ISOMETRISABLE GROUP ACTIONS

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ABSTRACT. Given a separable metrisable space $X$, and a group $G$ of homeomorphisms of $X$, we introduce a topological property of the action $G \curvearrowright X$ which is equivalent to the existence of a $G$-invariant compatible metric on $X$. This extends a result of Marjanović obtained under the additional assumption that $X$ is locally compact.

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INTRODUCTION

This paper grew out of the following question: given a metrisable topological space $X$, and a homeomorphism $g$ of $X$, how can one determine whether there exists a distance inducing the topology of $X$ and for which $g$ is an isometry? More generally, it is interesting to determine when there exists a compatible invariant distance for an action by homeomorphisms of some group $G$ on $X$. When this happens we say that the action $G \curvearrowright X$ is isometrisable.

When $X$ is compact, this problem is well understood, and various characterisations are available – for instance, in that case an action $G \curvearrowright X$ is isometrisable if and only if it is equicontinuous, in the sense that for any open $U \subseteq X \times X$ containing the diagonal $\Delta_X$, there exists an open $V \subseteq X \times X$ containing $\Delta_X$ and such that for all $g \in G$ one has $(g \times g)V \subseteq U$. One way to prove this is to note that, if the latter property holds, then the sets $G \cdot V$ form a countably generated uniformity which is compatible with the topology and admits a basis of invariant entourages, and such a uniformity comes from a $G$-invariant metric (as a general reference about uniformities and the basic facts about them used in this paper, the reader may consult Chapter 8 in [Eng89]). One could equivalently formalise the previous condition by saying that $G$ generates a relatively compact subgroup of the group of homeomorphisms of $X$, endowed with the topology of uniform convergence for some compatible distance on $X$ (here the nontrivial direction follows from the Arzelà-Ascoli theorem, or by considering averages of any compatible metric against the Haar measure of $G$).

Beyond that, only the locally compact case seems to be addressed in the literature. Marjanović [Mar69] appears to be the first with a significant result in this direction. In order to formulate it, we recall that, if $\mathcal{F}$ is a family of continuous maps from a topological space $X$ to a topological space $Y$, $\mathcal{F}$ is said to be evenly continuous if for all $x \in X$, all $y \in Y$ and all open $V \ni y$, there exists an open $U \ni x$ and an open $W$ with $y \in W \subseteq V$ and such that

$$\forall f \in \mathcal{F} \ f(x) \in W \Rightarrow f(U) \subseteq V.$$  

Theorem (Marjanović [Mar69]). Let $X$ be a locally compact separable metrisable space, and $f$ be homeomorphism of $X$. Then there is a compatible distance for which $f$ is an isometry if, and only if, the family $\{f^n : n \in \mathbb{Z}\}$ is evenly continuous from $X$ to its Alexandrov compactification.

This result was slightly extended by Borges [Bor71] and Kiang [Kia73]; it follows from Kiang’s work that Marjanović’s result extends to arbitrary groups acting on locally compact separable metrisable
Theorem. Let $X$ be a second countable metrisable space, and $G$ be a group acting on $X$ by homeomorphisms.

It is obvious that, if $G \curvearrowright X$ is isometrisable, then it is topologically equicontinuous. It is also not hard to check that, when $X$ is locally compact, even continuity of $G$ as a family of maps from $X$ to its Alexandrov compactification is equivalent to topological equicontinuity of $G$ as a family of maps from $X$ to itself. Topological equicontinuity is a strong assumption, and we discuss some consequences in the second section. It appears not to be sufficient for isometrisability of the action $G \curvearrowright X$, leading us to consider an even stronger property.

Definition. We say that $G \curvearrowright X$ is uniformly topologically equicontinuous if, for any $x, y \in X$ and any open subset $V \ni y$, there exists open subsets $W \ni x$ and $y \in U \subseteq V$ such that

$$\forall g \in G \ (gW \cap U \neq \emptyset) \Rightarrow gW \subseteq V.$$  

Our main result is the following.

Theorem. Let $X$ be a second countable metrisable space, and $G$ be a group acting on $X$ by homeomorphisms. Then the action $G \curvearrowright X$ is isometrisable if, and only if, it is uniformly topologically equicontinuous.

Rahter than directly defining a $G$-invariant compatible metric under the assumption of uniform topological equicontinuity, our argument proceeds by building a countably generated uniformity with a basis of $G$-invariant entourages, and finally using second countability to subsume this family into one $G$-invariant metric.

1. TOPOLOGICAL EQUICONTINUITY

Throughout the text $X$ stands for a separable metrisable space, and $G$ is a group of homeomorphisms of $X$.

Lemma 1.1. Assume that $G \curvearrowright X$ is topologically equicontinuous. Assume that $x_n \to x$ and $y_n \to y$ in $X$, and let $(g_n) \in G^\mathbb{N}$ be such $g_n x_n \to y$. Then $g_n^{-1} y_n \to x$.

In particular, $g_n x \to y$ if and only if $g_n^{-1} y \to x$; thus a topologically equicontinuous action is minimal if and only if it is topologically transitive.

Proof. Fix $U$ open containing $x$, and find $V$ open contained in $U$ and containing $x$, $W$ open containing $y$ such that $(gW \cap V \neq \emptyset) \Rightarrow gW \subseteq U$ for any $g \in G$. For $n$ large enough $x_n \in V$ and $y_n, g_n x_n \in W$, so $g_n^{-1} W \subseteq U$, and in particular $g_n^{-1} y_n \in U$, as desired.

To see why topological transitivity implies minimality, assume the action is topologically transitive (that is, for any nonempty open $U, V$ there exists $g \in G$ such that $gU \cap V \neq \emptyset$) and pick $x, y \in X$. By assumption, there exist $g_n \in G$ and $x_n \in X$ such that $x_n$ converges to $x$ and $g_n x_n$ converges to $y$. Hence $g_n^{-1} y$ converges to $x$, showing that the orbit of $y$ is dense. □

Proposition 1.2. Assume that $G \curvearrowright X$ is minimal. Then $G \curvearrowright X$ is isometrisable if, and only if, it is topologically equicontinuous.

Proof. One implication is clear. For the other, assume that $G \curvearrowright X$ is topologically equicontinuous, and denote by $\tau$ the topology of $X$. Consider the family of sets of the form $G \cdot U^2 \subseteq X^2$ where $U$ varies over all nonempty open sets in $X$.

Since the action is minimal, $G \cdot U^2$ contains the diagonal. Given such a set $G \cdot U^2$, find an open $\emptyset \neq V \subseteq U$ such that $gV \cap V \neq \emptyset \Rightarrow gV \subseteq U$. Assume now that $(x, y), (y, z) \in G \cdot V^2$, say $x, y \in h_1 V$ and $y, z \in h_2 V$. Then $h_1^{-1} y \in V \cap h_1^{-1} h_2 V$, hence $h_1^{-1} h_2 V \subseteq U$, so $h_1^{-1} z \in U$. Thus both $h_1^{-1} x$ and $h_1^{-1} z$ belong to $U$, and $(x, z) \in G \cdot U^2$.

Thus the sets $G \cdot U^2$ form a basis of entourages for a uniformity, which is metrisable by a $G$-invariant distance $d$ since it is countably generated by $G$-invariant entourages, and we claim that it is compatible...
Proposition 2.2. Let $X$ be a locally compact separable metrisable space, and $G$ a group acting on $X$ by homeomorphisms. There are continuous $G$-equivariant maps $\pi: X \to X/G$ for $G$-equivariant maps $\pi: X \to X/G$.

Proof. Let $U \subseteq X$ be open, $x \in U$ and $y \sim x$. Then $Gy \cap U \neq \emptyset$, or equivalently, $y \in GU$. It follows that the open set $GU$ is the $\sim$-saturation of $U$, so $\pi U$ is open.

2. Uniform topological equicontinuity and isometrisability

Metrizability of $X \bowtie G$ is obviously a necessary condition for the action $G \curvearrowright X$ to be isometrisable. Outside the realm of locally compact spaces, this seems to require a stronger hypothesis than mere topological equicontinuity.

Definition 2.1. We say that $G \curvearrowright X$ is uniformly topologically equicontinuous if for any $x \in X$ and any open $V \ni x$ there exists an open $U$ with $x \in U \subseteq V$ such that for all $y \in X$ there exists an open $W_y \ni y$ satisfying

$$\forall g \in G \ (gW_y \cap U \neq \emptyset) \Rightarrow gW_y \subseteq V.$$ 

When the conditions above are satisfied, we say that $U$ witnesses uniform topological equicontinuity for $x, V$.

This definition is obtained by inverting two quantifiers in the definition of topological equicontinuity, and is still a necessary condition for isometrisability of $G \curvearrowright X$.

Proposition 2.2. Assume that $G \curvearrowright X$ is uniformly topologically equicontinuous. Then $X \bowtie G$ is metrisable.

Proof. Since $X$ is second countable so is $X \bowtie G$, and it will suffice to prove that $X \bowtie G$ is regular. In other words, we need to prove that given a closed $G$-invariant $F \subseteq X$ and $x \notin F$, there exist open sets $U \ni x$ and $W \ni x$ such that $U \cap GW = \emptyset$. We choose $U$ which witnesses uniform topological equicontinuity for $x, X \setminus F$, and for each $y \in F$ we let $W_y \ni y$ be the corresponding neighbourhood. If there existed $y \in F$ and $g \in G$ such that $gW_y \cap U \neq \emptyset$ then $gW_y \subseteq X \setminus F$ and in particular $gy \notin F$, a contradiction. Therefore $U \cap \bigcup_{y \in F} GW_y = \emptyset$, which is enough.

Given Marjanović’s result recalled in the introduction, the following fact is worth mentioning. (If one merely wishes to prove that $X \bowtie G$ is metrisable when $X$ is locally compact and the action is topologically equicontinuous, a much shorter argument exists.)

Proposition 2.3. Let $X$ be a locally compact separable metrisable space, and $G$ a group acting on $X$ by homeomorphisms. The following conditions are equivalent.

(i) $G \curvearrowright X$ is uniformly topologically equicontinuous.

(ii) $G \curvearrowright X$ is topologically equicontinuous.

(iii) $G$, seen as a family of maps from $X$ to its Alexandrov compactification $X^*$, is evenly continuous.
Proof. Note that (iii) is equivalent to saying that, for all \( x \in X \) and \( y \in X^* \), if \((x_i)\) converges to \(x\) and \((g_i; x)\) converges to \(y\) then \((g_i; x_i)\) also converges to \(y\).

The implication \((i) \Rightarrow (ii)\) is by definition. To see that \((ii)\) implies \((iii)\), assume that there exists \( x \in X \) and a compact \( K \subseteq X \) such that for all open \( U \ni x \) and for all compact \( L \supseteq K \) there is \( g \in G \) such that \( g(x) \not\in L \) and \( g(U) \cap K = \emptyset \). From this we may build a sequence \((x_i)\) converging to \(x\) and elements \( g_i \in G \) such that \( g_i(x_i) \to \infty \) and \( g_i(x_i) \to k \in K \). This is incompatible with \((ii)\).

It remains to prove that \((iii) \Rightarrow (i)\). We again proceed by contradiction and assume that \( G \acts X \) is not uniformly topologically equicontinuous but \( G \) is an evenly continuous family of maps from \( X \) to \( X^* \). By assumption, there exists \( x \in X \) and an open \( V \ni y \) such that for any open \( U \ni y \in U \subseteq V \) there exists \( x \in X \) such that for all open \( W \ni x \) there exists \( g \in G \) with both \( gW \cap U \neq \emptyset \) and \( gW \not\subseteq V \). Letting \( U \) vary over a basis of open neighborhoods of \( y \), we obtain a sequence \((x_i)\) witnessing the above condition; up to extractions we see that there are two cases to consider:

- \((x_i)\) converges to some \( x \in X \). Then there exists sequences \((g_i)\) and \((y_i)\), \((z_i)\) converging to \(x\) such that \( g_iy_i \) converges to \(y\) and \( g_i; z_i \) lives outside \( V \). Up to some extraction, we may assume that \( g_i; x \) and \( g_i; z_i \) both converge in \( X^* \), and the fact that \( g_i; y_i \) and \( g_i; z_i \) have different limits shows that even continuity must be violated at \(x\).
- \((x_i)\) converges to \(\infty\), and for all compact \( K \) there exists \( I \) such that for all \( i \geq I \) and all \( g \) one has \( g; x_i \not\in K \) (otherwise, replacing \( x_i \) by some \( g_i; x_i \) and going to a subsequence we would be in the situation of the first case above). Letting \( U \) be a relatively compact neighborhood of \( y \), we see that for \( i \) large enough we have \( g; x_i \cap U = \emptyset \). Then the even continuity of \( G \) implies that there must exist some neighborhood \( W \) of \( x \) such that \( G; W \cap U = \emptyset \) (by essentially the same argument as above), which contradicts the choice of \( x_i \).

\[\blacksquare\]

**Definition 2.4.** Let \( U \) be an open cover of \( X \). We say that it is \( G\)-invariant if for any \( U \) one has \( gU \subseteq U \).

A \( G\)-basis of a \( G\)-invariant open cover \( U \) is a subset \( B \) such that all elements of \( U \) are of the form \( gB \) for some \( g \in B \).

We say that a \( G\)-invariant open cover \( U \) is \( G\)-locally finite if it admits a \( G\)-basis \( B \) such that for any \( x \in X \) there exists a neighborhood \( A \) of \( x \) (not necessarily belonging to \( U \)) such that \( \{ gB \cap A \neq \emptyset \} \) is finite.

**Lemma 2.5.** Assume that \( G \acts X \) is uniformly topologically equicontinuous. Then any \( G\)-invariant open cover admits a \( G\)-locally finite open refinement.

**Proof.** Let \( U \) be a \( G\)-invariant open cover. Let also \( \pi : X \to X \) denote the open quotient map. Since \( X \) is metrisable, it is paracompact (see e.g. Theorem 4.4.1 in [Eng89]), so we can find a locally finite refinement \( \nu \) of \( \pi U \). For any \( V \) we pick some \( U_{\nu} \subseteq U \) such that \( \pi U_{\nu} \supseteq V \), and set \( W_{\nu} \subseteq U_{\nu} \cap \pi^{-1}(V) \). Let \( W \) be the \( G\)-invariant open cover with \( G\)-basis \( \{ W_{\nu} : V \in \nu \} \). By construction \( W \) is an open cover and refines \( U \).

Now pick any \( x \in X \). There is an open neighborhood \( O \) of \( \pi(x) \) which meets only finitely many elements of \( \nu \). If \( W_{\nu} \) is such that \( gW_{\nu} \cap \pi^{-1}(O) \neq \emptyset \) for some \( g \in G \), then \( V \cap O \neq \emptyset \), so \( W \) is \( G\)-locally finite.

\[\blacksquare\]

**Notation 2.6.** To an open cover \( U \) of \( X \) we associate an entourage \( E(\sigma) = U_{\in U} U_2 \subseteq X^2 \).

**Lemma 2.7.** Assume that \( G \acts X \) is uniformly topologically equicontinuous. Let \( U \) be a \( G\)-invariant open cover of \( X \). Then there exists a \( G\)-invariant open refinement \( V \) of \( U \) with the property that for all \( x, y, z \in X \), if \((x, y), (y, z) \in E(\sigma) \) then \((x, z) \in E(\sigma) \).

**Proof.** Uniform topological equicontinuity of the action enables us to find a \( G\)-invariant open refinement \( W \) of \( U \) with the property that for all \( W \subseteq W \) there exists \( U(W) \subseteq U \) containing \( W \) such that for all \( y \in X \) there exists an open \( C_{W,y} \ni y \) satisfying \( gC_{W,y} \cap W \neq \emptyset \Rightarrow gC_{W,y} \subseteq U(W) \). Using Lemma 2.5 we may assume that \( W \) is \( G\)-locally finite and \( B \) is a \( G\)-basis of \( W \) witnessing that property. We let \( V \) consist of all open sets \( V \) such that for all \( g \in G \) and \( B \in B \): \( gV \cap B \neq \emptyset \Rightarrow gV \subseteq U(B) \).

Given \( x \in X \) there exists an open \( A \ni x \) such that \( B_A = \{ B : gB \cap A \neq \emptyset \} \) is finite, so \( x \in A \cap \bigcap_{B \in B_A} B_B \in V \). Thus \( V \) is a cover, and it is clearly \( G\)-invariant and refines \( U \).

Assume now that \((x, y), (y, z) \in E(V) \), say \( x, y \in V_1 \) and \( y, z \in V_2 \) where \( V_1 \subseteq V \). There exist some \( B \in B \) and \( g \in G \) such that \( gV_1 \subseteq B \) and \( x, z \in g^{-1}U(B) \subseteq U(B) \).
Lemma 2.8. Assume that $g \curvearrowright X$ is uniformly topologically equicontinuous, and fix $x \in X$. For any $G$-invariant open cover $U$ there exists a $G$-invariant open refinement of $U$ with $G$-basis $B$ and $B \in B$ such that for any $A \in B$ different from $B$ and any $g \in G$ one has $x \notin gA$.

Proof. Pick $U \in U$ such that $x \in U$. Using uniform topological equicontinuity, choose an open neighborhood $V$ of $x$ such that for any $y \in X \setminus gU$ there exists an open set $W_y$ satisfying $gW_y \cap V = \emptyset$ for all $g \in G$. Refining if necessary, we may assume that each $W_y$ is contained in some element of $U$; then $\{W_y : y \in X \setminus gU\} \cup \{U\}$ form a $G$-basis for a $G$-invariant open refinement of $U$ with the desired property.

Lemma 2.9. Assume that $g \curvearrowright X$ is uniformly topologically equicontinuous. Then for any $x \in X$ there exists a continuous $G$-invariant pseudometric $d_x$ such that $d_x(x, x) \rightarrow 0$ if and only if $(x_i)$ converges to $x$.

Proof. Fix $x \in X$. Using lemmas 2.8 and 2.7, we can build a sequence of $G$-invariant coverings $U_n$ of $X$ with $G$-basis $B_n$ with the following properties:

- For each $n$ there exists a unique $B_n \in B_n$ such that $x \in GB_n$, and $\{B_n\}_n$ forms a basis of neighborhoods of $x$.
- For all $n$, if $(y, z), (z, t) \in E_{n+1}$ then $(y, t) \in E_n$, where $E_n = E(U_n)$.

As in the proof of Proposition 1.2, the family of entourages $E_n$ gives rise to a uniformity which is more-over metrisable by $G$-invariant pseudo-metric $d_x$. Since all entourages are open, $d_x$ is continuous.

Assuming that $d_x(x, x) \rightarrow 0$, for all $n$ there must exist some $U_n \in U_n$ such that $x_{i_n} \in U_n$ for all $i$ large enough, and $U_n$ must be of the form $gB_n$. It follows that there exists a sequence $h_i$ with $h_i x_i \rightarrow x$ and $h_i x_i \rightarrow x$, so $x_i \rightarrow x$ by Lemma 1.1.

Theorem 2.10. Let $X$ be a second-countable metrisable space, and let $G$ act on $X$ by homeomorphisms. Then $G \curvearrowright X$ is isometrisable if and only if it is uniformly topologically equicontinuous.

Proof. One direction is clear, so we prove the other. Applying Lemma 2.9, we obtain a family of continuous $G$-invariant pseudometrics $(d_x)_{x \in X}$. Since $d_x(x, x) \rightarrow 0$ if and only if $x_i \rightarrow x$, for any open subset $U$ of $X$ and any $x \in U$, there exists $\epsilon > 0$ such that $d_x(x, y) < \epsilon \Rightarrow y \in U$. By the Lindelöf property, we obtain that for any open $U \subseteq X$ there exists a countable subset $A \subseteq U$ and a family $(\epsilon_a)_{a \in A}$ such that

$$U = \bigcup_{a \in A} \{x : d_a(x, a) < \epsilon_a\}.$$

Applying this to a countable basis for the topology of $X$, we obtain a countable family of $G$-invariant pseudometrics which generate the topology, and this countable family may be subsumed into a single $G$-invariant metric.

3. Complete and Incomplete Metrics

In this section we assume that $G \curvearrowright X$ is isometrisable, and $X$ admits a compatible complete metric. A natural question is then: must $G \curvearrowright X$ admit a complete invariant metric?

The following observation is immediate.

Proposition 3.1. Assume that $G \curvearrowright X$ is minimal. Then there exists a complete compatible $G$-invariant distance on $X$ if and only if all compatible $G$-invariant distances are complete.

Proof. Assume that a compatible complete $G$-invariant distance exists, fix $x \in X$, and let $d$ be another $G$-invariant complete distance. Then a sequence $(g_i x_i)$ is $d$-Cauchy if and only if for any neighborhood $V$ of $x$ there exists $N$ such that $g_{i,j}^{-1} g_i x_i \in V$ for all $i, j \geq N$. This property does not depend on $d$ but only on the topology of $X$, so any $d$-Cauchy sequence of the form $(g_i x_i)$ must converge.

Given any $d$-Cauchy sequence $(x_i)$, the minimality of $G \curvearrowright X$ enables us to find $g_i$ such that $d(g_i x_i, x_i) < 2^{-i}$ for all $i$. Then $(g_i x_i)$ is also $d$-Cauchy, hence convergent, and so is $(x_i)$.

The fact above is well-known in the particular case when $G$ is a Polish group acting by left translation on itself (and the proof is the same). All Polish groups admit left-invariant compatible metrics, but not all of them admit such metrics which are also complete (and, if one such metric is complete, all of them are). For instance, the group $S_\infty$ of all permutations of the integers, endowed with its usual Polish topology, does not admit a compatible left-invariant complete metric.

The following simple example was suggested by C. Rosendal.

Example 3.2. There exists a Polish space $X$ and a $\mathbb{Z}$-action on $X$ which is isometrisable but which admits no complete invariant distance.
Proof. Let \( r \) be an irrational rotation of the unit circle \( S \), and let \( X = S \setminus \{ r^i(1) : i \in \mathbb{Z} \} \). Then \( X \) is a \( G_\delta \) subset of \( S \), hence Polish, and the restriction of \( r \) to \( X \) generates an isometrisable \( \mathbb{Z} \)-action; the metric on \( X \) induced from the usual metric on \( S \) is both invariant and not complete, so there cannot exist an invariant complete metric on \( X \).

As it turns out, the minimal case contains essentially all the obstructions to the existence of a complete invariant metric.

**Theorem 3.3.** Assume that \( X \) is completely metrisable and the action \( G \actson X \) is isometrisable. Then there exists a compatible complete \( G \)-invariant distance on \( X \) if and only if there exists such a distance on the closure of each \( G \)-orbit.

**Proof.** The condition is clearly necessary. Now assume that there exists a compatible complete metric on the closure of each \( G \)-orbit, and let \( d \) be a \( G \)-invariant metric on \( X \). By Proposition 3.1 the restriction of \( d \) to each \( [x] = Gx \) is complete. Also, since the projection map \( X \to X / G \) is open and open maps with range in a metrisable space preserve complete metrisability (see for instance Exercise 5.5.8(d) p.341 in [Eng89]), there exists a complete distance \( \rho \) on \( X / G \). Consider the new metric \( d' \) defined by

\[
d'(x,y) = d(x,y) + \rho([x],[y]).
\]

Clearly \( d' \) is \( G \)-invariant and compatible with the topology of \( X \). Assume now that \( (x_n) \) is \( d' \)-Cauchy. Since \( \rho \) is complete, \( [x_n] \) must converge to \( [x] \) for some \( x \in X \), i.e., there exists a sequence \( (g_n) \) such that \( g_nx_n \to x \). By invariance, \( d(x_n, g_n^{-1}x) \to 0 \), so \( (g_n^{-1}x) \) is a \( d \)-Cauchy sequence in \( [x] \) which must converge to some \( y \). Therefore \( x_n \to y \) as well, concluding the proof. ■

**References**


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