Since this set is also disjoint from the orbit of ψ, we are done. ■

We now turn to a criterion for the existence of $\mathbf{G}[E]$.

Lemma 2.11. *Let E be a separable Banach space, and say that a type $\xi \in \mathrm{S}_{\mathbf{N}}(E)$ is a* Gurarij type *if it generates a Gurarij space. Then the set of Gurarij types over E is co-meagre in $\mathrm{S}_{\mathbf{N}}(E)$. Moreover, there exists a dense G_δ set $Z \subseteq \mathrm{S}_{\mathbf{N}}(E)$ such that if some $\xi \in Z$ generates F then F is Gurarij and $\{x_i\}_{i\in\mathbf{N}} \subseteq F$ is dense.*

In particular, the separable Gurarij space $\mathbf{G}$ exists.

Proof. Let $X = \{x_i\}_{i\in\mathbf{N}}$, so $\mathrm{S}_X(E) = \mathrm{S}_{\mathbf{N}}(E)$. Let y be a new variable symbol. For $k \in \mathbf{N}$, let $[\varnothing \to y] \colon E(X) \to E(X, y)$ denote $\mathrm{id}_{E(X)}$, let $[x_k \to y] \colon E(X) \to E(X, y)$ be defined as $\mathrm{id}_{E(X \smallsetminus \{x_k\})}$ together with $x_k \mapsto y$, and let $[y \to x_k] \colon E(X, y) \to E(X)$ be defined as $\mathrm{id}_{E(X)}$ together with $y \mapsto x_k$.

The space $\mathrm{S}_{X,y}(E)$ has a countable base of open sets $\{U_n\}_{n\in\mathbf{N}}$, and we may assume furthermore that each $U_n \neq \varnothing$ can be defined using only variables from among $\{x_0, \ldots, x_{n-1}, y\}$, so $[y \to x_n]_* U_n = [x_n \to y]^* U_n \neq \varnothing$. For each n we define $Z_n \subseteq \mathrm{S}_X(E) = \mathrm{S}_{\mathbf{N}}(E)$ to consist of all types ξ such that either

- $\xi \notin [\varnothing \to y]^* U_n$, which defines a closed set, since $[\varnothing \to y]^*$ is open, or
- there exists some k such that $\xi \in [y \to x_k]_* U_n$, which defines an open set.

Then Z_n is a G_δ set, and we claim that it is dense. Indeed, let $\varnothing \neq W \subseteq \mathrm{S}_X(E)$ be open. We may assume that $U_n \cap [\varnothing \to y]_* W \neq \varnothing$, since otherwise $W \cap Z_n \neq \varnothing$. Then there exist k such that $U_k \subseteq U_n \cap [\varnothing \to y]_* W$, so

$$\begin{aligned}\varnothing \neq [y \to x_k]_* U_k &\subseteq [y \to x_k]_* U_n \cap [y \to x_k]_* [\varnothing \to y]_* W \\ &= [y \to x_k]_* U_n \cap W \\ &\subseteq Z_n \cap W,\end{aligned}$$

proving our claim.

Now let $Z = \bigcap Z_n$, a dense G_δ set, and we claim that every $\xi \in Z$ is Gurarij. Indeed, let ξ generate F, and let $\varnothing \neq U \subseteq \mathrm{S}_1(F)$ be open. We define $\theta \colon \mathrm{S}_1(F) \to \mathrm{S}_{X,y}(E)$ as in the proof of Lemma 1.14 (working over E, whereas there we worked over 0), and there exists n such that $U \supseteq \theta^{-1}(U_n) \neq \varnothing$. Since $\theta^{-1}(U_n) \neq \varnothing$, we have $\xi \in [\varnothing \to y]^* U_n \cap Z_n$, and so for some k we have $[y \to x_k]^* \xi \in U_n$. This means exactly that $\mathrm{tp}(x_k/F) \in \theta^{-1}(U_n) \subseteq U$, showing that ξ is indeed Gurarij. Moreover, we have shown that every open set $\varnothing \neq U \subseteq \mathrm{S}_1(F)$ is realised in F by some x_i, from which it follows that $\{x_i\}_{i\in\mathbf{N}}$ is dense in F. ■

Notice that since a Banach space has no isolated points, if a sequence is dense there then every tail of the sequence is dense there as well.

Theorem 2.12 (Omitting Types Theorem for Gurarij spaces)**.** *Let E be a separable Banach space, and for each n, let $X_n \subseteq \mathrm{S}_n(E)$ be metrically open and topologically meagre. Then there exists a separable Gurarij space $\mathbf{G} \supseteq E$ such that in addition, for every n, no type in X_n is realised in $\mathbf{G}$ (we then say that $\mathbf{G}$* omits *all X_n). Moreover, the set of Gurarij types which generate such spaces is co-meagre.*

Proof. Let $Z \subseteq \mathrm{S}_{\mathbf{N}}(E)$ be the set produced by Lemma 2.11. For each n, let $[\mathbf{N}]^n = \{s \subseteq \mathbf{N} \colon |s| = n\}$. For $s \in [\mathbf{N}]^n$ can be enumerated uniquely as an increasing sequence $\{k_0, \ldots, k_{n-1}\}$, and we then define $[s] \colon E(n) \to E(\mathbf{N})$ by $x_i \mapsto x_{k_i}$ for $i < n$. Then $[s]^* \colon \mathrm{S}_{\mathbf{N}}(E) \to \mathrm{S}_n(E)$ is continuous and open, so $[s]_* X_n \subseteq \mathrm{S}_{\mathbf{N}}(E)$ is meagre. Since everything is countable,

$$Z_1 = Z \smallsetminus \bigcap_{n, s \in [\mathbf{N}]^n} [s]_* X_n$$

is co-meagre as well. All we need to show is that if $\xi \in Z_1$ generates $\mathbf{G}$ then $\mathbf{G}$ omits X_n. Indeed, assume that some $\xi \in X_n$ is realised in $\mathbf{G}$, say by $\bar{a}$. Since X_n is metrically open, there exists $r > 0$ such that $B(\xi, r) \subseteq X_n$. Since the sequence $\{x_i\}$ is dense in $\mathbf{G}$, there is an increasing sequence $k_0 < \ldots < k_{n-1}$ such that $\|x_{k_j} - a_j\| < r$. But then $\mathrm{tp}(x_{\bar{k}}/E) \in X_n$, so $\xi \in [\bar{k}]_* X_n$, contradicting the choice of ξ and completing the proof. ■

Corollary 2.13 (Criterion for primeness over E)**.** *Let G be a Gurarij space, and let $E \subseteq G$ be a separable subspace. Then the following are equivalent:*

(i) *The space G is* prime *over E, that is to say that it embeds isometrically over E in every Gurarij space containing an isometric copy of E.*

(ii) *The space G is separable and atomic over E, namely, $G = \mathbf{G}[E]$.*

Proof. It is immediate from Theorem 2.5 that $\mathbf{G}[E]$ is prime over E. For the other direction, assume that $\mathbf{G}$ is prime over E. Since E is separable, it embeds (by Theorem 2.5) in a separable Gurarij space, so $\mathbf{G}$ must be separable as well. Finally, assume toward a contradiction that $\mathbf{G}$ realises some non isolated type ξ. By Lemma 1.16 there exists $r > 0$ such that the closed metric ball $\overline{B}(\xi, r)$ has empty interior. Since the metric is lower semi-continuous, the closed metric ball is topologically closed, and is therefore meagre, as is the open ball $B(\xi, r)$. By Theorem 2.12, there exists a separable Gurarij space $\mathbf{G} \supseteq E$ which omits $B(\xi, r)$. Thus G cannot embed over E in $\mathbf{G}$, a contradiction. ■

Proposition 2.14. *Let E be a separable Banach space. Then $\mathbf{G}[E]$ exists if and only if, for each n, the set of isolated types in $\mathrm{S}_n(E)$ is dense.*

Proof. Assume first that the sets of isolated types are dense. For a given n, let $\mathscr{I}_n$ be the set of isolated types in $\mathrm{S}_n(E)$, and assume that it is dense. Then $B(\mathscr{I}_n, r)$ contains a dense open set, and $\bigcap_{r>0} B(\mathscr{I}_n, r) = \overline{\mathscr{I}}_n$ is co-meagre. By Lemma 1.16 we have $\mathscr{I}_n = \overline{\mathscr{I}}_n$, so $\mathrm{S}_n(E) \smallsetminus \mathscr{I}_n$ is meagre and metrically open. By Theorem 2.12, if $\mathscr{I}_n$ is dense for all n then an atomic separable Gurarij space over E exists.

Conversely, assume that $\mathbf{G}[E]$ exists. Then the set of n-types over E realised in $\mathbf{G}[E]$ is dense (by Lemma 2.2), and they are all isolated. ■

Model theorists will recognise Proposition 2.14 as the usual criterion for the existence of an atomic model, and as such it is in no way particular to Banach spaces. In the specific context of Banach spaces, however, it can be improves as follows.

Lemma 2.15. *For a type $\xi \in \mathrm{S}_{\bar{x}}(E)$ the following are equivalent*

(i) *The type ξ is isolated.*
(ii) *The type $\xi\restriction_{\bar{y}}$ is isolated for $\bar{y} \in E(\bar{x})^m$ (and every m).*
(iii) *The type $\xi\restriction_y$ is isolated for every $y \in E(\bar{x})$.*

Proof. (i) $\Longrightarrow$ (ii). When $\bar{y}$ are linearly independent over E, this follows from Lemma 1.14. Hence, for the general case, it is enough to consider the situation where $\bar{y}$, of length m, extends the original tuple of variables $\bar{x}$, of length n, and for $j < m$ let us write $y_j = a_j + \sum_{i<n} \lambda_{ij} x_i$. Given $r > 0$, there exists by hypothesis an open set U such that $\xi \in U \subseteq B(\xi, r)$, and let $V = (\cdot\restriction_{\bar{x}})^{-1} U \subseteq \mathrm{S}_{\bar{y}}(E)$. Intersecting V with the open sets defined by $\|y_j - \sum_{i<n} \lambda_{ij} y_i - a_j\| < r$ we obtain an open set V' with $\xi\restriction_{\bar{y}} \in V' \subseteq B(\xi\restriction_{\bar{y}}, r')$ for some $r' = r'(r, \bar{y})$ which goes to zero with r.

(ii) $\Longrightarrow$ (iii). Immediate.

(iii) $\Longrightarrow$ (i). We repeat the proof of Proposition 1.12 (in fact, that result is merely a special case of the present, alongside the fact that types in $\mathrm{S}_1(0)$ are trivially isolated). Indeed, for each N there exists by hypothesis a neighbourhood $U_N \ni \xi$ consisting of ζ such that

$$\forall y \in X_N \qquad d(\zeta\restriction_y, \xi\restriction_y) < 1/N.$$

Using Proposition 1.7 we conclude as for Proposition 1.12. ■

Theorem 2.16. *The following are equivalent for a separable Banach space E:*

(i) *The space $\mathbf{G}[E]$ exists.*
(ii) *For each n, the set of isolated types in $\mathrm{S}_n(E)$ is dense.*
(iii) *The set of isolated types in $\mathrm{S}_1(E)$ is dense.*

Proof. We only need to show that if the set of isolated 1-types is dense then $\mathbf{G}[E]$ exists. Indeed, proceeding as in the proof of Proposition 2.14 there exists a separable Gurarij space $\mathbf{G} \supseteq E$ which only realises isolated 1-types over E. By Lemma 2.15, $\mathbf{G}$ is atomic over E. ■

3. Isolated types over one-dimensional spaces

In this paper we characterise isolated types over arbitrary E. We start with the next-easiest case after $E = 0$, namely when $\dim E = 1$. Even though this case will be fully subsumed in the general case, it is technically significantly simpler and deserves some specific comments, so we chose to treat it separately.

Definition 3.1. A *norming linear functional* for $v \in E \smallsetminus \{0\}$ is a continuous linear functional $\lambda \in \partial B(E^*)$ such that $\lambda v = \|v\|$.

By the Hahn-Banach Theorem, a norming linear functional always exists. We say that v is *smooth* in E if the norming linear functional is unique.

Proposition 3.2. *Let E be a Banach space, and let $v \in E \setminus \{0\}$. Then E is atomic over v if and only if v is smooth in E.*

Proof. By Lemma 2.15, we may assume that $E = \langle v, u \rangle$ and show that $\mathrm{tp}(u/v)$ is isolated if and only if v is smooth in E. Assume first that for some $s, \varepsilon > 0$ and $D \in \mathbf{R}$ we have

$$\|v \pm su\| < \|v\| \pm sD + s\varepsilon.$$

It follows by the triangle inequality that

$$\|v\| \pm tD - t\varepsilon \leq \|v \pm tu\| < \|v\| \pm tD + t\varepsilon, \qquad 0 < t \leq s,$$

or equivalently,

$$\big| \| \pm rv + u\| - r\|v\| \mp D \big| < \varepsilon, \qquad r \geq s^{-1}.$$

If v is smooth, let λ be the unique norming functional, and let $D = \lambda u$. Then for any $\varepsilon > 0$ there exists s as above. Then $\xi = \mathrm{tp}(u/v)$ satisfies the open condition $\|v \pm sx\| < \|v\| \pm sD + s\varepsilon$, which in turn implies that $\big| \|rv - u\| - \|rv - x\| \big| \leq 2\varepsilon$ for all $|r| \geq s^{-1}$. Finitely many additional open condition can ensure that that the same holds for all r yielding an open set $\xi \in U \subseteq B(\xi, 3\varepsilon)$, showing that ξ is isolated.

Conversely, if v is not smooth then there are norming functionals $\lambda^{\pm}$, where $D^- = \lambda^- u < D^+ = \lambda^+ u$. Any neighbourhood of ξ contains one U which is defined by finitely many conditions of the form $\big| \|r_i v + x\| - \|r_i v + u\| \big| < \varepsilon$. We can construct a Banach space E' generated by $\{v, w\}$, with $\|v\|$ as in E, such that $\zeta = \mathrm{tp}(w/v) \in U$ and v is smooth in E', with unique norming functional being defined by $\mu w = D^-$. This means that for r big enough we have

$$\|rv + w\| \approx r\|v\| + D^- \leq \|rv + u\| + D^- - D^+,$$

so $d(\xi, \zeta) \geq D^+ - D^-$. Therefore $B(\xi, D^+ - D^-)$ is *not* a topological neighbourhood of ξ, and ξ is not isolated. ∎

We provided a fairly elementary argument to the "only if" part of Proposition 3.2. The machinery developed above provides us with a conceptually different argument, which in a sense we find preferable. First, let us recall that by Mazur [Maz33, Satz 2], the set of smooth points in the unit sphere of a separable Banach space is a dense G_δ. Assume now that E is atomic over v, and without loss of generality, say that $\|v\| = 1$, and let $u \in \mathbf{G}$ be smooth of norm one. By Theorem 2.5 there exists an isometric embedding of E in $\mathbf{G}$ sending v to u, so v must be smooth.

We obtain the following result of Lusky [Lus76].

Corollary 3.3. *The smooth points in the unit sphere of $\mathbf{G}$ form a single dense G_δ orbit under isometric automorphisms.*

Proof. Immediate from Proposition 3.2 and Theorem 2.6. ∎

It follows from Lemma 1.17 that if $E \subseteq F$, and the topology and metric coincided on $\mathrm{S}_n(F)$, then they would also coincide on $\mathrm{S}_n(E)$, which would mean that every type in $\mathrm{S}_n(E)$ is isolated. Given Proposition 3.2, not all types over a 1-dimensional E are isolated, and it follows that the metric strictly refines the topology on $\mathrm{S}_n(F)$ for every $F \neq 0$.

4. The Legendre-Fenchel transformation of 1-types

In this section we recall and develop a few technical tools which will be used later in order to characterise isolated types over arbitrary E. We start with the *Legendre-Fenchel transformation*. This being a duality construction, it will be convenient for us to put E and its dual E^* on equal footing.

Convention 4.1. Throughout, E will denote a locally convex topological vector space over $\mathbf{R}$. Its topological dual E^*, namely the space of continuous linear functionals, will always be equipped with the weak* topology, namely the least topology in which $\hat{v} \colon \lambda \mapsto \lambda v$ is continuous for each $v \in E$ (again a locally convex one).

This applies in particular when E is a normed space: we may refer to the dual norm via the sets $B(E^*)$ and $\partial B(E^*)$, or the corresponding properties $\|\lambda\| \leq 1$ and $\|\lambda\| = 1$, but the topology is always the weak* topology, so $B(E^*)$ is always compact.

The bi-dual E^{**} is canonically identified with E (which would not always be true if for a normed space E we calculated E^{**} with respect to the dual norm on E^*). This induces the *weak topology* on E which may be weaker than the original one, but gives rise to the same dual E^* $(= E^{***})$. The existence of two topologies on E will not pose much of a problem in the context of convexity.

Fact 4.2 (Hahn-Banach Theorem, see Brezis [Bre83]). *A closed convex subset of E is the intersection of the closed half-spaces which contain it, and is therefore weakly closed (a half-space of E is a set of the form $\{v : \lambda v \leq r\}$ where $\lambda \in E^*$ and $r \in \mathbf{R}$).*

Definition 4.3. Following Rockafellar [Roc70], we shall say that a convex function $f\colon E \to \mathbf{R} \cup \{\infty\}$ is *proper* if it is not identically ∞, and that it is *closed* if it is lower semi-continuous (equivalently, by Fact 4.2, lower semi-continuous in the weak topology). We then define its *domain* $\operatorname{dom} f = \{v \in E\colon f(v) < \infty\}$. In order to avoid special cases, we also allow the two constant functions $f = \pm\infty$ as *improper* closed convex functions.

For an arbitrary function $f\colon E \to [-\infty, +\infty]$ we define $f^*\colon E^* \to [-\infty, +\infty]$ by

$$f^*(\lambda) = \sup_{v \in E} \lambda v - f(v).$$

If f is closed and convex we call f^* its *conjugate*.

Fact 4.4. *For any $f\colon E \to [-\infty, +\infty]$, the function f^* is closed convex. If f is closed and convex then $f = f^{**}$ under the canonical identification $E = E^{**}$, and f^* is proper if and only if f is.*

Moreover, if g is another closed convex function, then $\|f - g\| = \|f^* - g^*\|$, where $\|\cdot\|$ denotes the supremum norm, possibly infinite, and we agree that $|\pm\infty \mp \infty| = 0$.

Proof. For the finite-dimensional case, see Rockafellar [Roc70, Section 12]. The general case is proved essentially in the same fashion, using Fact 4.2. The moreover part is easy to check directly. ■

Lemma 4.5. *Let $X \subseteq E$ and let $f\colon X \to \mathbf{R} \cup \{\infty\}$ not be identically ∞. Assume moreover that whenever $x \in X$ can be expressed as a limit of convex combinations $\sum_{i<\ell_k} t_{k,i} x_{k,i}$, where $x_{k,i} \in X$, we have $f(x) \leq \liminf_k \sum_{i<\ell_k} f_{k,i} f(x_{k,i})$. Then extending f by ∞ outside X, f^* is a proper closed convex function on E^* and $f = f^{**}{\restriction}_X$.*

Proof. Let $\operatorname{epi} f = \{(v, s) : s \geq f(v)\} \subseteq X \times \mathbf{R} \subseteq E^* \times \mathbf{R}$, the *epigraph* of f, let $Y = \overline{\operatorname{co}}(\operatorname{epi} f) \subseteq E^* \times \mathbf{R}$ be the closed convex hull and define $g(v) = \inf\{t : (v, t) \in Y\} \in \mathbf{R} \cup \{\infty\}$. Then g is a closed proper convex function, $g \leq f$, and the hypothesis implies that g agrees with f on X. Now $g^* \geq f^*$, so f^* is in particular proper (it is automatically closed and convex), and $g = g^{**} \leq f^{**} \leq f$. Therefore $f^{**}{\restriction}_X = f$. ■

We recall that if $X \subseteq E$ is convex then ∂X is defined as the set of all v such that, for some affine line L, v is one of two distinct boundary points of $L \cap X$ in L. The *relative interior*, sometimes denoted $\operatorname{ri}(X)$, is defined as $X \smallsetminus \partial X$: the set of all $v \in X$ such that for every affine line L going though v, either $L \cap X$ is a single point or contains v in its interior relative to L. When X generates a finite-dimensional affine subspace, this agrees with the usual topological notions as calculated in that space.

Lemma 4.6. *Let E be finite-dimensional and let $X \subseteq E$ be a compact convex subset. Let $f\colon X \to \mathbf{R}$ be closed and convex, and assume that $f{\restriction}_{\partial X}$ is continuous. Then f is continuous.*

Proof. We need to show that if $x_n \to x$ in X and $f(x_n) \to \alpha \in [-\infty, \infty]$ then $f(x) = \alpha$. Since f is closed, $f(x) \leq \alpha$, and let us assume that $f(x) < \alpha$. We may then assume that $f(x_n) > f(x) + \varepsilon$ for some $\varepsilon > 0$ and all n. Then the ray $R_n = x_n + \mathbf{R}^{\geq 0}(x_n - x)$ intersects ∂X at a single point, call it $y_n = x_n + s_n(x_n - x)$, and we may further assume that $y_n \to y$, where y is necessarily also on the boundary. Notice that $x_n = \frac{s_n x + y_n}{s_n + 1}$, so by convexity $f(y_n) \geq (s_n + 1) f(x_n) - s_n f(x) > f(x) + (s_n + 1)\varepsilon \geq f(x) + \varepsilon$. Since f is continuous on the boundary we must have $y \neq x$. But then $\|y_n - x\|$ is bounded away from zero, so $s_n \to \infty$, so f is unbounded on the boundary, even though ∂X is compact and f is continuous there, a contradiction. ■

The relevance of convex conjugation to our context comes from the following alternative characterisation of 1-types over a normed space E, introduced in [Ben14] (see also Katětov [Kat88] and Uspenskij [Usp08]). From now on, E denotes a normed space.

Definition 4.7. Let X be an arbitrary metric space. A *Katětov function* on X is a function $f\colon X \to \mathbf{R}$ satisfying $f(x) \leq f(y) + d(x, y)$ and $d(x, y) \leq f(x) + f(y)$ for all $x, y \in X$. The space of Katětov functions on X is denoted $K(X)$. As with type spaces, we equip $K(X)$ with a double structure, the topology of point-wise convergence and the metric of uniform convergence (i.e., the supremum metric).

If X is a normed space, or a convex subset thereof, we let $K_C(X)$ denote the space of *convex* Katětov functions on X, with the induced topometric structure.

Fact 4.8. *Let ξ be a 1-type over a normed space E, and let $f_\xi(a) = \|x - a\|^\xi$ for $a \in E$. Then*

(i) *The map* $\xi \mapsto f_\xi$ *defines a bijection between* $S_1(E)$ *and* $K_C(E)$*, whose inverse is given by*

$$\|\alpha x - a\|^\xi = \begin{cases} \|a\| & \alpha = 0 \\ |\alpha|\xi(a/\alpha) & \alpha \neq 0. \end{cases}$$

(ii) *This bijection is a topological homeomorphism and a metric isometry.*

Proof. The first item is [Ben14, Lemma 1.2]. For the second, that the bijection is homeomorphic (in the respective topologies of point-wise convergence) follows easily from the characterisation of the inverse, while the isometry is exactly Proposition 1.7 for 1-types. ■

Consequently, from now on we shall identify $K_C(E)$ with $S_1(E)$.

Fact 4.9. *Let* $X \subseteq Y$ *be metric spaces, and for* $f \in K(X)$ *and* $y \in Y$ *define*

$$\tilde{f}(y) = \inf_{x \in X} f(x) + d(x, y).$$

Then $\tilde{f} \in K(Y)$ *extends* f*, and the induced embedding* $K(X) \subseteq K(Y)$ *is isometric. When* $Y = E$ *is a normed space,* $X \subseteq E$ *is convex and* $f \in K_C(X)$*, the extension* $\tilde{f}$ *is convex as well, inducing an isometric embedding* $K_C(X) \subseteq K_C(E)$*.*

Proof. The first assertion goes back to Katětov [Kat88], and the second is [Ben14, Lemma 1.3(i)]. ■

Question 4.10. If $X \subseteq E$ is convex and compact (or totally bounded) then the topology and metric agree on $K_C(X)$, and it follows that the inclusion $K_C(X) \subseteq K_C(E)$ is also continuous, and therefore homeomorphic (since the restriction map is always continuous). At the other extremity, if $X = E$ then the inclusion is homeomorphic as well. What about general convex $X \subseteq E$?

A closed proper convex function $f\colon E \to \mathbf{R} \cup \{\infty\}$ is essentially the same thing as a closed convex function $f\colon X \to \mathbf{R}$, with convex domain X, such that $\liminf_{v \to u} f(v) = \infty$ for all $u \in \overline{X} \setminus X$. Indeed, we can get one from the other by restricting to the finite domain in one direction, or by extending by ∞ in the other. A special case of the second form is when $X \subseteq E$ is closed and convex and $f \in K_C(X)$. If X is merely convex, every $f \in K_C(X)$, being 1-Lipschitz, admits a unique extension to $\overline{f} \in K_C(\overline{X})$, so requiring X to be closed is not truly a constraint.

Lemma 4.11. *Let* $X \subseteq E$ *be closed and convex and let* $f \in K_C(X)$*. Then*

(i) *The domain* $\operatorname{dom} f^*$ *contains* $B(E^*)$*, and if* $\lambda \in \operatorname{dom} f^*$*,* $\|\lambda\| > 1$*, then* $f^*(\lambda) = \sup_{v \in \partial X} \lambda v - f(v)$*. In particular, if* $X = E$ *(so* $\partial X = \varnothing$*) then* $\operatorname{dom} f^*$ *is exactly the closed unit ball.*

(ii) *If* X *is bounded and* $\|\lambda\| = 1$ *then* $f^*(\lambda) = \sup_{v \in \partial X} \lambda v - f(v)$*.*

(iii) *Let* $\tilde{f} \in K_C(E)$ *be as per Fact 4.9. Then*

$$\tilde{f}^*(\lambda) = \begin{cases} f^*(\lambda) & \|\lambda\| \leq 1 \\ \infty & \|\lambda\| > 1 \end{cases} \quad \text{and} \quad \tilde{f}(v) = \sup_{\|\lambda\| \leq 1} \lambda v - f^*(\lambda).$$

In addition, if $v \notin X$*, then* $\tilde{f}(v) = \sup_{\|\lambda\|=1} \lambda v - f^*(\lambda)$*.*

(iv) *When* $X = E$*, we have* $f \in K_C(E) = S_1(E)$*. For* $\lambda \in \partial B(E^*)$*, the least possible value of a norm-preserving extension of* λ *at a realisation of* f *is* $f^*(\lambda)$*.*

(v) *Let* $g\colon E \to \mathbf{R} \cup \{\infty\}$ *be any closed proper convex function. Then* $g \in K_C(E)$ *if and only if* $\operatorname{dom} g^* = B(E^*)$ *and the* antipode inequality $g^*(\lambda) + g^*(-\lambda) \leq 0$ *holds for all* $\lambda \in \partial B(E^*)$*, or, equivalently, for all* $\lambda \in B(E^*)$*.*

(vi) *Assume* $g \in K_C(E)$ *is such that the* antipode identity $g^*(\lambda) + g^*(-\lambda) = 0$ *holds at some* $\lambda \in B(E^*)$*. Then* g^* *is continuous at* λ*.*

Proof. For (i), first let $\|\lambda\| \leq 1$, and let $u \in X$ be fixed. Then for all $v \in X$ we have $f(u) + \|u\| \geq \|v - u\| - f(v) + \|u\| \geq \lambda v - f(v)$, whereby $f^*(\lambda) \leq f(u) + \|u\| < \infty$. Now let $\|\lambda\| > 1$, say $\lambda w > \|w\|$, and assume that $\lambda \in \operatorname{dom} f^*$. Then for each $v \in \operatorname{dom} f$, the ray $\{v + \alpha w \colon \alpha \geq 0\}$ cannot be contained in X (or else $f^*(\lambda) = \infty$) and therefore intersects the boundary, say at v'. In this case $\lambda v - f(v) \leq \lambda v' - f(v')$, proving our assertion. When $\|\lambda\| = 1$ but X is assumed to be bounded, for every $v \in X$ we can find $v' \in \partial X$ with $\lambda v - f(v) \leq \lambda v' - f(v') + \varepsilon$ for ε arbitrarily small, whence (ii).

For (iii), we already know that $\operatorname{dom} \tilde{f}^*$ is exactly the closed unit ball. In addition, $\tilde{f} \leq f$ implies $\tilde{f}^* \geq f^*$, and if $\|\lambda\| \leq 1$, then for every $v \in E$ and $u \in X$:

$$\lambda v - \tilde{f}(v) = \lambda v - \inf_{u \in X} \big[f(u) + \|v - u\|\big] \leq \sup_{u \in X} \lambda u - f(u) = f^*(\lambda),$$

whereby $\tilde{f}^*(\lambda) \leq f^*(\lambda)$. This gives us the first identity, and then Fact 4.4 gives the second one. Now assume that $v \notin X$, so by Fact 4.2 there exists $\mu \in \partial B(E^*)$ such that $\mu\restriction_X < \mu v$. For any $\lambda \in B(E^*)$ there exists $\alpha \geq 0$ such that $\|\lambda+\alpha\mu\| = 1$. Then $f^*(\lambda+\alpha\mu) \leq f^*(\lambda)+\alpha\mu v$, or equivalently, $\lambda v - f^*(\lambda) \leq (\lambda + \alpha\mu)v - f^*(\lambda + \alpha\mu)$, whence it follows that $\tilde{f}(v) = \sup_{\|\lambda\|=1} \lambda v - f^*(\lambda)$.

Item (iv) is immediate. For (v), we have already seen that if $g \in K_C(E)$ then $\operatorname{dom} g^* = B(E^*)$, and the previous item implies that $g^*(\lambda)+g^*(-\lambda) \leq 0$ for $\lambda \in \partial B(E^*)$. Conversely, assume that $\operatorname{dom} g^* = B(E^*)$ and the antipode inequality holds. Then $g = g^{**}$ is necessarily 1-Lipschitz, so $\operatorname{dom} g = E$. For distinct $v, u \in E$, let $\lambda \in \partial B(E^*)$ norm $v - u$. Then

$$g(v) + g(u) \geq \lambda v - g^*(\lambda) - \lambda u - g^*(-\lambda) \geq \lambda(v - u) = \|v - u\|,$$

as desired.

For (vi), let $\lambda_\alpha \to \lambda$. Then $g^*(\lambda_\alpha) \leq -g^*(-\lambda_\alpha)$ by the antipode inequality and $g^*(\lambda) = -g^*(-\lambda_\alpha)$ by hypothesis. Since g^* is lower semi-continuous,

$$g^*(\lambda) \leq \liminf g^*(\lambda_\alpha) \leq \limsup g^*(\lambda_\alpha) \leq \limsup -g^*(-\lambda_\alpha) = -\liminf g^*(-\lambda_\alpha) \leq -g^*(-\lambda) = g^*(\lambda).$$

Therefore $\lim g^*(\lambda_\alpha) = g^*(\lambda)$, as desired. ■

Remark 4.12. Let $F \subseteq E$ be normed spaces and let $g \in K_C(F)$. Since F is convex in E, there may be some ambiguity about g^*, so let g^*_F denote the conjugate as a convex function on F and let g^*_E denote the conjugate of the extension by infinity to E^*. Also let $\tilde{g} \in K_C(E)$ denote the canonical extension of g. Then $g^*_E(\lambda) = g^*_F(\lambda\restriction_F)$ for $\lambda \in E^*$, and by Lemma 4.11(iii), if $\|\lambda\| \leq 1$ then this is further equal to $\tilde{g}^*(\lambda)$. Therefore, in what interests us, this ambiguity can never lead to any form of confusion.

5. Characterising isolated types over arbitrary spaces

In [Ben14] a special kind of "well behaved" convex Katětov functions is distinguished. These will play a crucial role here as well, and admit a natural characterisation in terms of their conjugate.

Definition 5.1. We say that a function $f \in K_C(E)$ is *local* if there are $f_k \in K_C(X_k)$, where each $X_k \subseteq E$ is convex and compact, such that $\tilde{f}_k \to f$ uniformly. The set of local functions in $K_C(E)$ was denoted in [Ben14] by $K_{C,0}(E)$.

Lemma 5.2. *Let E be a normed space, and let $f \in K_C(E)$. Then f is local if and only if f^* is continuous on $B(E^*)$.*

Proof. First let $X \subseteq E$ be compact and let $g \in K_C(X)$. If $X \subseteq \bigcup_{i<n} B(v_i, r)$ then $g^*(\lambda) - g^*(\mu) \leq 2r\|\lambda - \mu\| + \max_i(\lambda - \mu)v_i$, whence it follows that g^* is continuous on every bounded subset of E^*, and in particular on $B(E^*)$. Since a uniform limit of continuous functions is continuous, if f is local then f^* is continuous on $B(E^*)$.

Conversely, assume that f^* is continuous on $B(E^*)$. It follows by general topological arguments that there exists a separable $F \subseteq E$ such that $f^*(\lambda)$ only depends on $\lambda\restriction_F$, and the map $f' \colon F^*_{\leq 1} \to \mathbf{R}$, $\mu \mapsto f^*(\mu')$, where μ' is any norm-preserving extension of μ, is continuous. By Fact 4.4 there exists a proper closed convex function $g \colon F \to \mathbf{R} \cup \{\infty\}$ such that $g^* = f'$. By Lemma 4.11 we then have $g \in K_C(F)$ and by Remark 4.12 we have $f^* = \tilde{g}^*$, so $f = \tilde{g}$. We may therefore assume that E is separable, and choose an increasing sequence of compact convex subsets $X_k \subseteq E$ such that $\bigcup X_k$ is dense in E (take closed balls of increasing radius, of sub-spaces of increasing finite dimension). For each k let $f_k = f\restriction_{X_k}$. Then $\tilde{f}_k \searrow f$ point-wise, and for $\lambda \in B(E^*)$ we have

$$f^*(\lambda) = \sup_v \lambda v - f(v) = \sup_{v,k} \lambda v - f_k(v) = \sup_k f^*_k(\lambda),$$

i.e., $f^*_k \nearrow f^*$ point-wise on $B(E^*)$. Since each f^*_k is lower semi-continuous, f^* is continuous, and $B(E^*)$ is compact, this implies that $f^*_k \to f^*$ uniformly on $B(E^*)$, whereby $\tilde{f}_k \to f$ uniformly, and f is local. ■

A second ingredient is the following.

Definition 5.3. Let E be a normed space and let $r \geq 0$. We say that $f \in K_C(E)$ is *∂-r-extreme* (where ∂ could be pronounced *boundary*) if for every $g \in K_C(E)$ whenever $g \leq f$ (i.e., $g^* \geq f^*$) we have $g^* \leq f^* + r$ on $\partial B(E^*)$. If $r = 0$ we omit it and say that f is *∂-extreme*.

We observe that it does not matter whether we require that $f' \geq f$ on the sphere or on the entire ball, since, if $f' \geq f$ on the sphere, then we may take f'' to be the greatest convex function on the ball which agrees with f' on the sphere, and then $f'' \in K^*_C(E)$ as well, and $f'' \geq f$ on the entire ball.

Lemma 5.4. *Let $f, g \in K_C(E)$ with f ∂-r-extreme and $g \leq f$. Then g is ∂-r-extreme as well.*

Assume furthermore that $f = \widetilde{f\restriction_X}$ for some convex $X \subseteq E$. Then outside X we have $g \geq f - r$.

Proof. By hypothesis we have $g^* \geq f^*$, and we clearly obtain the first assertion, as well as $g^*(\lambda) \leq f^*(\lambda) + r$ for $\|\lambda\| = 1$. By Lemma 4.11(iii), if $v \notin X$ then

$$\tilde{f}(v) = \sup_{\|\lambda\|=1} \lambda v - f^*(\lambda) \leq \sup_{\|\lambda\|=1} \lambda v - g^*(\lambda) + r \leq g(v) + r,$$

as claimed. ■

Lemma 5.5. *Let E be a normed space, let $f \in K_C(E)$ be ∂-r-extreme, and let $\delta > 0$. Then $f + \delta$ is ∂-$(r + 2\delta)$-extreme.*

Proof. Assume not, so let $g \in K_C(E)$, $g \leq f + \delta$ (i.e., $g^* \geq f^* - \delta$), and let $\lambda \in \partial B(E^*)$ be such that $g^*(\lambda) > f^*(\lambda) + r + \delta$. By Fact 4.4 there exists a closed convex h on E such that $h^* = f^* \vee (g^* - \delta)$. For $\mu \in B(E^*)$ we have $f^*(\mu) + f^*(-\mu) \leq 0$, $g^*(\mu) - \delta + g^*(-\mu) - \delta \leq -2\delta < 0$ and $f^*(\mu) + g^*(-\mu) - \delta \leq g^*(\mu) + f^*(-\mu) \leq 0$. Thus $h^*(\mu) + h^*(-\mu) \leq 0$, and since the domain of h^* is the unit ball, $h \in K_C(E)$. Now, $h^* \geq f^*$ implies $h \leq f$, while on the other hand $h^*(\lambda) \geq g^*(\lambda) - \delta > f^*(\lambda) + r$, witnessing that f is not ∂-r-extreme. ■

Lemma 5.6. *Let E be a normed space, let $f \in K_C(E)$ be ∂-r-extreme and local, and let $r' > r$. Then f admits a neighbourhood $f \in W \subseteq K_C(E)$ such that $\operatorname{diam} W < r'$ and every $g \in W$ is ∂-r'-extreme.*

Proof. Let $\delta > 0$ be small enough. By locality, there exists a compact convex set $X \subseteq E$ such that $f + \delta > f' = \widetilde{f\restriction_X}$. Let $W \subseteq K_C(E)$ consist of all g such that $|f - g| < \delta$ on X. Since X is compact and Katětov functions are 1-Lipschitz, W is open, and we claim that it is as desired.

If $g \in W$ then $g < f' < f + \delta$. By Lemma 5.5, $f + \delta$ is ∂-$(r + 2\delta)$-extreme. By Lemma 5.4 so are f' and g, and since $f' = \widetilde{f'\restriction_X}$ we have $g \geq f' - r - 2\delta \geq f - r - 2\delta$ outside X. Since $g \geq f - \delta$ inside X, we conclude that $f - r - 2\delta \leq g < f + \delta$ throughout, which is enough. ■

Theorem 5.7. *Let $f \in K_C(E)$. Then f is isolated if and only if it is both local and ∂-extreme.*

Proof. One direction follows directly from Lemma 5.6, so let us assume that f is isolated. We can then construct a sequence of neighbourhoods $W_k \ni f$ such that $\operatorname{diam} W_k \to 0$, each defined using finitely many parameters. We let X_k be the (compact) convex hull of theses parameters and $f_k = f\restriction_{X_k}$. Then $\tilde{f}_k \in W_k$, so $\tilde{f}_k \to f$ uniformly and f is local.

Assume now that $g \in K_C(E)$, $g \leq f$. Let $W \ni f$ be a neighbourhood of small diameter, say $g \in W$ if and only if $|g(v_i) - f(v_i)| < \varepsilon$ for some v_i, $i < n$. We know that $f(v) = \sup_{\|\lambda\| \leq 1} \lambda v - f^*(\lambda)$, and since a closed convex function in dimension one is continuous on its domain, we have in fact $f(v) = \sup_{\|\lambda\| < 1} \lambda v - f^*(\lambda)$. Therefore, for each $i < n$ we may choose $\lambda_i \in E^*_{<1}$ such that $f(v_i) - \varepsilon < \lambda_i v_i - f^*(\lambda_i) \leq f(v_i)$. Let $g' = g \vee \bigvee_{i<n} \big(\lambda_i - f^*(\lambda_i)\big)$. Since $\|\lambda_i\| < 1$ for each i, g' agrees with g outside some ball. On the other hand, we have $g'(v_i) > f(v_i) - \varepsilon$ and $g' \leq f$, so $g' \in W$. Therefore $|f - g| \leq \operatorname{diam} W$ outside some ball. Since $\operatorname{diam} W$ can be taken arbitrarily small, $f = g$ asymptotically. ■

Before stating a few more corollaries, let us recall a few definitions and facts.

Definition 5.8. Let E be a locally convex space over and let $X \subseteq E$ be convex.

(i) A convex subset $F \subseteq X$ is called a *face* of X if a member of F cannot be expressed as a proper convex combination of points in X which are not both in F. A *proper face*, i.e., a face $F \neq X$, is always contained in the boundary ∂X.

(ii) A face consisting of a single point is called an *extreme point*. We shall denote the set of extreme points of X by $\mathcal{E}(X)$. We shall also denote by $\mathcal{E}_0(X)$ the set of $v \in \mathcal{E}(X)$ such that $\lambda v = \sup \lambda\restriction_X$ for some $\lambda \in E^* \setminus \{0\}$.

By the Krein-Milman Theorem [Bou81, Chapitre II.7, Théorème 1], if X is compact and convex then $X = \overline{\text{co}}\big(\mathcal{E}(X)\big)$. In addition, by [Bou81, Chapitre II.7, Proposition 2], if $v \in \mathcal{E}(X)$ then the family of sets $\{u \in X : \lambda u > r\}$, where $\lambda \in E^* \setminus \{0\}$ and $\lambda v > r \in \mathbf{R}$, forms a basis of neighbourhoods for v in X. Since every such neighbourhood contains a member of $\mathcal{E}_0(X)$ (any extreme point of $\{u \in X : \lambda u = \sup \lambda\restriction_X\}$ will do), we have $\mathcal{E}(X) \subseteq \overline{\mathcal{E}_0(X)}$. In the special case where $X = B(E^*)$, the set $\mathcal{E}_0\big(B(E^*)\big)$ consists exactly of those $\lambda \in \mathcal{E}\big(B(E^*)\big)$ which norm some vector $v \neq 0$.

Corollary 5.9. *Let E be a Banach space, $f \in K_C(E)$ isolated. Let $F_v = \{\lambda \in B(E^*) : \lambda v = 1\}$ where $\|v\| = 1$. Then $f^*\restriction_{F_v}$ is the greatest closed convex function less than $\lambda \mapsto -f^*(\lambda)$, i.e., $f^*(\lambda) = \sup_{u\in E} \inf_{\mu\in F_v} (\lambda - \mu)u - f^*(-\mu)$ for $\lambda \in F_v$.*

In particular, the antipode identity $f^*(\lambda) + f^*(-\lambda) = 0$ *holds at every* $\lambda \in \mathcal{E}\big(B(E^*)\big)$.

Proof. For $u \in E$ and $\alpha > 0$ define

$$h_u(\lambda) = \inf_{\mu\in F_v} (\lambda - \mu)u - f^*(-\mu), \qquad h_{u,\alpha}(\lambda) = \alpha(\lambda v - 1) + h_u(\lambda).$$

Since F_v is closed, applying Fact 4.4 we have $f^*(\lambda) = \sup_{u\in E} \inf_{\mu\in F_v} (\lambda-\mu)u + f^*(\mu) \le \sup_{u\in E} h_u(\lambda)$. For the converse inequality it will suffice to show that $f^* \ge h_u$ on F_v. Notice that h_u is linear and continuous, and in addition, for every $\lambda \in F_v$ we have $h(\lambda) + f^*(-\lambda) \le 0$. Fixing $\varepsilon > 0$, there exists an open set $V \supseteq F_v$ such that $h(\lambda) + f^*(-\lambda) < \varepsilon$ for all $\lambda \in V \cap B(E^*)$. By compactness of $B(E^*) \smallsetminus V$, we know that $\lambda \mapsto \lambda v$ is bounded there below some $r < 1$. We may therefore assume that $V = \{\lambda : \lambda v > r\}$, and that $r > 0$, so $V \cap -V = \varnothing$. For α big enough we have $h_{u,\alpha}(\lambda) \le \inf f^*$ for all $\lambda \in B(E^*) \smallsetminus V$. Having fixed such α, for $\lambda \in B(E^*) \cap V$ we have $h_{u,\alpha}(\lambda) - \varepsilon + f^*(-\lambda) \le h_u(\lambda) + f^*(-\lambda) - \varepsilon \le 0$. It follows that $f^* \vee (h_{u,\alpha} - \varepsilon)$ satisfies the antipode inequality, and is therefore of the form g^* for some $g \in K_C(E)$. But then $g \le f$, so $g^* = f^*$ on $\partial B(E^*)$ and in particular on F_v. Thus $f^* \ge h_{u,\alpha} - \varepsilon = h_u - \varepsilon$ on F_v. Since ε was arbitrary, $f^* \ge h_u$ on F_v, as desired.

It follows that the antipode identity holds on $\mathcal{E}_0\big(B(E^*)\big)$. By continuity of f^*, it holds throughout $\mathcal{E}\big(B(E^*)\big)$. ∎

Thus isolated types satisfy the antipode identity at some boundary points. It is worthwhile to point out that the antipode identity at a non-boundary point amounts to the type being realised.

Lemma 5.10. *Let E be a Banach space, $f \in K_C(E)$. Then the following are equivalent:*

(i) *The type f is realised in E, i.e., there exists $v \in E$ such that $f(x) = \|x - v\|$ for all $x \in E$.*
(ii) *The conjugate f^* satisfies the antipode identity throughout $B(E^*)$.*
(iii) *The conjugate f^* satisfies the antipode identity at some $\lambda \in B(E^*) \smallsetminus \partial B(E^*)$.*
(iv) *We have $f^*(0) = 0$.*

Proof. (i) $\Longrightarrow$ (ii). We have $f^*(\lambda) = \lambda v$.

(ii) $\Longrightarrow$ (iii). Clear.

(iii) $\Longrightarrow$ (iv). Assume $f^*(0) \neq 0$. Then necessarily $f^*(0) < 0$ and $\lambda \neq 0$. Let $\alpha = \|\lambda\| < 1$ and $\mu = \lambda/\alpha$, so $\lambda = \alpha\mu + (1-\alpha)0$. By convexity, $\alpha f^*(\mu) + \alpha f^*(-\mu) + 2(1-\alpha)f^*(0) \ge f^*(\lambda) + f^*(-\lambda) = 0$, contradicting the antipode inequality at μ.

(iv) $\Longrightarrow$ (i). We have $0 = f^*(0) = -\inf f$. Since f is Katětov, and sequence v_k such that $f(v_k) \to 0$ much be Cauchy, say with limit v. It follows that $f(v) = 0$ and consequently that f is realised by v. ∎

6. Existence and non-existence results

This section consists of examples of various cases where densely many isolated types are known to exist or not to exist. We do *not* have a full characterisation of separable spaces E such that isolated types over E are dense.

Definition 6.1. Let E be a Banach space. We say that $f \in K_C(E)$ is *strongly ∂-extreme* if

(i) The antipode identity $f^*(\lambda) + f^*(-\lambda) = 0$ holds on $\mathcal{E}(B(E^*))$.
(ii) The values of f^* on $\partial B(E^*)$ are maximal given the values at the extreme points and convexity.

Clearly, a strongly ∂-extreme type is in particular ∂-extreme.

Lemma 6.2. *Let E be a Banach space. Then the set of strongly ∂-extreme $f \in K_C(E)$ is dense.*

Consequently, if $\dim E < \infty$ *and $f^*\restriction_{\partial B(E^*)}$ is continuous whenever $f \in K_C(E)$ is strongly ∂-extreme, then the isolated types over E are dense. In particular, $\mathbf{G}[E]$ exists.*

Proof. In order to show that the strongly ∂-extreme types are dense, let U be open and $f \in U$. We may assume that U consists of all $g \in K_C(E)$ such that $|g(v_i) - f(v_i)| < \varepsilon$ for $i < n$. Let $X = \mathrm{co}(v_i : i < n) \subseteq E$, and replacing f with $\widetilde{f\restriction_X}$ we may assume that f is local, i.e., that f^* is continuous. For each $i < n$ fix $\lambda_i \in B(E^*)$ such that $f(v_i) + f^*(\lambda_i) < \lambda_i v_i + \varepsilon$, and we may require $\|\lambda_i\| < 1$. Define

$$\hat{f}(\lambda) = \begin{cases} \frac{f^*(\lambda) - f^*(-\lambda)}{2} & \lambda \in \mathcal{E}(B(E^*)), \\ f^*(\lambda_i) & \lambda = \lambda_i. \end{cases}$$

Then $\hat{f}$ satisfies the hypotheses of Lemma 4.5, and is therefore the restriction of $g^*\colon B(E^*) \to \mathbf{R}$ for some $g \in K_C(E)$. We have $g^* \geq f^*$, i.e., $g \leq f$, and $g(v_i) \geq \lambda_i v_i - g^*(\lambda_i) = \lambda_i v_i - f^*(\lambda_i) > f(v_i) - \varepsilon$, so $g \in U$, and g^* is strongly ∂-extreme.

The conclusion is by Lemma 4.6. ■

Remark 6.3. Over a reflexive Banach space E, every isolated type is strongly ∂-extreme, by Corollary 5.9 and the fact that every $\lambda \in \partial B(E^*)$ norms some $v \neq 0$.

Theorem 6.4. *Let E be a normed space of finite dimension. Then the isolated types in $K_C(E)$ are dense if any of the following holds:*

- (i) *Every face of $B(E^*)$ of dimension at most $\dim E - 2$ is a simplex. This holds in particular whenever $\dim E \leq 3$. Special cases of this include:*
  - (a) *Every face of $B(E^*)$ is a simplex. This holds in particular whenever $\dim E \leq 2$. In this case $\mathbf{G}$ is atomic over a copy $E \subseteq \mathbf{G}$ if and only if every $\lambda \in E^*$ admits a unique extension of the same norm to $\mathbf{G}$.*
  - (b) *The space E is smooth, i.e., every $v \in E$ is smooth. In this case $\mathbf{G}$ is atomic over a copy $E \subseteq \mathbf{G}$ if and only if every $v \in E$ is smooth in $\mathbf{G}$.*
- (ii) *The space E is polyhedral.*

Proof. In each case we apply Lemma 6.2, so we assume throughout that $f \in K_C(E)$ is strongly ∂-extreme.

In the first case, let $n = \dim E$ and let $X \subseteq \partial B(E^*)$ be the union of all faces of $\partial B(E^*)$ which are simplexes. Then $\partial B(E^*) \setminus X$ consists of the relative interiors of some faces of dimension $n-1$, so X is closed. On each face which is a simplex f^* is affine, and since it satisfies the antipode identity at the extreme points in satisfies it throughout X. By Lemma 4.11(vi), f^*, as a function on $B(E^*)$, is continuous at every $\lambda \in X$. It follows by Lemma 4.6 that the restriction of f^* to any face (be it a simplex or not) is continuous. Thus, if $\lambda_k \in \partial B(E^*)$, $\lambda_k \to \lambda$, then either λ belongs to the relative interior of some face of dimension $n-1$ or else belongs to X, and in any case $f^*(\lambda_k) \to f^*(\lambda)$.

In the first special case $X = \partial B(E^*)$ and the characterisation of $\mathbf{G}[E]$ follows from Lemma 4.11(iv). The second special case is clear.

If E is polyhedral, one prove by induction on m, using Lemma 4.6, that f^* is continuous on each face of dimension m and therefore on the union of all such faces. For $m = n-1$, this means that f^* is continuous on $\partial B(E^*)$. ■

Notice that Proposition 3.2 fits as a special case of all cases mentioned in Theorem 6.4. More generally, the case where $E = \ell_\infty(n)$, namely $\mathbf{R}^n$ equipped with the supremum norm, fits in all except the one of smooth E. This can be used to show there are infinitely many distinct orbits in the action of $\mathrm{Iso}_L(\mathbf{G}) \curvearrowright \partial B(\mathbf{G})$ (so far we merely knew there were at least two, since there are both smooth and non-smooth points).

Corollary 6.5. *For each $n \in \mathbf{N}$ there exists $v \in \partial B(\mathbf{G})$ such that $N(v) = \{\lambda \in \partial B(\mathbf{G}^*) : \lambda v = 1\}$ is a simplex of dimension n.*

Since the isomorphism type of $N(v)$ is invariant under the action of $\mathrm{Iso}_L(\mathbf{G})$ (say that two convex sets are isomorphic if there exists a homeomorphism between them respecting convex combinations), there exist infinitely many orbits.

Proof. By Theorem 6.4, $\mathbf{G}[\ell_\infty(n+1))]$ exists. Letting $v = (1, 1, \ldots) \in \ell(n+1)$, the set $N(v)$ is a simplex of dimension n. ■

Question 6.6. We know that vectors $v \in \partial B(\mathbf{G})$ for which $N(v)$ is a singleton (the smooth vectors) form an orbit. Do the vectors $v \in \partial B(\mathbf{G})$ for which $N(v)$ is a simplex of dimension n form a single orbit for $n \geq 1$ as well?

So far we can only show the following:

- Either $N(v)$ is a simplex of finite dimension or it generates an infinite-dimensional space.
- If $N(v)$ is a simple of dimension n then there exists an isometric embedding $\ell_\infty(n+1) \subseteq \mathbf{G}$ sending each of the standard basis vectors to smooth vectors in $\mathbf{G}$ whose sum is v. However, for $n \geq 1$ this does *not* imply that $\mathbf{G}$ is atomic over $\ell_\infty(n+1)$ (counter-examples are easy to produce for any $n \geq 1$), so the reasoning stops short of proving that the set of $v \in \partial B(\mathbf{G})$ for which $N(v)$ is a simplex of dimension n forms a single orbit.

Since the argument is fairly long for an incomplete result, we deem it inappropriate to include it.

Let us now give some examples in which the conclusion of Theorem 6.4 fails. This will show that while Theorem 6.4 is not necessarily optimal, none of the hypotheses can be simply done away with.

Example 6.7. We construct an example of a space E of dimension 4, such $\mathbf{G}[E]$ does not exist. Let $B_0 = \{\pm 1\}^4 \subseteq \mathbf{R}^4$, $B_1 = \{(x, y, 0, 0) : x^2 + y^2 = 2 \text{ and } x, y \neq \pm 1\}$ and $B = \mathrm{co}(B_0 \cup B_1)$. Then B is a compact symmetric convex neighbourhood of 0, so we may take $E^* = \mathbf{R}^4$ with $B(E^*) = B$. Moreover, the set of extreme points in B is exactly $B_0 \cup B_1$. It follows by Corollary 5.9 that if $f \in K_C(E)$ is isolated then f^* satisfies the antipode identity also at the four points $(\pm 1, \pm 1, 0, 0)$ (which are not extreme).

Let us construct a special $f \in K_C(E)$ by constructing f^*. At a point $(x, y, z, w) \in B_0$ we let $f^*(x, y, z, w) = xyz$, on B_1 we let f^* vanish, and the define f^* on B as the generated closed convex function. This function satisfies the antipode identity and therefore is indeed of the form f^* for some $f \in K_C(E)$. In addition, a direct calculation reveals that $f^*(\pm 1, \pm 1, 0, 0) = -1$.

Fix $\varepsilon > 0$ ($\varepsilon = 1/2$ will do). At each point $\lambda \in B_0$ there is some $v_\lambda \in E$ such that $f^*(\lambda) < \lambda v_\lambda - f(v_\lambda) + \varepsilon$. Let $U \subseteq K_C(E)$ consist of all g such that $|g(v_\lambda) - f(v_\lambda)| < \varepsilon$ for all $\lambda \in B_0$. Then U is a neighbourhood of f, and if $g \in U$ then at $\lambda \in B_0$ we have

$$g^*(\lambda) \geq \lambda v_\lambda - g(v_\lambda) > \lambda v_\lambda - f(v_\lambda) - \varepsilon > f^*(\lambda) - 2\varepsilon,$$

and therefore

$$g^*(\lambda) \leq -g(-\lambda) < -f^*(-\lambda) + 2\varepsilon = f^*(\lambda) + 2\varepsilon.$$

Thus $g^* < f^* + 2\varepsilon$ throughout the unit cube, and in particular $g^*(\pm 1, \pm 1, 0, 0) < 2\varepsilon - 1$. Thus, for $\varepsilon = 1/2$ or less, the antipode identity fails at $(\pm 1, \pm 1, 0, 0)$ for every $g \in U$, so U contains no isolated points.

If we want counter-examples consisting of smooth spaces we need to move to infinite dimension. In fact, we obtain a plethora of examples over which there are no isolated types other than the obvious ones.

Proposition 6.8. *Let E be a Banach space such that $\overline{\mathcal{E}(B(E^*))} \not\subseteq \partial B(E^*)$. Then the only isolated types over E are the realised ones.*

This holds in particular when $E = c_0$, $E^* = \ell_1$ or $E = \ell_p$, $E^* = \ell_q$ with $1 < p, q$, $\frac{1}{p} + \frac{1}{q} = 1$, since $\lambda_i \to 0$, where λ_i consists of a single 1 at position i and 0 elsewhere.

Proof. Indeed, let $f \in K_C(E)$ be isolated. By Corollary 5.9 f^* satisfies the antipode identity on $\mathcal{E}(B(E^*))$ and therefore at some non-boundary point. By Lemma 5.10 f is realised. ■

Over $E = \mathbf{G}$ the isolated types are again exactly the realised ones, but in this specific case they are dense and $\mathbf{G}[E] = E$.

Question 6.9. Is there any infinite-dimensional E other than $\mathbf{G}$ over which the isolated types are dense? Is there any infinite-dimensional E over which there are unrealised isolated types? Specifically, what happens in the case $E = \ell_1$, $E^* = \ell_\infty$, to which Proposition 6.8 does not apply?

7. Counting types

We conclude with a calculation of the size of the type-space over a separable Banach space E. By "size" we mean here the metric density character (since the cardinal $|\mathrm{S}_n(E)|$ is the continuum as soon as $n > 0$ and $E \neq 0$).

Theorem 7.1. *Let E be a separable Banach space.*

(i) *If E is finite-dimensional and polyhedral then $\mathrm{S}_n(E)$ is metrically separable.*
(ii) *Otherwise, $\mathrm{S}_n(E)$ has metric density character equal to the continuum for every $n \geq 1$.*

Proof. Assume first that E is finite-dimensional and polyhedral. Then by Melleray [Mel07, Remarks following Corollary 4.6], the space $K(E)$ is separable, and *a fortiori* so is $\mathrm{S}_1(E) = K_C(E)$. The passage from 1-types to n-types is done as in the proof of Lemma 2.15, and is left to the reader.

Now assume that E is not both finite-dimensional and polyhedral.. Then by Lindenstrauss [Lin64, Theorem 7.7] there exists a sequence $\{v_n\} \subseteq E$ such that for any $n \neq m$ and choice of signs:

$$\|v_n \pm v_m\| \leq \|v_n\| + \|v_m\| - 1.$$

Embed E (isometrically) in ℓ_∞, and for a sequence $\bar{\varepsilon} \in \{\pm 1\}^{\mathbf{N}}$, consider the family of closed balls $\overline{B}(\varepsilon_n v_n, \|v_n\| - \frac{1}{2})$. By hypothesis every two such balls intersect at a non empty set, and therefore there exists $v \in \ell_\infty$ which belongs to them all. In other words, there exists $\xi_{\bar{\varepsilon}} = \mathrm{tp}(v/E) \in \mathrm{S}_1(E)$ such that $\|x - \varepsilon_n v_n\|^{\xi_{\bar{\varepsilon}}} \leq \|v_n\| - \frac{1}{2}$. If $\bar{\varepsilon} \neq \bar{\varepsilon}'$ then $d(\xi_{\bar{\varepsilon}}, \xi_{\bar{\varepsilon}'}) \geq 1$, so the density character of $\mathrm{S}_1(E)$ is at least the continuum. The same holds *a fortiori* for $\mathrm{S}_n(E)$, $n \geq 1$. ■

Remark 7.2. Lindenstrauss's argument is quite elementary and yields a quick proof for Theorem 7.1(ii) which does not depend on the machinery developed in earlier sections. An argument closer to the spirit of the present paper can also be given.

Let Ξ be the set of lower semi-continuous functions $f_0 \colon \mathcal{E}(B(E^*)) \to \mathbf{R}$ which satisfy in addition $f_0(\lambda) + f_0(-\lambda) \leq 0$. Then E is not a finite-dimensional polyhedral space if and only if $\mathcal{E}(B(E^*))$ is infinite, in which case Ξ has density character continuum. If $f_0 \in \Xi$ and $f = f_0^*$ as in Lemma 4.5 then $f^*\restriction_{\mathcal{E}(B(E^*))} = f_0$ and $f^*(\lambda) + f^*(-\lambda) \leq 0$ throughout $B(E^*)$, so $f \in K_C(E)$ and we are done. Notice that this argument has the advantage of treating the two cases of "finite-dimensional, non polyhedral" and "infinite-dimensional" in the same manner, while the proof of [Lin64, Theorem 7.7] treats them separately, with the second one being significantly more involved.

Theorem 7.1(ii) answers Problem 2 of Avilés et al. [ACC⁺11, Section 4] in the negative (and we thank Wiesław Kubiś for having pointed this out to us). They say that a Banach space G is *of universal disposition for finite-dimensional spaces* if it satisfies a strengthening of Definition 2.1 with ψ being an isometry.

Corollary 7.3. *The density character of any space of universal disposition for finite-dimensional spaces is at least the continuum. In other words, the answer to Problem 2 of* [ACC⁺11, Section 4] *is negative.*

Proof. Assume that G is of universal disposition for finite-dimensional spaces. Then the Euclidean plane E embeds isometrically in G, and all types over E are realised in G, so the density character of G must be at least the metric density character of $S_1(E)$, namely the continuum. ■

On the other hand, say that a Gurarij space G is *strongly* $\aleph_1$*-homogeneous* if the following stronger version of Corollary 2.8 holds in G:

> For every separable $F \subseteq G$ and isometric embedding $\varphi \colon F \to G$ there exists an isometric automorphism $\psi \in \mathrm{Aut}(G)$ extending φ.

Clearly, a strongly $\aleph_1$-homogeneous Gurarij space is of universal disposition for finite-dimensional (and even separable) spaces. Moreover, there does exist such a space of density character the continuum. This is merely a special case of a general model theoretic result: for any cardinal κ and structure $\mathbf{M}$ of density character $\leq 2^\kappa$, in a language of cardinal $\leq \kappa$, there exists an elementary extension $\mathbf{M}' \succeq \mathbf{M}$ of density character still $\leq 2^\kappa$, which is moreover κ^+-saturated and strongly κ^+-homogeneous. Apply this to $\mathbf{M} = \mathbf{G}$ and $\kappa = \aleph_0$.

References

[ACC⁺11] Antonio Avilés, Félix Cabello Sánchez, Jesús M. F. Castillo, Manuel González, and Yolanda Moreno, *Banach spaces of universal disposition*, Journal of Functional Analysis **261** (2011), no. 9, 2347–2361, doi:10.1016/j.jfa.2011.06.011. 2, 19

[BBHU08] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with Applications to Algebra and Analysis, volume 2 (Zoé Chatzidakis, Dugald Macpherson, Anand Pillay, and Alex Wilkie, eds.), London Math Society Lecture Note Series, vol. 350, Cambridge University Press, 2008, pp. 315–427. 6

[Ben] Itaï Ben Yaacov, *Fraïssé limits of metric structures*, Journal of Symbolic Logic, to appear, arXiv:1203.4459. 7

[Ben08a] ———, *Continuous first order logic for unbounded metric structures*, Journal of Mathematical Logic **8** (2008), no. 2, 197–223, doi:10.1142/S0219061308000737, arXiv:0903.4957. 6

[Ben08b] ———, *Topometric spaces and perturbations of metric structures*, Logic and Analysis **1** (2008), no. 3–4, 235–272, doi:10.1007/s11813-008-0009-x, arXiv:0802.4458. 4

[Ben09] ———, *Modular functionals and perturbations of Nakano spaces*, Journal of Logic and Analysis **1** (2009), Paper 1, 42, doi:10.4115/jla.2009.1.1, arXiv:0802.4285. 6

[Ben14] ———, *The linear isometry group of the Gurarij space is universal*, Proceedings of the American Mathematical Society **142** (2014), no. 7, 2459–2467, doi:10.1090/S0002-9939-2014-11956-3, arXiv:1203.4915. 2, 12, 13, 14

[Bou81] Nicolas Bourbaki, *Espaces vectoriels topologiques. Chapitres 1 à 5*, new ed., Masson, Paris, 1981, Éléments de mathématique. 15

[Bre83] Haïm Brezis, *Analyse fonctionnelle*, Collection Mathématiques Appliquées pour la Maîtrise, Masson, Paris, 1983, Théorie et applications. 12

[BU07] Itaï Ben Yaacov and Alexander Usvyatsov, *On d-finiteness in continuous structures*, Fundamenta Mathematicae **194** (2007), 67–88, doi:10.4064/fm194-1-4. 6

[BU10] ———, *Continuous first order logic and local stability*, Transactions of the American Mathematical Society **362** (2010), no. 10, 5213–5259, doi:10.1090/S0002-9947-10-04837-3, arXiv:0801.4303. 6

[Gur66] Vladimir I. Gurarij, *Spaces of universal placement, isotropic spaces and a problem of Mazur on rotations of Banach spaces*, Sibirsk. Mat. Ž. **7** (1966), 1002–1013. 1

[Kat88] Miroslav Katětov, *On universal metric spaces*, General topology and its relations to modern analysis and algebra, VI (Prague, 1986), Res. Exp. Math., vol. 16, Heldermann, Berlin, 1988, pp. 323–330. 12, 13

[KS13] Wiesław Kubiś and Sławomir Solecki, *A proof of uniqueness of the Gurariĭ space*, Israel Journal of Mathematics **195** (2013), no. 1, 449–456, doi:10.1007/s11856-012-0134-9, arXiv:1110.0903. 7

[Lin64] Joram Lindenstrauss, *Extension of compact operators*, Memoirs of the American Mathematical Society **48** (1964), 112. 18, 19

[Lus76] Wolfgang Lusky, *The Gurarij spaces are unique*, Archiv der Mathematik **27** (1976), no. 6, 627–635. 1, 6, 7, 11

[Maz33] Stanisław Mazur, *Über konvexe mengen in linearen normierten räumen*, Studia Mathematica **4** (1933), 70–84. 1, 11

[Mel07] Julien Melleray, *On the geometry of Urysohn's universal metric space*, Topology and its Applications **154** (2007), no. 2, 384–403, doi:10.1016/j.topol.2006.05.005. 18

[Roc70] R. Tyrrell Rockafellar, *Convex analysis*, Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J., 1970. 12

[Usp08] Vladimir V. Uspenskij, *On subgroups of minimal topological groups*, Topology and its Applications **155** (2008), no. 14, 1580–1606, doi:10.1016/j.topol.2008.03.001, arXiv:math/0004119. 12

Itaï Ben Yaacov, Université Claude Bernard – Lyon 1, Institut Camille Jordan, CNRS UMR 5208, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France
URL: http://math.univ-lyon1.fr/~begnac/

C. Ward Henson, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801, USA
URL: http://www.math.uiuc.edu/~henson/