

THE LINEAR ISOMETRY GROUP OF THE GURARIJ SPACE IS UNIVERSAL

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ABSTRACT. We give a positive answer to a question of Uspenskij, whether the linear isometry group of the Gurarij space \mathbf{G} is a universal Polish group.

INTRODUCTION

Let $\text{Iso}(X)$ denote the isometry group of a metric space X , which we equip with the topology of point-wise convergence, rendering it, for complete separable X , a Polish group. We let $\text{Iso}_L(E)$ denote the linear isometry group of a normed space E . This is a closed subgroup of $\text{Iso}(E)$, and is therefore Polish as well when E is a separable Banach space.

It was shown by Uspenskij [Usp90] that the isometry group of the Urysohn space \mathbf{U} is a universal Polish group, following a construction of \mathbf{U} due to Katětov [Kat88]. The Gurarij space \mathbf{G} (see [Gur66, Lus76]) is, in *some ways*, the analogue of the Urysohn space in the category of Banach spaces. Either one is the unique separable, universal and approximately ultra-homogeneous object in its respective category, or equivalently, either one is the (necessarily unique) *Fraïssé limit* (see [Ben]) of the finitely generated objects in its respective category. It is therefore natural to compare properties of the two, and ask whether $\text{Iso}_L(\mathbf{G})$ is a universal Polish group as well.

Let us briefly recall Uspenskij's argument for the Urysohn space. Let X be a metric space. We recall that a real-valued function ξ on X is *Katětov* if $\xi(x) \leq \xi(y) + d(x, y)$ and $d(x, y) \leq \xi(x) + \xi(y)$ for all $x, y \in X$ (equivalently, if ξ is the distance function from points in X to some fixed point in a metric extension of X). Let $K(X)$ denote the set of Katětov functions on X . Equipped with the supremum distance, $K(X)$ is a complete metric space, endowed with a natural isometric embedding $X \hookrightarrow K(X)$, sending $x \mapsto d(x, \cdot)$. This construction is functorial, in the sense that an isometric embedding $Y \hookrightarrow X$ gives rise to a natural isometric embedding $K(Y) \hookrightarrow K(X)$, where ξ goes to $\hat{\xi}(x) = \inf_{y \in Y} d(x, y) + \xi(y)$, and everything composes and commutes as one would expect.

Even when X is separable, $K(X)$ need not be separable. More generally, if χ denotes the density character (which is always taken to be at least \aleph_0 , for convenience), we only have $\chi(K(X)) \leq |K(X)| \leq 2^{\chi(X)}$, and the bound can be attained. When $Y \subseteq X$, let us identify $K(Y)$ with its image in $K(X)$, and define

$$K_0(X) = \overline{\bigcup_{Y \subseteq X \text{ finite}} K(Y)}.$$

Then we have $X \hookrightarrow K_0(X)$ and this is functorial as for $K(X)$, and moreover $\chi(K_0(X)) = \chi(X)$. It will be useful to observe that $K_0(X) \supseteq \widehat{K(Y)}$ for every compact $Y \subseteq X$. We now define $X_0 = X$, $X_{n+1} = K_0(X_n)$ and $X_\omega = \widehat{\bigcup X_n}$. If X is separable then so is X_ω , and the latter is a Urysohn space. Moreover, since everything was functorial, we obtain continuous embeddings of the isometry groups $\text{Iso}(X) \subseteq \text{Iso}(X_\omega) = \text{Iso}(\mathbf{U})$, whence the desired result. We refer the reader to [Usp08, Section 3] for more details.

It turns out that the same strategy works for the Gurarij space as well. In Section 1 we identify the space of isomorphism types of one point extensions of a Banach space E (in model-theoretic terminology we would speak of *quantifier-free 1-types* over E) as the space $K_C(E)$ of *convex* Katětov functions. The passage from K to K_0 carries over essentially unchanged, in that $K_{C,0}(E) = K_C(E) \cap K_0(E)$ will do. The main technical difficulty lies in the fact that, while $K(X)$ is a metric space extension of X , $K_C(E)$ is *not* a Banach space extension of E . In Section 2 we show that $K_C(E)$ embeds canonically in a Banach space extensions of E . In fact we prove a more general result, generalising the Arens-Eells construction from pointed metric spaces to metric spaces over a normed space. We conclude in Section 3, constructing, for

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a separable Banach space E , a chain of separable Banach space extensions as above, with limit E_ω , and show that $E_\omega \cong \mathbf{G}$, with only marginally more complexity than for pure metric spaces.

1. CONVEX KATĚTOV FUNCTIONS

Let E be a normed space, $X \subseteq E$ convex, and define $K_C(X) \subseteq K(X)$ to consist of all convex Katětov functions. We observe that the inclusion $K(X) \subseteq K(E)$ restricts to $K_C(X) \subseteq K_C(E)$. For $\xi \in K(X)$ we define the generated convex function $\xi^C: X \rightarrow \mathbf{R}$ to be the greatest convex function lying below ξ , namely

$$\xi^C(a) = \inf_{\sum_{k < n} b_k = na} \sum \xi(b_k)/n.$$

We observe that ξ^C is Katětov as well, so $\xi^C \in K_C(X)$, and the map $\xi \mapsto \xi^C$ is 1-Lipschitz. Moreover, if $X \subseteq Y \subseteq E$ are all convex then there is no difference we first pass to ξ^C and then extend to Y , or the other way around, so we may refer to $\xi^C \in K_C(Y)$ without ambiguity.

For a convex set X (most often this will be the entire Banach space), we define $K_{C,0}(X) = K_0(X) \cap K_C(X)$. We claim that this intersection is not too small, and in particular, that it contains ξ^C for every $\xi \in K_0(X)$. More precisely, we claim that

$$K_{C,0}(X) = \overline{\bigcup_{Y \subseteq X \text{ convex compact}} K_C(Y)} = \{\xi^C : \xi \in K_0(X)\}.$$

Indeed, the first inclusion \supseteq follow immediately from our earlier observations, the second holds since $\xi \mapsto \xi^C$ is continuous and the convex hull of a finite set is compact, and $\{\xi^C : \xi \in K_0(X)\} \supseteq K_{C,0}(X)$ trivially.

Given $\xi \in K_C(E)$, let $E(x) = E \oplus \mathbf{R}x$, and for $\alpha x - a \in E(x)$ define

$$\|\alpha x - a\|^\xi = \begin{cases} |\alpha| \xi(a/\alpha) & \alpha \neq 0 \\ \|a\| & \alpha = 0. \end{cases}$$

Lemma 1.1. *The map $\xi \mapsto \|\cdot\|^\xi$ is a bijection between $K_C(E)$ and semi-norms on $E(x)$ extending $\|\cdot\|_E$, namely with the space of quantifier-free 1-types over E (see [BH]), whose inverse sends $\|\cdot\|$ to $\|x - \cdot\|$.*

Proof. All there is to show is that if $\xi \in K_C(E)$ then $\|\cdot\|$ is indeed a semi-norm, and for this it will suffice to show that $\|\alpha x - a\|^\xi + \|\beta x - b\|^\xi \geq \|(\alpha + \beta)x - (a + b)\|^\xi$. We consider several cases:

- (i) If both α and β are zero then there is nothing to show.
- (ii) If α and β are non zero with equal sign, say $0 > \alpha, \beta$ and $\alpha + \beta = 1$, then we use convexity: $\alpha \xi(a/\alpha) + \beta \xi(b/\beta) \geq \xi(\alpha a/\alpha + \beta b/\beta) = \xi(a + b)$.
- (iii) If $\alpha = -\beta$, say $\alpha = 1$, then this is $\xi(a) + \xi(-b) \geq \|a + b\| = d(a, -b)$, which follows from ξ being Katětov.
- (iv) The last case is when α and β have distinct signs and absolute values, say $\beta \leq 0 < \alpha$ and $\alpha + \beta = 1$. Then indeed, using the fact that ξ is Katětov,

$$\begin{aligned} \alpha \xi(a/\alpha) - \beta \xi(b/\beta) &= \xi(a/\alpha) - \beta [\xi(a/\alpha) + \xi(b/\beta)] \\ &\geq \xi(a/\alpha) - \beta \|a/\alpha - b/\beta\| \\ &= \xi(a/\alpha) + \|a/\alpha - (a + b)\| \geq \xi(a + b). \end{aligned} \quad \blacksquare_{1.1}$$

Given two (isometric) extensions $E \subseteq F_i$, $i = 0, 1$, there is a greatest (compatible) norm on $F_0 \oplus_E F_1$, namely $\|a - b\| = \inf_{c \in E} \|a - c\| + \|c - b\|$. With $F_0 = (E(x), \|\cdot\|^\xi)$ for some $\xi \in K_C(E)$ and $F_1 = F$ fixed, we obtain a canonical extension of $\|\cdot\|^\xi$ to $F(x) = E(x) \oplus_E F$, yielding an embedding $K_C(E) \subseteq K_C(F)$ which is compatible with $K(E) \subseteq K(F)$.

Lemma 1.2. *Let E be a normed space and $\xi_i \in K_C(E)$, $i = 0, 1$. Then for every $d(\xi_0, \xi_1) = r_0 \leq r \leq r_1 = \inf_{a \in E} \xi_0(a) + \xi_1(a)$ there exists a semi-norm $\|\cdot\|_r$ on $E(x_0, x_1) = E \oplus \mathbf{R}x_0 \oplus \mathbf{R}x_1$ whose restriction to $E(x_i)$ is $\|\cdot\|^{\xi_i}$, such that $\|x_i - y_i\|_r = r$.*

Proof. For $r = r_0$, this is a special case of a result of Henson, see [BH]. For $r = r_1$, just take the greatest norm on $E(x_0) \oplus_E E(x_1)$ compatible with $\|\cdot\|^{\xi_i}$ as above. Intermediate values can be obtained as convex combinations of the two extremal cases. $\blacksquare_{1.2}$

2. THE ARENS-EELLS SPACE OVER A NORMED SPACE

We recall the following facts from Weaver [Wea99, Chapter 2.2]:

Fact 2.1. *Let $(X, 0)$ be a pointed metric space. Then there exists a Banach space $\text{AE}(X)$, together with an isometric embedding $X \subseteq \text{AE}(X)$ sending $0 \mapsto 0$, having the following universal property: every Lipschitz map $\theta: X \rightarrow F$, where F is a Banach space, sending $0 \mapsto 0$, admits a unique continuous linear extension $\theta': \text{AE}(X) \rightarrow F$ with $\|\theta'\| = L(\theta)$, the Lipschitz constant of θ . This Banach space $\text{AE}(X)$ is unique up to a unique linear isometry, and is called the Arens-Eells space of X .*

Fact 2.2. *Let $(X, 0)$ be a pointed metric space, and let $\text{Lip}_0(X)$ denote the space of real-valued Lipschitz functions on X sending $0 \mapsto 0$, equipped with $\|\xi\| = L(\xi)$, the Lipschitz constant of ξ . Then $\text{Lip}_0(X)$ is a Banach space. Moreover, for each $\lambda \in \text{AE}(X)^*$, its restriction to X belongs to $\text{Lip}_0(X)$, and this restriction map is an isometric isomorphism between $\text{AE}(X)^*$ and $\text{Lip}_0(X)$.*

We shall generalise this as follows:

Definition 2.3. Let E be a fixed normed space. By a *metric space over E* we mean a pair (X, φ) where X is a metric space, $\varphi: E \rightarrow X$ is isometric, and for each $x \in X$ the function $a \mapsto d(\varphi a, x)$ is convex. Most of the time φ will just be an inclusion map, in which case it is replaced with E or simply omitted. We also define $\text{Lip}(X, E)$ to consist of all Lipschitz functions on X which are linear on E .

For the trivial normed space 0 , a metric space over 0 is the same thing as a pointed metric space. Also, every isometric inclusion of normed spaces $E \subseteq F$ renders F metric over E .

Theorem 2.4. *Let X be a metric space over a normed space E . Then there exists a Banach space $\text{AE}(X, E)$, together with an isometric embedding $X \subseteq \text{AE}(X, E)$ linear on E , having the following universal property: every Lipschitz map $\theta: X \rightarrow F$, where F is a Banach space, which is linear on E , admits a unique continuous linear extension $\theta': \text{AE}(X, E) \rightarrow F$ with $\|\theta'\| \leq L(\theta)$. This Banach space $\text{AE}(X, E)$ is unique up to a unique linear isometry, and we shall call it the Arens-Eells space of X over E . Its dual $\text{AE}(X, E)^*$ is isometrically isomorphic to $\text{Lip}(X, E)$ via restriction to X .*

Proof. Let $\iota: E \rightarrow \text{AE}(X)$ denote the inclusion map, so that we may distinguish between algebraic operations in E and in $\text{AE}(X)$. Let $F_0 \subseteq \text{AE}(X)$ be the closed subspace generated by all expressions of the form $\iota(a+b) - \iota a - \iota b$ for $a, b \in E$, observing that it also contains $\iota \alpha a - \alpha \iota a$ for $a \in E$ and $\alpha \in \mathbf{R}$. Define $\text{AE}(X, E)$ as the completion of the quotient $\text{AE}(X)/F_0$, and let $\psi: X \rightarrow \text{AE}(X, E)$ be the natural map. Then ψ is linear and 1-Lipschitz, and both the universal property and characterisation of the dual hold, so all we need to show is that ψ is isometric.

Indeed, let $x_i \in X$, $i = 0, 1$, and let $\xi_i = d(\cdot, x_i) \in K_C(E)$. Let $r_0 \leq r_1$ be as in Lemma 1.2, and observe that by the triangle inequality $r_0 \leq d(x_0, x_1) \leq r_1$. Therefore, by Lemma 1.2 there exists a semi-norm $\|\cdot\|$ on $E(x_0, x_1)$ which induces the same distance on $E \cup \{x_0, x_1\}$ as X does. By the Hahn-Banach Theorem, there exists a linear functional $\lambda \in E(x_0, x_1)^*$ which norms $x_0 - x_1$: $\|\lambda\| = 1$ and $\lambda(x_0 - x_1) = d(x_0, x_1)$. The restriction of λ to $E \cup \{x_0, x_1\}$ is 1-Lipschitz, and by [Wea99, Theorem 1.5.6], it extends to a 1-Lipschitz function $\lambda': X \rightarrow \mathbf{R}$. Then $\lambda' \in \text{Lip}(X, E) = \text{AE}(X, E)^* \subseteq \text{Lip}_0(X)$ and $\|\lambda'\| = 1$, witnessing that $\|\psi x_0 - \psi x_1\| \geq d(x_0, x_1)$, completing the proof. $\blacksquare_{2.4}$

Similarly, it is shown in [Wea99] that if $Y \subseteq X$ then $\text{AE}(Y)$ is naturally identified with the subspace of $\text{AE}(X)$ generated by Y , and the same can be deduced for metric spaces over a normed space E .

Corollary 2.5. *Let E be a normed space and $S \subseteq K_C(E)$. Then there exists a Banach space $E[S]$ together with an isometric embedding $E \cup S \subseteq E[S]$ (here we identify E with its image in $K_C(E)$) such that $E \subseteq E[S]$ is a Banach space extension, and the following universal property holds: every Lipschitz map $\theta: E \cup S \rightarrow F$, where F is a Banach space, which is linear on E , admits a unique continuous linear extension $\theta': E[S] \rightarrow F$ with $\|\theta'\| \leq L(\theta)$.*

In particular, $E[S]$ is uniquely determined by this universal property, and is generated as a Banach space by $E \cup S$.

Proof. Just take $E[S] = \text{AE}(S \cup E, E)$. $\blacksquare_{2.5}$

3. MAIN THEOREM

Definition 3.1. For a Banach space E we define

- $E' = E[K_{C,0}(E)]$ as per Corollary 2.5,
- $E_0 = E$, $E_{n+1} = E'_n$, $E_\omega = \overline{\bigcup E_n}$.

In order to show that E_ω is a Gurarij space, it will be easiest to use the following criterion, which may simply be taken as the definition.

Fact 3.2. *A separable Banach space E is a Gurarij space if and only if for every $\xi \in K_C(E)$, every finitely many points $a_0, \dots, a_{n-1} \in E$ and every $\varepsilon > 0$ there exists $b \in E$ such that $|||b - a_i|| - \xi(a_i)| < \varepsilon$ for all $i < n$. Moreover, we may restrict the a_i to some dense subset of E .*

Proof. The main assertion is a restatement of [BH, Lemma 2.2], and the moreover part follows. ■_{3.2}

Fact 3.3. *The Gurarij space \mathbf{G} exists and is unique.*

Proof. Existence is due to Gurarij [Gur66], as well as almost isometric uniqueness. Isometric uniqueness is due to Lusky [Lus76], with a recent plethora of more “elementary” (that is to say, in the author’s view, more model-theoretic) proofs [KS, BH, Ben]. ■_{3.3}

We recall from Uspenskij [Usp08],

Definition 3.4. An isometric embedding of metric spaces $X \subseteq Y$ is said to be a *g-embedding* if there exists a continuous homomorphism $\Theta: \text{Iso}(X) \rightarrow \text{Iso}(Y)$ such that $\Theta\varphi$ extends φ for each $\varphi \in \text{Iso}(X)$.

Fact 3.5 (Mazur-Ulam Theorem). *For a normed space E , $\text{Iso}(E)$ is the group of affine isometries. Stated equivalently, $\text{Iso}_L(E) = \{\varphi \in \text{Iso}(E) : \varphi 0 = 0\}$.*

Proof. [FJ03, Theorem 1.3.5]. ■_{3.5}

It follows that an isometric embedding of Banach spaces $E \subseteq F$ is a *g-embedding* if and only if there exists a continuous homomorphism $\Theta: \text{Iso}_L(E) \rightarrow \text{Iso}_L(F)$ such that $\Theta\varphi$ extends φ for all $\varphi \in \text{Iso}_L(E)$. Indeed, given $\Theta: \text{Iso}(E) \rightarrow \text{Iso}(F)$ we have $(\Theta\varphi)0 = \varphi 0 = 0$, so Θ restricts to $\Theta': \text{Iso}_L(E) \rightarrow \text{Iso}_L(F)$, and conversely, given $\Theta: \text{Iso}_L(E) \rightarrow \text{Iso}_L(Y)$ we can define $\Theta': \text{Iso}(E) \rightarrow \text{Iso}(F)$ by $(\Theta'\varphi)x = [\Theta(\varphi - \varphi 0)]x + \varphi 0$.

Lemma 3.6. *Let E be a normed space. Then E' and E_ω are Banach space extensions of E with $\chi(E) = \chi(E') = \chi(E_\omega)$, and the embeddings $E \subseteq E' \subseteq E_\omega$ are g-embeddings.*

Moreover, if E is separable then E_ω is the Gurarij space.

Proof. Since $E[S]$ is generated by the image of S , we have $\chi(E[S]) = \chi(E) + \chi(S)$, whence the first assertion. For the second, every $\varphi \in \text{Iso}_L(E)$ of E extends to an isometry $\hat{\varphi}: \xi \mapsto \xi \circ \varphi^{-1}$ of $K_{C,0}(E)$ and by the universal property of $E[S]$, to an isometry $\varphi' \in \text{Iso}_L(E')$. Proceeding by induction we obtain an extension $\varphi_\omega \in \text{Iso}(E_\omega)$. We leave it to the reader to check that the maps $\varphi \mapsto \varphi'$ and therefore $\varphi \mapsto \varphi_\omega$ are continuous homomorphisms.

For the moreover part, it is clear that E_ω satisfies the criterion of Fact 3.2, with the a_i restricted to the dense set $\bigcup E_n$. ■_{3.6}

Theorem 3.7. *The group $\text{Iso}_L(\mathbf{G})$ of linear isometries of the Gurarij space is a universal Polish group. Moreover, every separable normed space is g-embeddable in \mathbf{G} .*

Proof. The moreover part follows immediately from Lemma 3.6. Now let H be a Polish group. Then it admits a compatible left-invariant distance d , which we may assume is bounded by 1. Adjoining a distinguished point $*$ to H with $d(*, h) = 1$ (or even $\frac{1}{2}$) for all $h \in H$ we obtain a pointed metric space H^* . The left action of $h \in H$ on H^* , fixing $*$, is an isometry, which extends by the universal property to a linear isometry $h' \in \text{Iso}_L(\text{AE}(H^*, *))$, and the map $h \mapsto h'$ is a continuous embedding. We conclude by the moreover part. ■_{3.7}

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