AN INDEPENDENCE THEOREM FOR NTP_2 THEORIES

ITAÏ BEN YAACOV AND ARTEM CHERNIKOV

Abstract. We establish several results regarding dividing and forking in NTP_2 theories.

We show that dividing is the same as array-dividing. Combining it with existence of strictly invariant sequences we deduce that forking satisfies the chain condition over extension bases (namely, the forking ideal is $S1$, in Hrushovski's terminology). Using it we prove an independence theorem over extension bases (which, in the case of simple theories, specializes to the ordinary independence theorem). As an application we show that Lascar strong type and compact strong type coincide over extension bases in an NTP_2 theory.

We also define the dividing order of a theory – a generalization of Poizat's fundamental order from stable theories – and give some equivalent characterizations under the assumption of NTP_2. The last section is devoted to a refinement of the class of strong theories and its place in the classification hierarchy.

Introduction

The class of NTP_2 theories, namely theories without the tree property of the second kind, was introduced by Shelah [She80] and is a natural generalization of both simple and NIP theories containing new important examples (e.g. any ultra-product of p-adics is NTP_2, see [Che]).

The realization that it is possible to develop a good theory of forking in the NTP_2 context came from the paper [CK12], where it was demonstrated that the basic theory can be carried out as long as one is working over an extension base (a set is called an extension base if every complete type over it has a global non-forking extension, e.g. any model or any set in a simple, o-minimal or C-minimal theory is an extension base).

Here we establish further important properties of forking, thus demonstrating that a large part of simplicity theory can be seen as a special case of the theory forking in NTP_2 theories.

In Section 1 we consider the notion of *array dividing*, which is a multi-dimensional generalization of dividing. We show that in an NTP_2 theory, dividing coincides with array dividing over an arbitrary set (thus generalizing a corresponding result of Kim for the class of simple theories).

Section 2 is devoted to a property of forking called the *chain condition*. We say that forking in T satisfies the chain condition over a set A if for any A-indiscernible sequence $(a_i)_{i\in\omega}$ and any formula $\varphi(x,y)$, if $\varphi(x,a_0)$ does not fork over A, then $\varphi(x,a_0)\wedge\varphi(x,a_1)$ does not fork over A. This property is equivalent to requiring that there are no anti-chains of unbounded size in the partial order of formulas non-forking over A ordered by implication (hence the name, see Section 2 for more equivalences and the history of the notion). The following question had been raised by Adler and by Hrushovski:

Question 0.1. What are the implications between NTP_2 and the chain condition?

We resolve it by showing that:

(i) Forking in NTP_2 theories satisfies the chain condition over extension bases (Theorem 2.9, our proof combines the equality of dividing and array-dividing with the existence of universal Morley sequences from [CK12]).
(ii) There is a theory with TP_2 in which forking satisfies the chain condition (Section 2.3).

In his work on approximate subgroups, Hrushovski [Hru12] reformulated the independence theorem for simple theories with respect to an arbitrary invariant $S1$-ideal. In Section 3 we observe that the chain condition means that the forking ideal is $S1$. Using it we prove a independence theorem for forking over an arbitrary

First author supported by the Institut Universitaire de France.

Second author supported by the Marie Curie Initial Training Network in Mathematical Logic - MALOA - From MAthematical LOgic to Applications, PITN-GA-2009-238381.

extension base in an NTP_2 theory (Theorem 3.3), which is a natural generalization of the independence theorem of Kim and Pillay for simple theories. As an application we show that Lascar type coincides with compact strong type over an extension base in an NTP_2 theory.

In Section 4 we discuss a possible generalization of the fundamental order of Poizat which we call the *dividing order*. We prove some equivalent characterizations and connections to the existence of universal Morley sequences in the case of NTP_2 theories, and make some conjectures.

In the final section we define burden^2 and strong^2 theories (which coincide with strongly^2 dependent theories under the assumption of NIP, just as Adler's strong theories specialize to strongly dependent theories). We establish some basic properties of burden^2 and prove that NTP_2 is characterized by the boundedness of burden^2.

Preliminaries. We assume some familiarity with the basics of forking and dividing (e.g. [CK12, Section 2]), simple theories (e.g. [Wag00]) and NIP theories (e.g. [Adla]).

As usual, T is a complete first-order theory, $\mathbb{M} \vDash T$ is a monster model. We write $a \mathop{\smile\!\!\!\!\!\mid}_C b$ when $\mathrm{tp}(a/bC)$ does not fork over C and $a \mathop{\smile\!\!\!\!\!\mid}^d_C b$ when $\mathrm{tp}(a/bC)$ does not divide over C. In general these relations are not symmetric. We say that a global type $p(x) \in S(\mathbb{M})$ is *invariant* (*Lascar-invariant*) over A if whenever $\varphi(x,a) \in p$ and $b \equiv_A a$ (resp. $b \equiv^{\mathrm{L}}_A a$, see Definition 3.1), then $\varphi(x,b) \in p$.

We use the plus sign to denote concatenation of sequences, as in $I + J$, or $a_0 + I + b_1$ and so on.

Definition 0.2. Recall that a formula $\varphi(x,y)$ is TP_2 if there are $(a_{ij})_{i,j\in\omega}$ and $k \in \omega$ such that:

- $\{\varphi(x,a_{ij})\}_{j\in\omega}$ is k-inconsistent for each $i \in \omega$,
- $\left\{\varphi\left(x,a_{if(i)}\right)\right\}_{i\in\omega}$ is consistent for each $f : \omega \to \omega$.

A formula is NTP_2 if it is not TP_2, and a theory T is NTP_2 if it implies that every formula is NTP_2.

1. Array dividing

For the clarity of exposition (and since this is all that we will need) we only deal in this section with 2-dimensional arrays. All our results generalize to n-dimensional arrays by an easy induction (or even to λ-dimensional arrays for an arbitrary ordinal λ, by compactness; see [Ben03, Section 1]).

Definition 1.1.
(i) We say that $(a_{ij})_{i,j\in\kappa}$ is an *indiscernible array* over A if both $\left((a_{ij})_{j\in\kappa}\right)_{i\in\kappa}$ and $\left((a_{ij})_{i\in\kappa}\right)_{j\in\kappa}$ are indiscernible sequences. Equivalently, all $n\times n$ sub-arrays have the same type over A, for all $n<\omega$. Equivalently, $\mathrm{tp}(a_{i_0j_0}a_{i_0j_1}...a_{i_nj_n}/A)$ depends just on the quantifier-free types of $\{i_0,...,i_n\}$ and $\{j_0,...,j_n\}$ in the language of order and equality. Notice that, in particular, $\left(a_{if(i)}\right)_{i\in\kappa}$ is an A-indiscernible sequence of the same type for any strictly increasing function $f:\kappa\to\kappa$.
(ii) We say that an array $(a_{ij})_{i,j\in\kappa}$ is *strongly indiscernible* over A if it is an indiscernible array over A, and in addition its rows are mutually indiscernible over A, i.e. $(a_{ij})_{j\in\kappa}$ is indiscernible over $(a_{i'j})_{i'\in\kappa\smallsetminus\{i\},j\in\kappa}$ for each $i\in\kappa$.

Definition 1.2. We say that $\varphi(x,a)$ *array-divides* over A if there is an A-indiscernible array $(a_{ij})_{i,j\in\omega}$ such that $a_{00}=a$ and $\{\varphi(x,a_{ij})\}_{i,j\in\omega}$ is inconsistent.

Definition 1.3.
(i) Given an array $\mathbf{A}=(a_{i,j})_{i,j\in\omega}$ and $k\in\omega$, we define:
  (a) $\mathbf{A}^k=\left(a'_{i,j}\right)_{i,j\in\omega}$ with $a'_{i,j}=a_{ik,j},a_{ik+1,j},\ldots,a_{ik+k-1,j}$.
  (b) $\mathbf{A}^{\mathrm{T}}=(a_{j,i})_{i,j\in\omega}$, namely the transposed array.
(ii) Given a formula $\varphi(x,y)$, we let $\varphi^k(x,y_0\ldots y_{k-1})=\bigwedge_{i<k}\varphi(x,y_i)$.
(iii) Notice that with this notation $\left(\mathbf{A}^k\right)^l=\mathbf{A}^{kl}$ and $\left(\varphi^k\right)^l=\varphi^{kl}$.

Lemma 1.4.
(i) *If $\mathbf{A}$ is a B-indiscernible array, then $\mathbf{A}^k$ (for any $k\in\omega$) and $\mathbf{A}^{\mathrm{T}}$ are B-indiscernible arrays.*
(ii) *If $\mathbf{A}$ is a strongly indiscernible array over B, then $\mathbf{A}^k$ is a strongly indiscernible array over B (for any $k\in\omega$).*

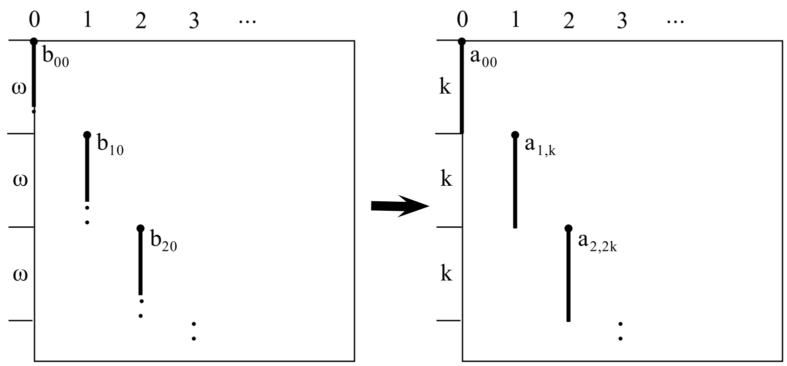
Lemma 1.5. *Assume that T is* NTP_2 *and let $(a_{ij})_{i,j\in\omega}$ be a strongly indiscernible array. Assume that the first column $\{\varphi(x,a_{i0})\}_{i\in\omega}$ is consistent. Then the whole array $\{\varphi(x,a_{ij})\}_{i,j\in\omega}$ is consistent.*

Proof. Let $\varphi(x,y)$ and a strongly indiscernible array $\mathbf{A}=(a_{ij})_{i,j\in\omega}$ be given. By compactness, it is enough to prove that $\{\varphi(x,a_{ij})\}_{i<k,j\in\omega}$ is consistent for every $k\in\omega$. So fix some k, and let $\mathbf{A}^k=(b_{ij})_{i,j\in\omega}$ — it is still a strongly indiscernible array by Lemma 1.4. Besides $\left\{\varphi^k(x,b_{i0})\right\}_{i\in\omega}$ is consistent. But then $\left\{\varphi^k(x,b_{ij})\right\}_{j\in\omega}$ is consistent for some $i\in\omega$ (as otherwise φ^k would have TP_2 by the mutual indiscernibility of rows), thus for $i=0$ (as the sequence of rows is indiscernible). Unwinding, we conclude that $\{\varphi(x,a_{ij})\}_{i<k,j\in\omega}$ is consistent. $\blacksquare_{1.5}$

Lemma 1.6. *Assume that T is* NTP_2 *and let $\mathbf{A}=(a_{ij})_{i,j\in\omega}$ be an indiscernible array and assume that the diagonal $\{\varphi(x,a_{ii})\}_{i\in\omega}$ is consistent. Then for any $k\in\omega$, if $\mathbf{A}^k=(b_{ij})_{i,j\in\omega}$ then the diagonal $\left\{\varphi^k(x,b_{ii})\right\}_{i\in\omega}$ is consistent.*

Proof. By compactness we can extend our array $\mathbf{A}$ to $(a_{ij})_{i\in\omega\times\omega,j\in\omega}$ and let $b_{ij}=a_{i\times\omega+j,i}$.

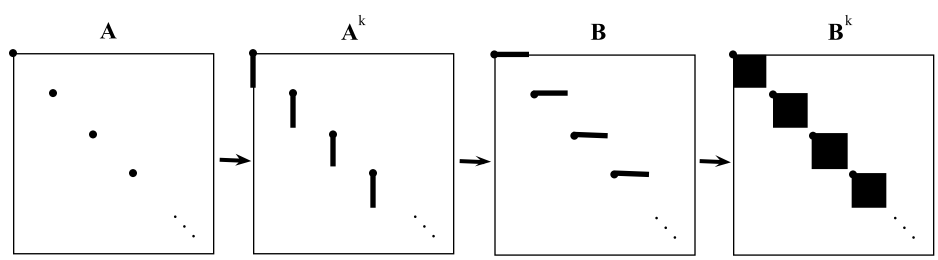
It then follows that $(b_{ij})_{i,j\in\omega}$ is a strongly indiscernible array and that $\{\varphi(x,b_{i0})\}_{i\in\omega}$ is consistent. But then $\{\varphi(x,b_{ij})\}_{i,j\in\omega}$ is consistent by Lemma 1.5 , and we can conclude by indiscernibility of $\mathbf{A}$.



$\blacksquare_{1.6}$

Proposition 1.7. *Assume T is* NTP_2. *If $(a_{ij})_{i,j\in\omega}$ is an indiscernible array and the diagonal $\{\varphi(x,a_{ii})\}_{i\in\omega}$ is consistent, then the whole array $\{\varphi(x,a_{ij})\}_{i,j\in\omega}$ is consistent. Moreover, this property characterizes* NTP_2.

Proof. Let $\kappa\in\omega$ be arbitrary. Let $\mathbf{A}^k=(b_{ij})_{i,j\in\omega}$, then its diagonal $\left\{\varphi^k(x,b_{ii})\right\}_{i\in\omega}$ is consistent by Lemma 1.6. As $\mathbf{B}=\left(\mathbf{A}^k\right)^T$ has the same diagonal, using Lemma 1.6 again we conclude that if $\mathbf{B}^k=(c_{ij})_{i,j\in\omega}$, then its diagonal $\left\{\varphi^{k^2}(x,c_{ii})\right\}_{i\in\omega}$ is consistent. In particular $\{\varphi(x,a_{ij})\}_{i,j<k}$ is consistent. Conclude by compactness.



"Moreover" follows from the fact that if T has TP_2, then there is a strongly indiscernible array witnessing this. $\blacksquare_{1.7}$

Corollary 1.8. *Let T be* NTP_2. *Then $\varphi(x,a)$ divides over A if and only if it array-divides over A.*

Proof. If $(a_{ij})_{i,j\in\omega}$ is an A-indiscernible array with $a_{00}=a$, then $\{\varphi(x,a_{ii})\}_{i\in\omega}$ is consistent since $(a_{ii})_{i\in\omega}$ is indiscernible over A and $\varphi(x,a)$ does not divide over A, apply Proposition 1.7. $\blacksquare_{1.8}$

Remark 1.9. Array dividing was apparently first considered for the purposes of classification of Zariski geometries in [HZ96]. Kim [Kim96] proved that in simple theories dividing equals array dividing. Later the first author used it to develop the basics of simplicity theory in the context of compact abstract theories [Ben03], and Adler used it in his presentation of thorn-forking in [Adl09].

2. The chain condition

2.1. The chain condition.

Definition 2.1. We say that forking in T satisfies the *chain condition* over A if whenever $I=(a_i)_{i\in\omega}$ is an indiscernible sequence over A and $\varphi(x,a_0)$ does not fork over A, then $\varphi(x,a_0)\wedge\varphi(x,a_1)$ does not fork over A. It then follows that $\{\varphi(x,a_i)\}_{i\in\omega}$ does not fork over A.

Lemma 2.2. *The following are equivalent for any theory T and a set A:*

(i) *Forking in T satisfies the chain condition over A.*
(ii) *Let $\kappa=(2^{|T|+|A|})^+$. Then for every $p(x)\in S(A)$, whenever $(p(x)\cup\{\varphi_i(x,a_i)\})_{i<\kappa}$ is a family of partial types non-forking over A, there are $i<j<\kappa$ such that $p(x)\cup\{\varphi_i(x,a_i)\}\cup\{\varphi_j(x,a_j)\}$ does not fork over A.*
(iii) *The previous item holds for* some *κ. In other words, there are no anti-chains of unbounded size in the partial order of non-forking types over A.*
(iv) *If $b\mathop{\smile\!\!\!\!\!\!\;\;|}_A a_0$ and $I=(a_i)_{i\in\omega}$ is indiscernible over A, then there is $I'\equiv_{Aa_0}I$, indiscernible over Ab and such that $b\mathop{\smile\!\!\!\!\!\!\;\;|}_A I'$.*

Proof. (i) $\Longrightarrow$ (ii). Follows from the fact that in every set S with elements of size λ, if $|S|>2^{\lambda+|T|}$ then some two different elements appear in an indiscernible sequence (see e.g. [Cas03, Proposition 3.3]).

(ii) $\Longrightarrow$ (iii). Obvious.

(iii) $\Longrightarrow$ (iv). We may assume that I is of length κ, long enough. Let $p(x,a_0)=\mathrm{tp}(b/a_0A)$. It follows from (iii) by compactness that $\bigcup_{i<\kappa}p(x,a_i)$ does not fork over A. Then there is b' realizing it, such that in addition $b'\mathop{\smile\!\!\!\!\!\!\;\;|}_A I$. By Ramsey, automorphism and compactness we find an I' as wanted.

(iv) $\Longrightarrow$ (i). Assume that the chain condition fails, let I and $\varphi(x,y)$ witness this, so $\varphi(x,a_0)\wedge\varphi(x,a_1)$ forks over A. Let $b\vDash\varphi(x,a_0)\wedge\varphi(x,a_1)$. It is clearly not possible to find I' as in (4). $\blacksquare_{2.2}$

Remark 2.3. The term "chain condition" refers to Lemma 2.2(iii) interpreted as saying that there are no antichains of unbounded size in the partial order of non-forking formulas (ordered by implication). The chain condition was introduced and proved by Shelah with respect to weak dividing, rather than dividing, for simple theories in the form of (ii) in [She80]. Later [GIL02, Theorem 4.9] presented a proof due to Shelah of the chain condition with respect to dividing for simple theories using the independence theorem, again in the form of (ii). The chain condition as defined here was proved for simple theories by Kim [Kim96]. It was further studied by Dolich [Dol04], Lessmann [Les00], Casanovas [Cas03] and Adler [Adlb] establishing the equivalence of the first three forms. In the case of NIP theories, the chain condition follows immediately from the fact that non-forking is equivalent to Lascar-invariance (see Lemma 2.11).

Of course, the chain condition need not hold in general.

Example 2.4. Let T be the model completion of the theory of triangle-free graphs. It eliminates quantifiers. Let $M\vDash T$ and let $(a_i)_{i\in\omega}$ be an M-indiscernible sequence such that $\vDash\neg Ra_ib$ for any i and $b\in M$. Notice that by indiscernibility $\vDash\neg Ra_ia_j$ for $i\neq j$. It is easy to see that Rxa_0 does not divide over M. On the other hand, $Rxa_0\wedge Rxa_1$ divides over M.

2.2. NTP$_2$ implies the chain condition.

We will need some facts about forking and dividing in NTP$_2$ theories established in [CK12]. Recall that a set C is an *extension base* if every type in $S(C)$ does not fork over C.

Definition 2.5. We say that $(a_i)_{i\in\kappa}$ is a *universal Morley sequence* in $p(x)\in S(A)$ when:

- it is indiscernible over A with $a_i\vDash p(x)$

- for any $\varphi(x,y) \in L(A)$, if $\varphi(x,a_0)$ divides over , then $\{\varphi(x,a_i)\}_{i\in\kappa}$ is inconsistent.

Fact 2.6. [CK12] *Assume that T is* NTP$_2$.

(i) *Let M be a model. Then for every $p(x) \in S(M)$, there is a universal Morley sequence in it.*
(ii) *Let C be an extension base. Then $\varphi(x,a)$ divides over C if and only if $\varphi(x,a)$ forks over C.*

First we observe that the chain condition always implies equality of dividing and array dividing:

Proposition 2.7. *If T satisfies the chain condition over C, and forking equals dividing over C, then $\varphi(x,a)$ divides over C if and only if it array-divides over C.*

Proof. Assume that $\varphi(x,a)$ does not divide over C. Let $(a_{ij})_{i,j\in\omega}$ be a C-indiscernible array and $a_{00}=a$. It follows by the chain condition and compactness that $\{\varphi(x,a_{i0})\}_{i\in\omega}$ does not divide over C. But as $\left((a_{ij})_{i\in\omega}\right)_{j\in\omega}$ is also a C-indiscernible sequence, applying the chain condition and compactness again we conclude that $\{\varphi(x,a_{ij})\}_{i,j\in\omega}$ does not divide over C, so in particular it is consistent. $\blacksquare_{2.7}$

And in the presence of universal Morley sequences witnessing dividing, the converse holds:

Proposition 2.8. *Let T be* NTP$_2$ *and $M \vDash T$. Then forking satisfies the chain condition over M.*

Proof. Let κ be very large compared to $|M|$, assume that $\bar{a}_0 = (a_{0i})_{i\in\kappa}$ is indiscernible over M, $\varphi(x,a_{00})$ does not divide over M, but $\varphi(x,a_{00}) \wedge \varphi(x,a_{01})$ does. By Fact 2.6, let $(\bar{a}_i)_{i\in\omega}$ be a universal Morley sequence in $\mathrm{tp}(\bar{a}_0/M)$. By the universality and indiscernibility of $\bar{a}_0$, $\{\varphi(x,a_{ij_1}) \wedge \varphi(x,a_{ij_2})\}_{i\in\omega}$ is inconsistent for any $j_1 \neq j_2$. We can extract an M-indiscernible sequence $\left((a'_{ij})_{i\in\omega}\right)_{j\in\omega}$ from $\left((a_{ij})_{i\in\omega}\right)_{j\in\kappa}$, such that type of every finite subsequence over M is already present in the original sequence. It follows that $\left(a'_{ij}\right)_{i,j\in\omega}$ is an M-indiscernible array and that $\left\{\varphi(x,a'_{ij})\right\}_{i,j\in\omega}$ is inconsistent, thus $\varphi(x,a_{00})$ array-divides over M, thus divides over M by Corollary 1.8 — a contradiction. $\blacksquare_{2.8}$

Theorem 2.9. *If T is* NTP$_2$*, then it satisfies the chain condition over extension bases.*

Proof. Let C be an extension base and $\bar{a} = (a_i)_{i\in\omega}$ be a C-indiscernible sequence. As C is an extension base, we can find $M \supseteq C$ such that $M \mathop{\smile\!\!\!\!\!\mid}_C \bar{a}$. It follows that for any $n \in \omega$, $\bigwedge_{i<n} \varphi(x,a_i)$ divides over C if and only if it divides over M. It follows from Proposition 2.8 that if $\varphi(x,a_0)$ does not divide over C, then $\{\varphi(x,a_i)\}_{i\in\omega}$ does not divide over C. $\blacksquare_{2.9}$

Corollary 2.10. *If T is* NTP$_2$*, A is an extension base, $(a_{ij})_{i,j\in\omega}$ is an A-indiscernible array, and $\varphi(x,a_{00})$ does not divide over A, then $\{\varphi(x,a_{ij})\}_{i,j\in\omega}$ does not divide over A.*

2.3. **The chain condition does not imply NTP$_2$.**

Lemma 2.11. *Let T be a theory satisfying:*

- *For every set A and a global type $p(x)$, it does not fork over A if and only if it is Lascar-invariant over A.*

Then T satisfies the chain condition.

Proof. Let $\bar{a} = (a_i)_{i\in\omega}$ be an A-indiscernible sequence and assume that $\varphi(x,a_0)$ does not fork over A. Then there is a global type $p(x)$ containing $\varphi(x,a_0)$ and non-forking over A, thus Lascar-invariant over A. Taking $c \vDash p|_{\bar{a}A}$, it follows by Lascar-invariance that $c \vDash \{\varphi(x,a_i)\}_{i\in\omega}$. $\blacksquare_{2.11}$

In [CKS12, Section 5.3] the following example is constructed:

Fact 2.12. *There is a theory T such that:*

(i) *T has* TP$_2$.
(ii) *A global type does not fork over a small set A if and only if it is finitely satisfiable in A (therefore, if and only if it is Lascar-invariant over A).*

It follows from Lemma 2.11 that this T satisfies the chain condition.

3. The independence theorem and Lascar types

Definition 3.1. As usual, we write $a \equiv^{\mathrm{L}}_C b$ to denote that a and b have the same *Lascar type* over C. That is, if any of the following equivalent properties holds:

(i) a and b are equivalent under every C-invariant equivalence relation with a bounded number of classes.

(ii) There are $n \in \omega$ and $a = a_0, ..., a_n = b$ such that a_i, a_{i+1} start a C-indiscernible sequence for each $i < n$.

We let $d_C(a, b)$ be the *Lascar distance*, that is the smallest n as in (2) or ∞ if it does not exist.

Now we will use the chain condition in order to deduce a independence theorem over an extension base.

Lemma 3.2. *Assume that $d_A(b, b') = 1$ and $a \mathop{\smile\!\!\!\!\!\!|}_{Ab} b'$. Then there exists a sequence $(a_i b_i)_{i\in\omega}$ indiscernible over A and such that $a_0 b_0 b_1 = abb'$.*

Proof. Standard. $\blacksquare_{3.2}$

Theorem 3.3. *Let T be* NTP$_2$ *and A an extension base. Assume that $c \mathop{\smile\!\!\!\!\!\!|}_A ab$, $a \mathop{\smile\!\!\!\!\!\!|}_A bb'$ and $b \equiv^{\mathrm{L}}_A b'$. Then there is c' such that $c' \mathop{\smile\!\!\!\!\!\!|}_A ab'$, $c'a \equiv_A ca$, $c'b' \equiv_A cb$.*

Proof. Let us first consider the case $d_A(b, b') = 1$. Since $a \mathop{\smile\!\!\!\!\!\!|}_{Ab} b'$, by Lemma 3.2 we can find $(a_i b_i)_{i\in\omega}$ indiscernible over A and such that $a_0 b_0 b_1 = abb'$. As $c \mathop{\smile\!\!\!\!\!\!|}_A a_0 b_0$, it follows by the chain condition that there exists $c' \equiv_{A a_0 b_0} c$ such that $c' \mathop{\smile\!\!\!\!\!\!|}_A (a_i b_i)_{i\in\omega}$ and $(a_i b_i)_{i\in\omega}$ is indiscernible over $c'A$. In particular $c' \mathop{\smile\!\!\!\!\!\!|}_A ab'$, $c'a \equiv_A ca$ and $c'b' \equiv_A c'b \equiv_A cb$, as desired.

For the general case, assume that $d_A(b, b') \leq n$, namely that there are $b_0, ..., b_n$ be such that $b_i b_{i+1}$ start an A-indiscernible sequence for all $i < n$ and $b_0 = b$, $b_n = b'$. We may assume that $a \mathop{\smile\!\!\!\!\!\!|}_A b_0 ... b_n$.

By induction on $i \leq n$ we choose c_i such that:

(i) $c_i \mathop{\smile\!\!\!\!\!\!|}_A ab_i$,
(ii) $c_i a \equiv_A ca$,
(iii) $c_i b_i \equiv_A cb_0$.

Let $c_0 = c$, it satisfies (1)–(3) by hypothesis. Given c_i, by the Lascar distance 1 case there is some $c_{i+1} \mathop{\smile\!\!\!\!\!\!|}_A ab_{i+1}$ such that $c_{i+1} a \equiv_A c_i a \equiv_A ca$ and $c_{i+1} b_{i+1} \equiv_A c_i b_i \equiv_A cb_0$ (by the inductive assumption).

It follows that $c' = c_n$ is as wanted. $\blacksquare_{3.3}$

Remark 3.4. For simplicity of notation, let us work over $A = \varnothing$.

(i) It is easy to see that the usual statement of the independence theorem for simple theories implies this one. Indeed, let c_1 be such that $c_1 b' \equiv^{\mathrm{L}} cb$. Then $c_1 \mathop{\smile\!\!\!\!\!\!|} b'$, $c \mathop{\smile\!\!\!\!\!\!|} a$, $a \mathop{\smile\!\!\!\!\!\!|} b'$ and $c_1 \equiv^{\mathrm{L}} c$. By the independence theorem we find c' such that $c' \mathop{\smile\!\!\!\!\!\!|} ab'$, $c'a \equiv ca$ and $c'b' \equiv c_1 b' \equiv cb$.

(ii) Conversely, in a simple theory, the usual independence theorem follows from ours by a direct forking calculus argument. Indeed, assume that we are given $d_1 \mathop{\smile\!\!\!\!\!\!|} e_1$, $d_2 \mathop{\smile\!\!\!\!\!\!|} e_2$, $d_1 \equiv^{\mathrm{L}} d_2$ and $e_1 \mathop{\smile\!\!\!\!\!\!|} e_2$. Using symmetry and Lemma 3.10 we find $e_1' d_2'$ such that $e_1' d_2' \mathop{\smile\!\!\!\!\!\!|} e_1 e_2$ and $e_1' d_2' \equiv^{\mathrm{L}} e_1 d_1$. It is easy to check that all the assumptions of Theorem 3.3 are satisfied with $c = d_2'$, $b = e_1'$, $a = e_2$ and $b' = e_1$. Applying it we find some d such that $d \mathop{\smile\!\!\!\!\!\!|} e_1 e_2$, $de_1 \equiv d_2' e_1' \equiv d_1 e_1$ and $de_2 \equiv d_2 e_2$.

We observe that the chain condition means precisely that the ideal of forking formulas is S1, in the terminology of Hrushovski [Hru12]. Combining Proposition 2.7 with [Hru12, Theorem 2.18] we can slightly relax the assumption on the independence between the elements, at the price of assuming that some type has a global invariant extension:

Proposition 3.5. *Let T be* NTP$_2$ *and A an extension base. Assume that $c \mathop{\smile\!\!\!\!\!\!|}_A ab$, $b \mathop{\smile\!\!\!\!\!\!|}_A a$, $b' \mathop{\smile\!\!\!\!\!\!|}_A a$, $b \equiv_A b'$ and $\mathrm{tp}(a/A)$ extends to a global A-invariant type. Then there exists $c' \mathop{\smile\!\!\!\!\!\!|}_A ab'$ and $c'b' \equiv_A cb$, $c'a \equiv_A ca$.*

Using Theorem 3.3, we can show that in NTP$_2$ theories Lascar types coincide with Kim-Pillay strong types over extension bases.

Corollary 3.6. *Assume that T is* NTP$_2$ *and A is an extension base. Then $d \equiv^{\mathrm{L}}_A e$ if and only if $d_A(d, e) \leq 3$.*

Proof. Let $d \equiv^{\mathrm{L}}_A e$ and let $(d_i)_{i\in\omega}$ be a Morley sequence over A starting with $d = d_0$. As $d_{\geq 1} \mathop{\smile\!\!\!\!\!\!|}_A d_0$, we may assume that $d_{\geq 1} \mathop{\smile\!\!\!\!\!\!|}_A d_0 e$.

We have:

- $d_{>1} \mathop{\smile\!\!\!\!\!\!|}_A d_0 d_1$
- $d_1 \mathop{\smile\!\!\!\!\!\!|}_A d_0 e$
- $d_0 \equiv^{\mathrm{L}}_A e$

Applying Theorem 3.3 (with $a = d_1$, $b = d_0$, $b' = e$ and $c = d_{>1}$) we get some $d'_{>1}$ such that $d_1 d'_{>1} \equiv_A d_1 d_{>1}$ (thus $d_1 + d'_{>1}$ is an A-indiscernible sequence) and $e d'_{>1} \equiv_A d_0 d_{>1}$ (thus $e + d'_{>1}$ is an A-indiscernible sequence). It follows that $d_A(d,e) \leq 3$ along the sequence d, d_1, d'_2, e. $\blacksquare_{3.6}$

Remark 3.7. Consider the standard example [CLPZ01, Section 4] showing that the Lascar distance can be exactly n for any $n \in \omega$. It is easy to see that this theory is NIP, as it is interpretable in the real closed field. However, $\varnothing$ is not an extension base.

It is known that both in simple theories (for arbitrary A) and in NIP theories (for A an extension base), $a \equiv_A b$ implies that $d_A(a,b) \leq 2$ ([HP11, Corollary 2.10(i)]), while our argument only gives an upper bound of 3. Thus it is natural to ask:

Question 3.8. Is there an NTP_2 theory T, an extension base A and tuples a,b such that $d_A(a,b) = 3$?

Definition 3.9. Let $a \equiv'_A b$ be the transitive closure of the relation "a,b start a Morley sequence over A, or b,a starts a Morley sequence over A". This is an A-invariant equivalence relation refining $\equiv^{\mathrm{L}}_A$.

The proof of Corollary 3.6 demonstrates in particular that if A is an extension base in an NTP_2 theory, then $a \equiv^{\mathrm{L}}_A b$ if and only if $a \equiv'_A b$. We show that in fact this holds in a much more general setting.

Let T be an arbitrary theory. We call a type $p(x) \in S(A)$ *extensible* if it has a global extension non-forking over A, equivalently if it does not fork over A (thus A is an extension base if and only if every type over it is extensible).

Lemma 3.10. *Let* $\mathrm{tp}(a/A)$ *be extensible. Then for any* b *there is some* a' *such that* $a' \equiv'_A a$ *and* $a' \mathop{\smile\!\!\!\!\!\!|}_A b$.

Proof. Let $(a_i)_{i\in\omega}$ be a Morley sequence over A starting with a_0. It follows that $a_{\geq 1} \mathop{\smile\!\!\!\!\!\!|}_A a_0$. Then there is $a'_{\geq 1} \mathop{\smile\!\!\!\!\!\!|}_A a_0 b$ and such that $a_{\geq 1} \equiv_{a_0 A} a'_{\geq 1}$. In particular $a_0 + a'_{\geq 1}$ is still a Morley sequence over A, thus $a'_1 \equiv'_A a_0$, and $a'_1 \mathop{\smile\!\!\!\!\!\!|}_A b$ as wanted. $\blacksquare_{3.10}$

Proposition 3.11. *Let* p *be an extensible type. Then* $a \equiv^{\mathrm{L}}_A b$ *if and only if* $a \equiv'_A b$, *for any* $a, b \vDash p(x)$.

Proof. By Definition 3.1(1) it is enough to show that $\equiv'_A$ has boundedly many classes on the set of realizations of p.

Assume not, and let κ be large enough. We will choose $\equiv'$-inequivalent $(a_i)_{i\in\kappa}$ such that in addition $a_i \mathop{\smile\!\!\!\!\!\!|}_A a_{<i}$. Suppose we have chosen $a_{<j}$ and let us choose a_j. Let $b \vDash p$ be $\equiv'_A$-inequivalent to a_i for all $i < j$. By Lemma 3.10, there exists $a_j \equiv'_A b$ such that $a_j \mathop{\smile\!\!\!\!\!\!|}_A a_{<j}$. In particular $a_j \not\equiv'_A a_i$ for all $i < j$ as desired.

With κ sufficiently large, we may extract an A-indiscernible sequence $\bar{b} = (b_i)_{i\in\omega}$ from $(a_i)_{i\in\kappa}$ — a contradiction, as then $\bar{b}$ is a Morley sequence over A but $b_i \not\equiv'_A b_j$ for any $i \neq j$. $\blacksquare_{3.11}$

4. The dividing order

In this section we suggest a generalization of the fundamental order of Poizat [Poi85] in the context of NTP_2 theories. For simplicity of notation, we only consider 1-types, but everything we do holds for n-types just as well.

Given a partial type $r(x)$ over A, we let $S^{\mathrm{EM},r}(A)$ be the set of Ehrenfeucht-Mostowski types of A-indiscernible sequences in $r(x)$. We will omit A when $A = \varnothing$ and omit r when it is "$x = x$".

Definition 4.1. Given $p \in S^{\mathrm{EM}}(A)$, let $\mathrm{cl}^{\mathrm{div}}(p)$ be the set of all $\varphi(x,y) \in L(A)$ such that for some (any) infinite A-indiscernible sequences $\bar{a} \vDash p$, the set $\{\varphi(a_i,y)\}_{i\in\omega}$ is consistent. For $p,q \in S^{\mathrm{EM}}(A)$, we say that $p \sim^{\mathrm{div}}_A q$ (respectively, $p \leq^{\mathrm{div}}_A q$) if $\mathrm{cl}^{\mathrm{div}}(p) = \mathrm{cl}^{\mathrm{div}}(q)$ (respectively, $\mathrm{cl}^{\mathrm{div}}(p) \supseteq \mathrm{cl}^{\mathrm{div}}(q)$). We obtain a partial order $\left(S^{\mathrm{EM}}(A)/\sim^{\mathrm{div}}_A, \leq^{\mathrm{div}}_A\right)$.

Proposition 4.2. *Let T be stable. Then $p \sim^{\mathrm{div}} q$ if and only if $p = q$, and $\left(S^{\mathrm{EM}}, \leq^{\mathrm{div}}\right)$ is isomorphic to the fundamental order of T.*

Proof. For a type p over a model M we let $\mathrm{cl}(p)$ denote its fundamental class, namely the set of formulas $\varphi(x,y)$ such that there exists an instance $\varphi(x,b) \in p(x)$. We denote the fundamental order of T by $\left(S/\sim^{\mathrm{fund}}, \leq^{\mathrm{fund}}\right)$ where S is the set of all types over all models of T, $p \leq^{\mathrm{fund}} q$ if $\mathrm{cl}(p) \supseteq \mathrm{cl}(q)$ and $\sim^{\mathrm{fund}}$ is the corresponding equivalence relation. Given $p \in S(M)$, let $p^{(\omega)} \in S_\omega(M)$ be the type of its Morley sequence over M. By stability $p^{(\omega)}$ is determined by p. Let p^{EM} be the Ehrenfeucht-Mostowski type over the empty set of $\bar{a} \vDash p^{(\omega)}|_M$. Let $f : S \to S^{\mathrm{EM}}$, $f : p \mapsto p^{\mathrm{EM}}$.

(i) Given $p \in S(M)$, let $\bar{a} \vDash p^{(\omega)}$, and let us show that $\varphi(x,y) \in \mathrm{cl}(p)$ if and only if $\{\varphi(a_i, y)\}_{i\in\omega}$ is consistent. Indeed, by stability, either condition is equivalent to:$\varphi(a_0, y)$ does not divide over M. In other words, $\mathrm{cl}(p) = \mathrm{cl}^{\mathrm{div}}(f(p))$, so $p \leq^{\mathrm{fund}} q \Leftrightarrow f(p) \leq^{\mathrm{div}} f(q)$.

(ii) We show that f is onto. Let $P \in S^{\mathrm{EM}}$ be arbitrary, and let $(a_i)_{i\in 2\omega}$ be an indiscernible sequence with P as its EM type. Let M be a model containing $I = (a_i)_{i\in\omega}$, such that $J = (a_{\omega+i})_{i\in\omega}$ is indiscernible over M. Then J is a Morley sequence in $p(x) = \mathrm{tp}(a_\omega/M)$, and $f(p) = P$, as wanted.

(iii) To conclude, let $P, Q \in S^{\mathrm{EM}}$, $P \sim^{\mathrm{div}} Q$, and let us show that they are equal. Let $p \in S(M)$ and $q \in S(N)$ be sent by f to P and Q, respectively. Since $Th(M) \subseteq \mathrm{cl}^{\mathrm{div}}(P)$ and similarly for N, Q, we have $M \equiv N$. Taking non-forking extensions of p, q, we may therefore assume that $M = N$ is a monster model. Since $\mathrm{cl}(p) = \mathrm{cl}(q)$, the types of (the parameters of) their definitions are the same, so there exists an automorphism sending one definition to the other, and therefore sending $p \mapsto q$. Since $f(p)$ does not involve any parameters, it follows that $P = f(p) = f(q) = Q$.

$\blacksquare_{4.2}$

Remark 4.3. A couple of remarks on the existence of the greatest element in the dividing order in NTP_2 theories.

(i) Given a type $r(x_1, x_2) \in S(A)$, assume that $p\left((x_{1j}, x_{2j})_{j\in\omega}\right)$ is the greatest element in $S^{\mathrm{EM},r}(A)$ (modulo $\sim^{\mathrm{div}}_A$). Then for $i = 1, 2$, $p_i\left((x_{ij})_{j\in\omega}\right) = p|_{(x_{ij})_{j\in\omega}}$ is the greatest element in $S^{\mathrm{EM},r_i}(A)$ with $r_i = r|_{x_i}$.

(ii) If for every $r \in S(A)$ there is a $\leq^{\mathrm{div}}$-greatest element in $S^{\mathrm{EM},r}(A)$, then a formula $\varphi(x,a)$ forks over A if and only if it divides over A.

(iii) If T is NTP_2 then for every extension base A and $r \in S(A)$ there is a $\leq^{\mathrm{div}}$-greatest element in $S^{\mathrm{EM},r}(A)$.

Proof. (i) Clear as e.g. given an A-indiscernible sequence $(a_{1j})_{j\in\omega}$ in $r_1(x_1)$, by compactness and Ramsey we can find $(a_{2j})_{j\in\omega}$ such that $(a_{1j}a_{2j})_{j\in\omega}$ is an A-indiscernible sequence in $r(x_1, x_2)$.

(ii) Assume that $\varphi(x,a) \vdash \bigvee_{i<k} \varphi_i(x, a_i)$ and $\varphi_i(x, a_i)$ divides over A for each $i < k$. Let $r(xx_0 \dots x_{k-1}) = \mathrm{tp}(aa_0 \dots a_{k-1}/A)$, let $p(\bar{x}\bar{x}_0 \dots \bar{x}_{k-1})$ be the greatest element in $S^{\mathrm{EM},r}(A)$ and let $\left(a_j a_{0j} \dots a_{(k-1)j}\right)_{j\in\omega}$ realize it. As $\{\varphi(x, a_j)\}_{j\in\omega}$ is consistent, it follows that $\{\varphi_i(x, a_{ij})\}_{j\in\omega}$ is consistent for some $i < k$ — contradicting the assumption that $\varphi_i(x, a_i)$ divides by (i).

(iii) Let $a \vDash r$. As A is an extension base, let $M \supseteq A$ be a model such that $M \mathop{\smile\!\!\!\!\!\!|}_A a$. Let $I = (a_i)_{i\in\omega}$ be a universal Morley sequence in $\mathrm{tp}(a/M)$ which exists by Fact 2.6. Then $\mathrm{tp}(I/A)$ is the greatest element in $S^{\mathrm{EM},r}(A)$. Indeed, $\varphi(x,a)$ divides over $A \Leftrightarrow \varphi(x,a)$ divides over $M \Leftrightarrow \{\varphi(x, a_i)\}_{i\in\omega}$ is inconsistent.

$\blacksquare_{4.3}$

Definition 4.4. For $p, q \in S^{\mathrm{EM}}$, we write $p \leq^{\#} q$ if there is an array $(a_{ij})_{i,j\in\omega}$ such that:

- $(a_{ij})_{j\in\omega} \vDash p$ for each $i \in \omega$,
- $\left(a_{if(i)}\right)_{i\in\omega} \vDash q$ for each $f : \omega \to \omega$.

Proposition 4.5. *Let $p, q \in S^{\mathrm{EM}}$.*

(i) *If $p \leq^{\mathrm{div}} q$, then $p \leq^{\#} q$.*

(ii) *If T is NTP_2 and $p \leq^{\#} q$, then $p \leq^{\mathrm{div}} q$.*

Proof. (i) We show by induction that for each $n \in \omega$ we can find $(\bar{a}_i)_{i\in n}$ and $\bar{b}$ such that: $\bar{a}_i \models p$ and $a_{0j_0} + ... + a_{(n-1)j_{n-1}} + \bar{b} \models q$ for any $j_0, \ldots, j_{n-1} \in \omega$. Assume we have found $(\bar{a}_i)_{i<n}$ and $\bar{b}$, without loss of generality $\bar{b} = \bar{b}' + \bar{b}'' = (b'_i)_{i\in\omega} + (b''_i)_{i\in\omega}$. Consider the type

$$\begin{array}{rcl} r(\bar{x}_0...\bar{x}_{n-1}, y, \bar{z}) & = & \bigcup_{i<n} p(\bar{x}_i) \cup q(\bar{z}) \cup \\ \cup \quad \bigcup_{j_0,\ldots,j_{n-1}\in\omega} & & \text{"}x_{0j_0} + x_{1j_1} + ... + x_{(n-1)j_{n-1}} + y + \bar{z} \text{ is indiscernible"} \end{array}$$

For every finite $r' \subset r$, $\{r'(\bar{x}_0...\bar{x}_{n-1}, y_i, \bar{z})\}_{i\in\omega} \cup q(\bar{y})$ is consistent — since by the inductive assumption $\models r'(\bar{a}_0...\bar{a}_{n-1}, b'_i, \bar{b}'')$ for all $i \in \omega$. Together with $p \leq^{\text{div}} q$ this implies that $\{r'(\bar{x}_0...\bar{x}_{n-1}, y_i, \bar{z})\}_{i\in\omega} \cup p(\bar{y})$ is consistent. By compactness we find $\bar{a}_0, ..., \bar{a}_{n-1}, \bar{a}_n, \bar{b}$ realizing it, and they are what we were looking for.

(ii) Follows from the definition of TP_2.

$\blacksquare_{4.5}$

Definition 4.6. We write $p \leq^+ q$[1] if there is $\bar{a} = (a_i)_{i\in\mathbb{Z}} \models q$ and $\bar{b} = (b_i)_{i\in\mathbb{Z}} \models p$ such that $a_0 = b_0$ and $\bar{b}$ is indiscernible over $(a_i)_{i\neq 0}$.

Remark 4.7. In any theory, $p \leq^{\#} q$ implies $p \leq^+ q$ (and so $p \leq^{\text{div}} q$ implies $p \leq^+ q$).

Proof. If $p \leq^{\#} q$, then by compactness and Ramsey we can find an array $(c_{ij})_{i,j\in\mathbb{Z}}$ such that:

- $\bar{c}_i$ is indiscernible over $\bar{c}_{\neq i}$,
- $(\bar{c}_i)_{i\in\mathbb{Z}}$ is an indiscernible sequence,
- $\bar{c}_i \models p$ for all $i \in \omega$,
- $\left(c_{if(i)}\right)_{i\in\omega} \models q$ for all $f : \omega \to \omega$.

Then take $\bar{a} = (c_{0j})_{j\in\mathbb{Z}}$ and $\bar{b} = (c_{i0})_{i\in\mathbb{Z}}$. $\blacksquare_{4.7}$

It is much less clear, however, if the converse implication holds.

Definition 4.8. We say that T is *resilient*[2] if we cannot find indiscernible sequences $\bar{a} = (a_i)_{i\in\mathbb{Z}}$, $\bar{b} = (b_j)_{i\in\mathbb{Z}}$ and a formula $\varphi(x, y)$ such that:

- $a_0 = b_0$,
- $\bar{b}$ is indiscernible over $(a_i)_{i\neq 0}$,
- $\{\varphi(x, a_i)\}_{i\in\mathbb{Z}}$ is consistent,
- $\{\varphi(x, b_j)\}_{j\in\mathbb{Z}}$ is inconsistent.

Remark 4.9. (i) It follows by compactness that we get an equivalent definition replacing $\mathbb{Z}$ by $\mathbb{Q}$ for either of i or j (or both), and replacing $\mathbb{Z}$ by ω for j.

(ii) If T is resilient and A is a set of constants, then $T(A)$ is resilient.

Lemma 4.10. *The following are equivalent:*

(i) *T is resilient.*
(ii) *For every $p, q \in S^{\text{EM}}$, $p \leq^+ q$ implies $p \leq^{\text{div}} q$.*
(iii) *For any indiscernible sequence $\bar{a} = (a_i)_{i\in\mathbb{Z}}$ and $\varphi(x, y) \in L$, if $\varphi(x, a_0)$ divides over $(a_i)_{i\neq 0}$, then $\{\varphi(x, a_i)\}_{i\in\mathbb{Z}}$ is inconsistent.*
(iv) *There is no array $(a_{ij})_{i,j\in\omega}$, $\varphi(x, y) \in L$ and $k \in \omega$ such that $\{\varphi(x, a_{i0})\}_{i\in\omega}$ is consistent, $\{\varphi(x, a_{ij})\}_{j\in\omega}$ is k-inconsistent for each $i \in \omega$ and $\bar{a}_i = (a_{ij})_{j\in\omega}$ is indiscernible over $(a_{j0})_{j\neq i}$ for each $i \in \omega$.*

---

[1]Note that "#" and "+" are supposed to graphically represent the combinatorial configuration which we are using in the definition of the order.

[2]The term was suggested by Hans Adler as a replacement for "NTP_2" but we prefered to use it for a (possibly) smaller class of theories.

Proof. (i) is equivalent to (ii) Assume that $p \leq^+ q$, i.e. there is $\bar{a} = (a_i)_{i \in \mathbb{Z}} \models q$ and $\bar{b} = (b_i)_{i \in \mathbb{Z}} \models p$ such that $a_0 = b_0$ and $\bar{b}$ is indiscernible over $(a_i)_{i \neq 0}$. For any $\varphi(x, y)$, if $\{\varphi(x, b_i)\}_{i \in \omega}$ is inconsistent, then $\{\varphi(x, a_i)\}_{i \in \omega}$ is inconsistent by resilience, which means precisely that $p \leq^{\mathrm{div}} q$. The converse is clear.

(i) is equivalent to (iii) If $\varphi(x, a_0)$ divides over $a_{\neq 0}$, then there is a sequence $(b_i)_{i \in \mathbb{Z}}$ indiscernible over $a_{\neq 0}$ and such that $b_0 = a_0$ and $\{\varphi(x, b_i)\}_{i \in \mathbb{Z}}$ is inconsistent. It follows by resilience that $\{\varphi(x, a_i)\}_{i \in \mathbb{Z}}$ is inconsistent. On the other hand, assume that $\{\varphi(x, a_i)\}_{i \in \mathbb{Z}}$ is inconsistent. By compactness we can extend our indiscernible sequence to $\bar{a}' + \bar{a} + \bar{a}'' = (a'_i)_{i \in \omega^*} + (a_i)_{i \in \mathbb{Z}} + (a''_i)_{i \in \omega}$. But then $\bar{a}$ witnesses that $\varphi(x, a_0)$ divides over $\bar{a}'\bar{a}''$. Sending $\bar{a}'$ to $a_{\leq -1}$ and $\bar{a}''$ to $a_{\geq 1}$ by an automorphism fixing a_0 we conclude that $\varphi(x, a_0)$ divides over $a_{\neq 0}$.

(i) is equivalent to (iv) Let $\bar{a}$, $\bar{b}$ and $\varphi(x, y)$ witness that T is not resilient. Then we let $\bar{a}_0 = \bar{b}$ and we let $\bar{a}_i$ be an image of $\bar{b}$ under some automorphism sending $(\dots, a_{-1}, a_0, a_1, \dots)$ to $(\dots, a_{i-1}, a_i, a_{i+1}, \dots)$ by indiscernibility. It follows that $(a_{ij})_{i,j \in \omega}$ is an array as wanted.

Conversely, if we have an array as in (iv), by compactness we may assume that it is of the form $(a_{ij})_{i \in \mathbb{Z}, j \in \omega}$ and that in addition $(a_{i0})_{i \in \mathbb{Z}}$ is indiscernible. Then $\bar{a} = (a_{i0})_{i \in \mathbb{Z}}$, $\bar{b} = (a_{0j})_{j \in \omega}$ and $\varphi(x, y)$ contradict resilience (in view of Remark 4.9).

$\blacksquare_{4.10}$

Proposition 4.11. (i) *If T is* NIP*, then it is resilient.*
(ii) *If T is simple, then it is resilient.*
(iii) *If T is resilient, then it is* NTP_2*.*

Proof. (i) Fix $\varphi(x, y)$ and assume that $\{\varphi(x, a_i)\}_{i \in \mathbb{Q}}$ is consistent. Then by NIP there is a maximal $k \in \omega$ such that $\{\neg\varphi(x, a_i)\}_{i \in s} \cup \{\varphi(x, a_i)\}_{i \notin s}$ is consistent, for $s = \{1, 2, ..., k\} \subseteq \mathbb{Q}$. Let d realize it. If $\{\varphi(x, b_i)\}_{i \in \mathbb{Q}}$ was inconsistent, then we would have $\neg\varphi(d, b_i)$ for some $i \in \mathbb{Q}$, and thus $\{\neg\varphi(x, a_i)\}_{i \in s \cup \{k+1\}} \cup \{\varphi(x, a_i)\}_{i \notin s \cup \{k+1\}}$ would be consistent, by all the indiscernibility around — a contradiction to the maximality of k. Thus, $\{\varphi(x, b_i)\}_{i \in \mathbb{Q}}$ is consistent.
(ii) It is easy to see that $(a_i)_{i > 0}$ is a Morley sequence over $A = (a_i)_{i < 0}$ by finite satisfiability. If $\varphi(x, a_0)$ divides over $a_{\neq 0}$, then by Kim's lemma $\{\varphi(x, a_i)\}_{i \in \mathbb{Z}}$ is inconsistent.
(iii) By Erdős-Rado and compactness we can find a strongly indiscernible array $(c_{ij})_{i,j \in \mathbb{Z}}$ witnessing TP_2 for $\varphi(x, y)$. Set $a_i = c_{i0}$ for $i \in \omega$ and $b_j = b_{0j}$ for $j \in \omega$. Then $\bar{a}$, $\bar{b}$ and $\varphi(x, y)$ witness that T is not resilient.

$\blacksquare_{4.11}$

Claim. *Let T be resilient, A an extension base, and let $\bar{a} = (a_i)_{i \in \mathbb{Z}}$ be indiscernible over A, say in and $r = \mathrm{tp}(a_0/A) \in S(A)$. Then the following are equivalent:*
(i) *The EM type $\mathrm{tp}^{\mathrm{EM}}(\bar{a}/A)$ is $\leq^{\mathrm{div}}_A$-greatest in $S^{\mathrm{EM},r}(A)$.*
(ii) *$\mathrm{tp}(a_{\neq 0}/a_0 A)$ does not divide over A.*

Proof. We may assume that $A = \varnothing$.

(i) implies (ii) in any theory: Let $\models \varphi(a_{\neq 0}, a_0)$. By indiscernibility and compactness $\{\varphi(x, a_i)\}_{i \in \mathbb{Z}}$ is consistent, so by (i) $\varphi(x, a_0)$ does not divide.

(ii) implies (i): Assume that $\varphi(x, a_0)$ divides. As $\mathrm{tp}(a_{\neq 0}/a_0)$ does not divide, it follows that $\varphi(x, a_0)$ divides over $a_{\neq 0}$. But then by Lemma 4.10(iii) we have that $\{\varphi(x, a_i)\}_{i \in \mathbb{Z}}$ is inconsistent, hence (i). $\blacksquare_{4.11}$

Remark 4.12. Similar observation in the context of NIP theories based on [She09] is made in [KU].

Recall that a theory is called *low* if for every formula $\varphi(x, y)$ there is $k \in \omega$ such that for any indiscernible sequence $(a_i)_{i \in \omega}$, $\{\varphi(x, a_i)\}_{i \in \omega}$ is consistent if and only if it is k-consistent. The following is a generalization of [BPV03, Lemma 2.3].

Proposition 4.13. *Let T be resilient. Then the following are equivalent:*
(i) *$\varphi(x, y)$ is low.*
(ii) *The set $\{(c, d) : \varphi(x, c)$ divides over $d\}$ is type-definable (where d is allowed to be of infinite length).*

Proof. (i) implies (ii) holds in any theory, and we show that (ii) implies (i).

Assume that $\varphi(x,y)$ is not low. Then for every $i \in \omega$ we have a sequence $\bar{a}_i = (a_{ij})_{j\in\mathbb{Z}}$ such that $\{\varphi(x,a_{ij})\}_{j\in\mathbb{Z}}$ is i-consistent, but inconsistent. In particular $\varphi(x,a_{i0})$ divides over $(a_{ij})_{j\neq 0}$ for each i.

If (ii) holds, then by compactness we can find a sequence $\bar{a} = (a_j)_{j\in\omega}$ such that $\{\varphi(x,a_j)\}_{j\in\omega}$ is consistent and $\varphi(x,a_0)$ still divides over $a_{\neq 0}$. But this is a contradiction to resilience by Lemma 4.10(iii). $\blacksquare_{4.13}$

However, the main question remains unresolved:

Question 4.14. (i) Does NTP_2 imply resilience?
(ii) Is resilience preserved under reducts?
(iii) Does type-definability of dividing imply lowness in NTP_2 theories?

5. On a strengthening of strong theories

Recently several attempts have been made to define weight outside of the familiar context of simple theories. First Shelah had defined strongly dependent theories and several notions of dp-rank in [She09, She]. The study of dp-rank was continued in [OU11]. After that Adler [Adlc] had introduced *burden*, a notion based on the invariant κ_{inp} of Shelah [She90] which generalizes simultaneously dp-rank in NIP theories and weight in simple theories. In this section we are going to add yet another version of measuring weight. First we recall the notions mentioned above.

For notational convenience we consider an extension Card^* of the linear order on cardinals by adding a new maximal element ∞ and replacing every limit cardinal κ by two new elements κ_- and κ_+. The standard embedding of cardinals into Card^* identifies κ with κ_+. In the following, whenever we take a supremum of a set of cardinals, we will be computing it in Card^*.

Definition 5.1. [Adlc] Let $p(x)$ be a (partial) type.

(i) An inp-pattern of depth κ in $p(x)$ consists of $(\bar{a}_i, \varphi_i(x,y_i), k_i)_{i\in\kappa}$ with $\bar{a}_i = (a_{ij})_{j\in\omega}$ and $k_i \in \omega$ such that:
   - $\{\varphi_i(x,a_{ij})\}_{j\in\omega}$ is k_i-inconsistent for every $i \in \kappa$,
   - $p(x) \cup \{\varphi_i(x,a_{if(i)})\}_{i\in\kappa}$ is consistent for every $f : \kappa \to \omega$.

(ii) The *burden* of a partial type $p(x)$ is the supremum (in Card^*) of the depths of inp-patterns in it. We denote the burden of p as $\mathrm{bdn}(p)$ and we write $\mathrm{bdn}(a/A)$ for $\mathrm{bdn}(\mathrm{tp}(a/A))$.

(iii) We get an equivalent definition by taking supremum only over inp-patterns with mutually indiscernible rows.

(iv) It is easy to see by compactness that T is NTP_2 if and only if $\mathrm{bdn}("x=x") < \infty$, if and only if $\mathrm{bdn}("x=x") < |T|^+$.

(v) A theory T is called *strong* if $\mathrm{bdn}(p) \leq (\aleph_0)_-$ for every finitary type p (equivalently, there is no inp-pattern of infinite depth). Of course, if T is strong then it is NTP_2.

Fact 5.2. [Adlc]

(i) *Let T be* NIP*. Then* $\mathrm{bdn}(p) = \text{dp-rk}(p)$ *for any p.*

(ii) *Let T be simple. Then the burden of p is the supremum of weights of its complete extensions.*

Some basics of the theory of burden were developed by the second author in [Che].

Fact 5.3. [Che] *Let T be an arbitrary theory.*

(i) *The following are equivalent:*
   (a) $\mathrm{bdn}(p) < \kappa$.
   (b) *For any $(\bar{a}_i)_{i\in\kappa}$ mutually indiscernible over A and $b \models p$, there is some $i \in \kappa$ and $\bar{a}'_i$ such that $\bar{a}'_i$ is indiscernible over bA and $\bar{a}'_i \equiv_{Aa_{i0}} \bar{a}_i$.*

(ii) *Assume that* $\mathrm{bdn}(a/A) < \kappa$ *and* $\mathrm{bdn}(b/aA) < \lambda$*, with κ and λ finite or infinite cardinals. Then* $\mathrm{bdn}(ab/A) < \kappa \times \lambda$.

(iii) *In particular, in the definition of strong (or* NTP_2*) it is enough to look at types in one variable.*

In [KOU] it is proved that dp-rank is sub-additive, so burden in NIP theories is sub-additive as well. The sub-additivity of burden in simple theories follows from Fact 5.2 and the sub-additivity of weight in simple theories. It thus becomes natural to wonder if burden is sub-additive in general, or at least in NTP_2 theories.

Now we are going to define a refinement of the class of strong theories.

Definition 5.4. Let $p(x)$ be a partial type.

(i) An inp^2-pattern of depth κ in $p(x)$ consists of formulas $(\varphi_i(x, y_i, z_i))_{i\in\kappa}$, mutually indiscernible sequences $(\bar{a}_i)_{i\in\kappa}$ and $b_i \subseteq \bigcup_{j<i} \bar{a}_j$ such that:
  (a) $\{\varphi_i(x, a_{i0}, b_i)\}_{i\in\omega} \cup p(x)$ is consistent,
  (b) $\{\varphi_i(x, a_{ij}, b_i)\}_{j\in\omega}$ is inconsistent for every $i \in \omega$.

(ii) An inp^3-pattern of depth κ in $p(x)$ is defined exactly as an inp^2-pattern of depth κ, but allowing $b_i \subseteq \bigcup_{j\in\kappa, j\neq i} \bar{a}_j$. It is then clear that every inp^2-pattern is an inp^3-pattern of the same depth, but the opposite is not true.

(iii) The *burden*^{2} (*burden*^{3}) of a partial type $p(x)$ is the supremum (in Card^*) of the depths of inp^2-patterns (resp. inp^3-patterns) in it. We denote the burden^2 of p as $\mathrm{bdn}^2(p)$ and we write $\mathrm{bdn}^2(a/A)$ for $\mathrm{bdn}^2(\mathrm{tp}(a/A))$ (and similarly for bdn^3).

(iv) A theory T is called *strong*^{2} if $\mathrm{bdn}^2(p) \leq (\aleph_0)_-$ for every finitary type p (that is, there is no inp^2-pattern of infinite depth). Similarly for *strong*^{3}.

In the following proposition we sum up some of the properties of bdn^2 and bdn^3.

Proposition 5.5. (i) *For any partial type $p(x)$, $\mathrm{bdn}(p) \leq \mathrm{bdn}^2(p) \leq \mathrm{bdn}^3(p)$.*

(ii) *Strong^3 implies strong^2 implies strong.*

(iii) *In fact, T is strong^2 if and only if it is strong^3.*

(iv) *T is strongly^2 dependent if and only if it is NIP and strong^2 (we recall from* [KS12, Definition 2.2] *that T is called strongly^2 dependent when there are no $\left(\varphi_i(x, y_i, z_i), \bar{a}_i = (a_{ij})_{j\in\omega}, b_i \subseteq \bigcup_{j<i} \bar{a}_j\right)_{i\in\omega}$ such that $(\bar{a}_i)_{i\in\omega}$ are mutually indiscernible and the set $\{\varphi_i(x, a_{i0}, b_i) \wedge \neg\varphi_i(x, a_{i1}, b_i)\}_{i\in\omega}$ is consistent.).*

(v) *If T is supersimple, then it is strong^2.*

(vi) *There are strong^2 stable theories which are not superstable.*

(vii) *There are strong stable theories which are not strong^2.*

(viii) *We still have that T is NTP_2 if and only if every finitary type has bounded burden^3.*

Proof. (i) is immediate by comparing the definitions, and (ii) follows from (i).

(iii) Assume that T is not strong^3, witnessed by $(\varphi_i(x, y_i, z_i), \bar{a}_i, b_i)_{i\in\omega}$. For $i \in \omega$, let $f(i)$ be the smallest $j \in \omega$ such that $b_i \in \bar{a}_{<j}$. Now for $i \in \omega$ we define inductively:

- $\alpha_0 = 0$, $\alpha_{i+1} = f(\alpha_i)$,
- $b_i' = b_{\alpha_i} \cap \bar{a}_{\in\{\alpha_0,\alpha_1,\ldots,\alpha_{i-1}\}}$ and $b_i'' = b_{\alpha_i} \cap \bar{a}_{\in\{0,1,\ldots,\alpha_{i+1}-1\}\setminus\{\alpha_0,\alpha_1,\ldots,\alpha_i\}}$, so we may assume that $b_{\alpha_i} = b_i'' b_i'$.
- $a_{ij}' = a_{\alpha_i j} b_i''$ for $j \in \omega$,
- $\varphi_i'(x, a_{ij}', b_i') = \varphi_i(x, a_{ij}, b_i)$.

It is now easy to check that $(\bar{a}_i')_{i\in\omega}$ are mutually indiscernible, $b_i' \in \bar{a}_{<i}'$, $\{\varphi_i'(x, a_{i0}', b_i')\}_{i\in\omega}$ is consistent and $\{\varphi_i'(x, a_{ij}', b_i')\}_{j\in\omega}$ is inconsistent for every $i \in \omega$. This gives us an inp^2-pattern of infinite depth, witnessing that T is not strong^2.

(iv) Let $(\varphi_i(x, y_i, z_i), \bar{a}_i, b_i)_{i\in\omega}$ witness that T is not strong^2 and let $c \models \{\varphi_i(x, a_{i0}, b_i)\}_{i\in\omega}$, it follows from the inconsistency of $\{\varphi(x, a_{ij}, b_i)\}_{j\in\omega}$'s that for each $i \in \omega$ there is some $k_i \in \omega$ such that $c \models \{\varphi_i(x, a_{i0}, b_i) \wedge \neg\varphi_i(x, a_{ik_i}, b_i)\}_{i\in\omega}$. Define $a_{ij}' = a_{i,k_i\times j} a_{i,k_i\times j+1} \ldots a_{i,k_i\times(j+1)-1}$ and $\varphi'(x, a_{ij}', b_i) = \varphi(x, a_{i,k_i\times j}, b_i)$. Then $(\bar{a}_i')_{i\in\omega}$ are mutually indiscernible, $b_i \in \bigcup_{j<i} \bar{a}_j'$ and $c \models \{\varphi_i(x, a_{i0}', b_i) \wedge \neg\varphi_i(x, a_{i1}', b_i)\}_{i\in\omega}$ — witnessing that T is not strongly^2 dependent.

On the other hand, let $(\varphi_i(x, y_i, z_i), \bar{a}_i, b_i)_{i\in\omega}$ witness that T is not strongly^2 dependent and assume that T is NIP. Let $\varphi_i'(x, y_i', z_i) = \varphi_i(x, y_i^0, z_i) \wedge \neg\varphi_i(x, y_i^1, z_i)$, $a_{ij}' = a_{i(2j)} a_{i(2j+1)}$ for all $i, j \in \omega$. We then have that $(\bar{a}_i')_{i\in\omega}$ are still mutually indiscernible and $b_i \in \bigcup_{j<i} \bar{a}'$, $\{\varphi_i'(x, a_{i0}', b_i)\}_{i\in\omega}$ is consistent and $\{\varphi_i'(x, a_{ij}', b_i)\}_{j\in\omega}$ is inconsistent (otherwise let c realize it, it follows that $\varphi_i(c, a_{ij}, b_i)$ holds if and only if j is even, contradicting NIP). But this shows that T is not strong^2.

(v) Let T be supersimple, and assume that T is not strong^2, witnessed by $(\varphi_i(x, y_i, z_i), \bar{a}_i, b_i)_{i\in\omega}$ and let $A = \bigcup_{i,j\in\omega} a_{ij}$. Let $c \vDash \{\varphi_i(x, a_{i0}, b_i)\}_{i\in\omega}$. By supersimplicity, there has to be some finite $A_0 \subset A$ such that $\operatorname{tp}(c/A)$ does not divide over A_0. It follows that there is some $i' \in \omega$ such that $A_0 \subset \bigcup_{i<i', j\in\omega} a_{ij}$. But then $c \vDash \varphi_{i'}(x, a_{i'0}, b_{i'})$, $(a_{i'j}b_{i'})_{j\in\omega}$ is indiscernible over A_0 and $\{\varphi(x, a_{i'j}, b_{i'})\}_{j\in\omega}$ is inconsistent, so $\operatorname{tp}(c/A)$ divides over A_0 — a contradiction.

(vi) It is easy to see that the theory of an infinite family of refining equivalence relations with infinitely many infinite classes satisfies the requirement.

(vii) In [She, Example 2.5] Shelah gives an example of a strongly stable theory which is not strongly^2 stable. In view of (3) this is sufficient. Besides, there are examples of NIP theories of burden 1 which are not strongly^2 dependent (e.g. $(\mathbb{Q}_p, +, \cdot, 0, 1)$ or $(\mathbb{R}, <, +, \cdot, 0, 1)$).

(viii) We remind the statement of Fodor's lemma.

Fact (Fodor's lemma). If κ is a regular, uncountable cardinal and $f : \kappa \to \kappa$ is such that $f(\alpha) < \alpha$ for any $\alpha \neq 0$, then there is some γ and some stationary $S \subseteq \kappa$ such that $f(\alpha) = \gamma$ for any $\alpha \in S$.

If T has TP_2, then clearly $\operatorname{bdn}^3(T) = \infty$, and we prove the converse. Assume that $\operatorname{bdn}^3(T) \geq |T|^+$ and let $\kappa = |T|^+$. Then we can find $(\varphi_i(x, y_i, z_i), \bar{a}_i, b_i)_{i\in\kappa}$ with $(\bar{a}_i)_{i\in\kappa}$ mutually indiscernible, finite $b_i \in \bigcup_{j\in\kappa, j\neq i} \bar{a}_j$ such that $\{\varphi_i(x, a_{i0}, b_i)\}_{i\in\kappa}$ is consistent and $\{\varphi_i(x, a_{ij}, b_i)\}_{j\in\omega}$ is inconsistent for every $i \in \kappa$. For each $i \in \kappa$, let $f(i)$ be the largest $j < i$ such that $\bar{a}_j \cap b_i \neq \varnothing$ and let $g(i)$ be the largest $j \in \kappa$ such that $\bar{a}_j \cap b_i \neq \varnothing$. By Fodor's lemma there is some stationary $S \subseteq \kappa$ and $\gamma \in \kappa$ such that $f(i) = \gamma$ for all $i \in S$.

By induction we choose an increasing sequence $(i_\alpha)_{\alpha\in\kappa}$ from S such that $i_0 > \gamma$ and $i_\alpha > g(i_\beta)$ for $\beta < \alpha$. Now let $a'_{\alpha j} = a_{i_\alpha j} b_{i_\alpha}$ and $\varphi'_\alpha(x, y'_\alpha) = \varphi_{i_\alpha}(x, y_{i_\alpha}, z_{i_\alpha})$. It follows by the choice of i_α's that $(\bar{a}'_\alpha)_{\alpha\in\kappa}$ are mutually indiscernible, $\{\varphi'_\alpha(x, a'_{\alpha 0})\}_{\alpha\in\kappa}$ is consistent and $\left\{\varphi'_\alpha\left(x, a'_{\alpha j}\right)\right\}_{j\in\omega}$ is inconsistent for each $\alpha \in \kappa$. It follows that we had found an inp-pattern of depth $\kappa = |T|^+$ — so T has TP_2. $\blacksquare_{5.5}$

We are going to give an analogue of Fact 5.3(1) for burden^{2,3}, but first a standard lemma.

Lemma 5.6. *Let $\bar{a} = (a_i)_{i\in\omega}$ be indiscernible over A and let $p(x, a_0) = \operatorname{tp}(c/a_0 A)$. Assume that $\{p(x, a_i)\}_{i\in\omega}$ is consistent. Then there is $\bar{a}' \equiv_{a_0 A} \bar{a}$ which is indiscernible over cA.*

Lemma 5.7. *Let $p(x)$ be a partial type over A:*

(i) *The following are equivalent:*
   (a) $\operatorname{bdn}^3(p) < \kappa$.
   (b) *For any $(\bar{a}_i)_{i\in\kappa}$ mutually indiscernible over A and $c \vDash p(x)$ there is some $i \in \kappa$ and $\bar{a}'_i$ such that:*
   - $\bar{a}'_i \equiv_{a_{i0}\bar{a}_{\neq i}A} \bar{a}_i$,
   - *$\bar{a}'_i$ is indiscernible over $c\bar{a}_{\neq i}A$.*

(ii) *The following are equivalent:*
   (a) $\operatorname{bdn}^2(p) < \kappa$.
   (b) *For any $(\bar{a}_i)_{i\in\kappa}$ mutually indiscernible over A and $c \vDash p(x)$ there is some $i \in \kappa$ and $\bar{a}'_i$ such that:*
   - $\bar{a}'_i \equiv_{a_{i0}\bar{a}_{<i}A} \bar{a}_i$,
   - *$\bar{a}'_i$ is indiscernible over $c\bar{a}_{<i}A$.*

Proof. (i): (a) implies (b): Let $(\bar{a}_i)_{i\in\kappa}$ mutually indiscernible over A and $c \vDash p(x)$ be given. Define $p_i(x, a_{i0}) = \operatorname{tp}(c/a_{i0}\bar{a}_{\neq i}A)$. By Lemma 5.6 it is enough to show that $\bigcup_{j\in\omega} p_i(x, a_{ij})$ is consistent for some $i \in \kappa$.

Assume not, but then by compactness for each $i \in \kappa$ we have some $\varphi_i(x, a_{i0}, b_i d_i) \in p_i(x, a_{i0})$ with $b_i \in \bar{a}_{\neq i}$ and $d_i \in A$ such that $\{\varphi_i(x, a_{ij}, b_i d_i)\}_{j\in\omega}$ is inconsistent. Let $\varphi'_i\left(x, a'_{ij}, b'_i\right) = \varphi_i(x, a_{ij}, b_i d_i)$ with $a'_{ij} = a_{ij} d_i$ and $b'_i = b_i$. It follows that $(\bar{a}'_i)_{i\in\kappa}$ are mutually indiscernible, $c \vDash \{\varphi'_i(x, a'_{i0}, b'_i)\}_{i\in\kappa} \cup p(x)$ and $\left\{\varphi'_i\left(x, a'_{ij}, b'_i\right)\right\}_{j\in\omega}$ is inconsistent for each $i \in \kappa$, thus witnessing that $\operatorname{bdn}^3(p) \geq \kappa$ — a contradiction.

(b) implies (a): Assume that $\operatorname{bdn}^3(p) \geq \kappa$, witnessed by an inp^3-pattern $(\varphi_i(x, y_i, z_i), \bar{a}_i, b_i)_{i\in\kappa}$ in $p(x)$. Let $c \vDash \{\varphi_i(x, a_{i0}, b_i)\}_{i\in\kappa}$ and take $A = \varnothing$. It is then easy to check that (2) fails.

(ii): Similar. $\blacksquare_{5.7}$

References

[Adla] Hans Adler, *An introduction to theories without the independence property*, Archive for Mathematical Logic, to appear.

[Adlb] ______, *Pre-independence relations*, preprint.

[Adlc] ______, *Strong theories, burden, and weight*, preprint.

[Adl09] ______, *Thorn-forking as local forking*, Journal of Mathematical Logic **9** (2009), no. 1, 21–38, doi:10.1142/S0219061309000823.

[Ben03] Itaï Ben Yaacov, *Simplicity in compact abstract theories*, Journal of Mathematical Logic **3** (2003), no. 2, 163–191, doi:10.1142/S0219061303000297.

[BPV03] Itaï Ben Yaacov, Anand Pillay, and Evgueni Vassiliev, *Lovely pairs of models*, Annals of Pure and Applied Logic **122** (2003), no. 1-3, 235–261, doi:10.1016/S0168-0072(03)00018-6.

[Cas03] Enrique Casanovas, *Dividing and chain conditions*, Archive for Mathematical Logic **42** (2003), no. 8, 815–819, doi:10.1007/s00153-003-0192-0.

[Che] Artem Chernikov, *Theories without the tree property of the second kind*, reprint, arXiv:1204.0832.

[CK12] Artem Chernikov and Itay Kaplan, *Forking and dividing in NTP_2 theories*, Journal of Symbolic Logic **77** (2012), no. 1, 1–20, arXiv:0906.2806.

[CKS12] Artem Chernikov, Itay Kaplan, and Saharon Shelah, *On non-forking spectra*, preprint, 2012, arXiv:1205.3101.

[CLPZ01] Enrique Casanovas, Daniel Lascar, Anand Pillay, and Martin Ziegler, *Galois groups of first order theories*, Journal of Mathematical Logic **1** (2001), no. 2, 305–319, doi:10.1142/S0219061301000119.

[Dol04] Alfred Dolich, *Weak dividing, chain conditions, and simplicity*, Archive for Mathematical Logic **43** (2004), no. 2, 265–283, doi:10.1007/s00153-003-0176-0.

[GIL02] Rami Grossberg, José Iovino, and Olivier Lessmann, *A primer of simple theories*, Archive for Mathematical Logic **41** (2002), no. 6, 541–580, doi:10.1007/s001530100126.

[HP11] Ehud Hrushovski and Anand Pillay, *On NIP and invariant measures*, Journal of the European Mathematical Society (JEMS) **13** (2011), no. 4, 1005–1061, doi:10.4171/JEMS/274.

[Hru12] Ehud Hrushovski, *Stable group theory and approximate subgroups*, Journal of the American Mathematical Society **25** (2012), no. 1, 189–243, doi:10.1090/S0894-0347-2011-00708-X.

[HZ96] Ehud Hrushovski and Boris Zilber, *Zariski geometries*, Journal of the American Mathematical Society **9** (1996), no. 1, 1–56, doi:10.1090/S0894-0347-96-00180-4.

[Kim96] Byunghan Kim, *Simple first order theories*, Ph.D. thesis, University of Notre Dame, 1996, p. 96.

[KOU] Itay Kaplan, Alf Onshuus, and Alexander Usvyatsov, *Additivity of the dp-rank*, Transactions of the American Mathematical Society, to appear, arXiv:1109.1601.

[KS12] Itay Kaplan and Saharon Shelah, *Chain conditions in dependent groups*, preprint, 2012, arXiv:1112.0807.

[KU] Itay Kaplan and Alexander Usvyatsov, *Strict independence in dependent theories*, In preparation.

[Les00] Olivier Lessmann, *Counting partial types in simple theories*, Colloquium Mathematicum **83** (2000), no. 2, 201–208.

[OU11] Alf Onshuus and Alexander Usvyatsov, *On dp-minimality, strong dependence and weight*, Journal of Symbolic Logic **76** (2011), no. 3, 737–758, doi:10.2178/jsl/1309952519.

[Poi85] Bruno Poizat, *Cours de théorie des modèles*, Nur al-Mantiq wal-Ma'rifah, Lyon, 1985, Une introduction à la logique mathématique contemporaine.

[She] Saharon Shelah, *Strongly dependent theories*, preprint, arXiv:math.LO/0504197.

[She80] ______, *Simple unstable theories*, Annals of Mathematical Logic **19** (1980), no. 3, 177–203, doi:10.1016/0003-4843(80)90009-1.

[She90] ______, *Classification theory and the number of nonisomorphic models*, second ed., Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam, 1990.

[She09] ______, *Dependent first order theories, continued*, Israel Journal of Mathematics **173** (2009), 1–60, doi:10.1007/s11856-009-0082-1.

[Wag00] Frank O. Wagner, *Simple theories*, Kluwer Academic Publishers, 2000.

Itaï Ben Yaacov, Université Claude Bernard – Lyon 1, Institut Camille Jordan, CNRS UMR 5208, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France

URL: `http://math.univ-lyon1.fr/~begnac/`

Artem Chernikov, Einstein Institute of Mathematics, Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel

URL: `http://chernikov.me`