LIPSCHITZ FUNCTIONS ON TOPOMETRIC SPACES

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Abstract. We study functions on topometric spaces which are both (metrically) Lipschitz and (topologically) continuous, using them in contexts where, in classical topology, ordinary continuous functions are used.

(i) We define *normal* topometric spaces and characterise them by analogues of Urysohn's Lemma and Tietze's Extension Theorem.
(ii) We define *completely regular* topometric spaces and characterise them by the existence of a topometric Stone-Čech compactification.
(iii) For a compact topological space X, we characterise the subsets of $\mathcal{C}(X)$ which can arise as the set of continuous 1-Lipschitz functions with respect to a topometric structure on X.

Introduction

Topometric spaces are spaces equipped both with a metric and a topology, *which need not agree*. To be precise,

Definition 0.1. A *topometric space* is a triplet $(X, \mathscr{T}, d)$, where $\mathscr{T}$ is a topology and d a metric on X, satisfying:

(i) The distance function $d\colon X^2 \to [0, \infty]$ is lower semi-continuous in the topology.
(ii) The metric refines the topology.

Compact topometric spaces were first defined in [BU10] as a formalism for various global and local type spaces arising in the context of continuous first order logic, allowing for some kind of (topometric) Cantor-Bendixson analysis in spaces which, from a purely topological point of view, are possibly even perfect. General topometric spaces (i.e., non compact) were defined studied further from an abstract point of view in [Ben08b], where the formalism is shown to be further useful for the analysis of perturbation structures on type spaces. The same idea was also shown to be useful in the context of (very non compact) Polish groups, which may admit "topometric ample generics" even when no purely topological ample generics need exist, see [BBM].

The aim of this paper is to study some basic properties of the class of (topologically) continuous and (metrically) Lipschitz functions on a topometric space. These are naturally linked with separation axioms. In Section 1 we discuss topometric normality, which we related to existence results such as Urysohn's Lemma and Tietze's Extension Theorem. As a consequence, we obtain a Lipschitz Morleyisation result, the unique model-theoretic result of this paper. In Section 2 we construct the Stone-Čech compactification of a topometric space and relate it to topometric complete regularity. To conclude, Section 3 characterises the bare minimum that the set of Lipschitz functions needs to satisfy.

Lipschitz functions on an ordinary metric spaces, and algebras thereof, are extensively studied in Weaver [Wea99]. This is some natural resemblance between our object of study here and that of Weaver, with the increased complexity due to the additional topological structure. The reader may wish to compare, for example, our version of Tietze's Extension Theorem (Theorem 1.10) with [Wea99, Theorem 1.5.6] (as well as with the classical version of Tietze's Theorem, see Munkres [Mun75]).

We follow the convention that unless explicitly qualified, the vocabulary of general topology (compact, continuous, etc.) refers to the topological structure, while the vocabulary of metric spaces (Lipschitz function, etc.) refers to the metric structure. Excluded from this convention are separation axioms: we assimilate the lower semi-continuity of the distance function to the Hausdorff separation axiom, and stronger axioms, such as normality and complete regularity, will be defined for topometric spaces below.

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We shall also make frequent reference to two classes of examples, arising from the embedding of the categories of (Hausdorff) topological spaces and of metric spaces in the category of topometric spaces. By a *maximal* topometric space we mean one equipped with the discrete $0/1$ distance, which can be identified, for all (or most) intents and purposes, with its underlying pure topological structure. Similarly, a *minimal* topometric space is one in which the metric and topology agree, which may be identified with its underlying metric structure. These sometimes serve as first sanity checks (e.g., when we define a normal topometric space we must check that a maximal one is normal if and only if it is normal as a pure topological space, and that minimal ones are always normal).

1. Normal topometric spaces and Urysohn–Tietze results

For two topometric spaces X and Y we define $\mathcal{C}_{\mathcal{L}(1)}(X,Y)$ to be the set of all continuous 1-Lipschitz functions from X to Y. An important special case is $\mathcal{C}_{\mathcal{L}(1)}(X) = \mathcal{C}_{\mathcal{L}(1)}(X,\mathbf{C})$, where $\mathbf{C}$ is equipped with the standard metric and topology (i.e., with the standard minimal topometric structure), which codes information both about the topology and about the metric structure of X. In the present paper we seek conditions under which $\mathcal{C}_{\mathcal{L}(1)}(X)$ codes the entire topometric structure, as well as analogues of classical results related to separation axioms, in which $\mathcal{C}(X)$ would be replaced with $\mathcal{C}_{\mathcal{L}(1)}(X)$. As discussed in [Ben08b], we consider the lower semi-continuity of the distance function to be a topometric version of the Hausdorff separation axiom, so we may expect other classical separation axioms to take a different form in the topometric setting. We start with normality.

Definition 1.1. Let X be a topometric space. We say that a closed set $F \subseteq X$ has *closed metric neighbourhoods* if for every $r > 0$ the set $\overline{B}(F,r) = \{x \in X : d(x,F) \leq r\}$ is closed in X.

We say that X *admits closed metric neighbourhoods* if all closed subsets of X do.

It was shown in [Ben08b] that compact sets always have closed metric neighbourhoods, so a compact topometric space admits closed metric neighbourhoods. Indeed, the first definition of a *compact* topometric space in [BU10] was given in terms of closed metric neighbourhoods. While this property seems too strong to be part of the definition of a non compact topometric space, it will play a crucial role in this section.

Definition 1.2. A *normal topometric space* is a topometric space X satisfying:

(i) Every two closed subset $F, G \subseteq X$ with positive distance $d(F,G) > 0$ can be separated by disjoint open sets.
(ii) The space X admits closed metric neighbourhoods.

One checks that a maximal topometric space X (i.e., equipped with the discrete $0/1$ distance) is normal if and only if it is so as a topological space. Similarly, a minimal topometric space (i.e., equipped with the metric topology) is always normal. Also, every compact topometric space is normal (since it admits closed metric neighbourhoods and the underlying topological space is normal).

We contend that our definition of a normal topometric space is the correct topometric analogue of the classical notion of a normal topological space. This will be supported by analogues of Urysohn's Lemma and of Tietze's Extension Theorem. The technical core of the proofs (and indeed, the only place where the definition of a normal topometric space is used) lies in the following Definition and Lemma.

Definition 1.3. Let X be a topometric space, $c > 0$ a constant, $S \subseteq \mathbf{R}$ and $\Xi_S = \{(F_\alpha, G_\alpha) : \alpha \in S\}$ a sequence of pairs of closed sets $F_\alpha, G_\alpha \subseteq X$.

(i) We say that Ξ_S is an *approximation of a strictly c-Lipschitz partial continuous function on X*, or simply a *partial c-Lipschitz approximation*, if $d(F_\alpha, G_\beta)c > \beta - \alpha$ for $\alpha < \beta$ in S.
(ii) It is a *(total) approximation* if in addition $F_\alpha \cup G_\alpha = X$ for all $\alpha \in S$ (so particular $G_\alpha^c \subseteq F_\alpha \subseteq G_\beta^c \subseteq F_\beta$ for $\alpha < \beta$).
(iii) We say that Ξ_S is an approximation of a function $f : X \to \mathbf{R}$ if $f{\restriction}_{F_\alpha} \leq \alpha$ and $f{\restriction}_{G_\alpha} \geq \alpha$ for $\alpha \in S$.

If $f : X \to \mathbf{R}$ is c-Lipschitz and $S \subseteq \mathbf{R}$ then the sequence $\{(F_\alpha, G_\alpha) : \alpha \in S\}$ defined by $F_\alpha = \{x : f(x) \leq \alpha\}$, $G_\alpha = \{x : f(x) \geq \alpha\}$ is a c'-Lipschitz approximation f for all $c' > c$.

Lemma 1.4. *Let $\{(F_\alpha, G_\alpha) : \alpha \in S\}$ be a finite partial c-Lipschitz approximation in a normal topometric space, and let $\beta \in S$. Then there are $F'_\beta, G'_\beta \subseteq X$ such that*

- $F'_\beta \supseteq F_\beta$, $G'_\beta \supseteq G_\beta$.
- $F'_\beta \cup G'_\beta = X$.

- *Letting $F'_\alpha = F_\alpha$ and $G'_\alpha = G_\alpha$ for $\alpha \neq \beta$ then $\{(F'_\alpha, G'_\alpha) \colon \alpha \in S\}$ is a partial c-Lipschitz approximation.*

Proof. Since the partial approximation is finite it is also c'-Lipschitz for some $c' < c$. Define:

$$K = \bigcup_{\alpha \in S, \alpha < \beta} \overline{B}(F_\alpha, (\beta - \alpha)/c'), \qquad L = \bigcup_{\alpha \in S, \alpha > \beta} \overline{B}(G_\alpha, (\alpha - \beta)/c').$$

By construction $d(K, L) > 0$ and both are closed as finite unions of closed sets. Since X is normal we can find disjoint open sets $U \supseteq K$ and $V \supseteq L$.

We claim that $F'_\beta = F_\beta \cup V^c$ and $G'_\beta = G_\beta \cup U^c$ will do. The first two items are trivially verified, so we only need to check the last one. So assume that $\alpha < \beta$. We already know by hypothesis that $d(F_\alpha, G_\beta)c > \beta - \alpha$. We also know by construction that $U \supseteq \overline{B}(F_\alpha, (\beta - \alpha)/c')$, whereby $d(F_\alpha, U^c)c > d(F_\alpha, U^c)c' \geq \beta - \alpha$. Thus $d(F_\alpha, G'_\beta)c > \beta - \alpha$. We show similarly that if $\beta < \alpha$ then $d(F'_\beta, G_\alpha)c > \alpha - \beta$, and we are done. $\blacksquare_{1.4}$

Lemma 1.5. *Let X be a normal topometric space, $\Xi_S = \{(F_\alpha, G_\alpha) \colon \alpha \in S\}$ a finite c-Lipschitz approximation. Then for every $\beta \in \mathbf{R}$ there is a c-Lipschitz approximation $\Xi'_{S \cup \{\beta\}} \supseteq \Xi_S$.*

Proof. We may assume that $\beta \notin S$, and let $F_\beta = G_\beta = \varnothing$. Then $\{(F_\alpha, G_\alpha) \colon \alpha \in S \cup \{\beta\}\}$ is a partial c-Lipschitz approximation and Lemma 1.4 (with the same β) we obtain the required approximation $\Xi'_{S \cup \{\beta\}} = \{(F'_\alpha, G'_\alpha)\}_{\alpha \in S \cup \{\beta\}}$. $\blacksquare_{1.5}$

Proposition 1.6. *In a normal topometric space every finite approximation of a c-Lipschitz continuous function approximates such a function.*

Proof. Let X be a normal topometric space, $\{(F_\alpha, G_\alpha) \colon \alpha \in S\}$ a finite c-Lipschitz approximation. Since S is finite its convex hull is a compact interval $I \subseteq \mathbf{R}$. Let $T \subseteq I$ be a countable dense subset containing S. By repeated applications of Lemma 1.5 one can extend the given approximation into a c-Lipschitz approximation $\{(F_\alpha, G_\alpha) \colon \alpha \in T\}$. Letting $f(x) = \sup\{\alpha \in I \colon x \in G_\alpha\} = \inf\{\alpha \in I \colon x \in F_\alpha\}$ (here $\inf \varnothing = \sup I$ and $\sup \varnothing = \inf I$) one obtains a continuous, c-Lipschitz function $f \colon X \to I$ which is approximated by $\{(F_\alpha, G_\alpha) \colon \alpha \in S\}$. $\blacksquare_{1.6}$

The topometric analogue of Urysohn's Lemma is obtained as an easy corollary.

Corollary 1.7 (Urysohn's Lemma for topometric spaces)**.** *Let X be a normal topometric space, $F, G \subseteq X$ closed sets, $0 < r < d(F, G)$. Then there exists a 1-Lipschitz continuous function $f \colon X \to [0, r]$ equal to 0 on F and to r on G.*

Conversely, every topometric space in which this property holds is normal.

Proof. Apply Proposition 1.6 to $S = \{0, r\}$, $F_0 = F$, $G_r = G$, $G_0 = F_r = X$.

Assume now that the first property holds in X. Then closed sets of positive distance can be separated by a 1-Lipschitz continuous function, and therefore by open sets. Also, if $F \subseteq X$ is closed and $d(x, F) > r$ then we may separate F and x by a 1-Lipschitz continuous function such that $f{\restriction}_F = 0$ and $f(x) > r$. Then $\{y \colon f(y) \leq r\}$ is a closed set containing $\overline{B}(F, r)$ but not x. If follows that $\overline{B}(F, r)$ is closed. $\blacksquare_{1.7}$

Corollary 1.8. *Let X be a compact topometric space. Then the family of continuous Lipschitz functions on X is dense in $\mathcal{C}(X)$.*

Proof. It will be enough to show that the family $\mathcal{C}_\mathcal{L}(X, \mathbf{R})$ of real-valued continuous Lipschitz functions on X is uniformly dense in $\mathcal{C}(X, \mathbf{R})$. Since X is compact, it is normal. The family $\mathcal{C}_\mathcal{L}(X, \mathbf{R})$ forms a lattice, and in addition, for every two distinct points $x, y \in \mathrm{S}_n(\mathcal{L})$ and values $r, s \in \mathbf{R}$, there exists by Urysohn's Lemma $f \in \mathcal{C}_\mathcal{L}(X, \mathbf{R})$ such that $f(x) = s$ and $f(y) = r$. By the lattice version of the Stone-Weierstrass theorem, $\mathcal{C}_\mathcal{L}(X, \mathbf{R})$ is uniformly dense in $\mathcal{C}(X, \mathbf{R})$. $\blacksquare_{1.8}$

Lemma 1.9. *Let X be a normal topometric space, $Y \subseteq X$ closed. Then for every finite c-Lipschitz approximation $\{(F_\alpha, G_\alpha) \colon \alpha \in S\}$ in Y there is one $\{(F'_\alpha, G'_\alpha) \colon \alpha \in S\}$ in X such that $F'_\alpha \supseteq F_\alpha$, $G'_\beta \supseteq G_\beta$.*

Proof. Observe that $\{(F_\alpha, G_\alpha) \colon \alpha \in S\}$ is a partial c-Lipschitz approximation on X, so we may apply Lemma 1.4 to each $\alpha \in S$ and obtain the required approximation. $\blacksquare_{1.9}$

Observe that the forced limit operator $\mathcal{F}\lim \colon [0, 1]^\mathbf{N} \to [0, 1]$ defined in [BU10] is 1-Lipschitz where $[0, 1]^\mathbf{N}$ is equipped with the supremum metric.

Theorem 1.10 (Tietze's Extension Theorem for topometric spaces)**.** *Let X be a normal topometric space. Then for every $c < c'$ every continuous c-Lipschitz function $f\colon Y \to [0,1]$ on a closed subset $Y \subseteq X$ extends to a continuous c'-Lipschitz function $g\colon X \to [0,1]$.*

Moreover, for an arbitrary topometric space the following are equivalent:

(i) *X is a normal topometric space.*
(ii) *Tietze's Extension Theorem for topometric spaces (i.e., the statement above) holds in X.*
(iii) *The statement of Proposition 1.6 holds in X.*
(iv) *Urysohn's Lemma (the main assertion of Corollary 1.7) holds in X.*

Proof. Let $Y \subseteq X$ be closed, $f\colon Y \to [0,1]$ be continuous and c-Lipschitz. For $\alpha \in [0,1]$ let $F_\alpha = f^{-1}([0,\alpha])$ and $G_\alpha = f^{-1}([\alpha,1])$. For $n \in \mathbf{N}$ let $S_n = \{k2^{-n} \colon 0 \leq k \leq 2^n\}$, and $\Xi_n = \{(F_\alpha, G_\alpha) \colon \alpha \in S_n\}$. Then Ξ_n is a c'-Lipschitz approximation on Y for any $c' > 0$.

By Lemma 1.9 it admits an extension $\Xi'_n = \{(F'_{n,\alpha}, G'_{n,\alpha}) \colon \alpha \in S_n\}$ to X (which may depend on n) which is c'-Lipschitz as well. By Proposition 1.6 there exists a continuous c'-Lipschitz function $g_n\colon X \to [0,1]$ approximated by Ξ'_n, and let $g = \mathcal{F}\lim g_n$. Notice that if $y \in Y$ and $k2^{-n} \leq f(y) \leq (k+1)2^{-n}$ then $y \in F_{(k+1)2^{-n}} \cap G_{k2^{-n}} \subseteq F'_{n,(k+1)2^{-n}} \cap G'_{n,k2^{-n}}$, whereby $k2^{-n} \leq g_n(y) \leq (k+1)2^{-n}$ as well. Thus $|g_n{\restriction}_Y - f| \leq 2^{-n}$ for all n whereby $g{\restriction}_Y = f$. Also, a forced limit of a family of continuous c'-Lipschitz functions is continuous and c'-Lipschitz.

For the moreover part, we have seen that if X is normal then (ii)-(iv) hold. Conversely, each of (ii) and (iii) clearly implies (iv), and by Corollary 1.7 (iv) implies that X is normal. $\blacksquare_{1.10}$

This proof of Tiezte's theorem is fairly different from other the author managed to find in the literature. Indeed none of the more common proofs seems to be capable of preserving the Lipschitz condition.

For the last result of this section we shall assume some familiarity with continuous logic. A language for continuous logic was defined in [BU10] to consist of a collection of symbols equipped with uniform continuity moduli, which their interpretations are required to respect. Arbitrary continuity moduli are allowed since, first, this extra generality creates no additional difficulties, and second, even if we had required all symbols to be, say, 1-Lipschitz, arbitrary definable predicates would still merely be uniformly continuous, creating an inconvenient discrepancy. That said, we can now show that in many situations one may assume that the language is indeed 1-Lipschitz.

Theorem 1.11 (Lipschitz Morleyisation)**.** *Let $\mathcal{L}$ be any continuous language. Then the family of Lipschitz $\mathcal{L}$-definable predicates is uniformly dense in the family of all definable predicates and witness distances between types.*

It follows that there exists a 1-Lipschitz relational language $\mathcal{L}'$ of cardinality as $|\mathcal{L}| + \aleph_0$ such that the class of $\mathcal{L}$-structures stands in a bidefinable bijection with an elementary class of $\mathcal{L}'$-structures, and moreover, for any two n-types p and q in this elementary class we have $d(p,q) \leq r$ if and only if $|P^p - P^q| \leq r$ for all n-ary predicate symbols in $\mathcal{L}'$. This bijection necessarily respects elementary embeddings, ultra-products, elementary sub-classes, and so on.

Proof. For the first assertion, for each n the space $\mathrm{S}_n(\mathcal{L})$ of complete n-types in $\mathcal{L}$ is compact, so we may apply Corollary 1.8, observing that the n-ary $\mathcal{L}$-definable predicates are in a natural bijection with the continuous function on $\mathrm{S}_n(\mathcal{L})$, and that this bijection respects uniform distance and uniform continuity moduli. The second assertion follows. $\blacksquare_{1.11}$

2. Completely regular topometric spaces and Stone-Čech compactification

Let $\{X_i \colon i \in I\}$ be a family of topometric spaces. We equip the set $\prod_{i\in I} X_i$ with the product topology and the supremum metric $d(\bar{x}, \bar{y}) = \sup\{d(x_i, y_i) \colon i \in I\}$. One verifies easily the result is indeed a topometric space which we call the *product topometric structure*.

In particular we obtain large compact topometric spaces of the form $[0,\infty]^I$, and we claim that these are in some sense universal, meaning that every compact topometric space embeds in one of those. Similarly, every bounded compact topometric (i.e., of finite diameter) can be embedded in $[0,M]^I$, and up to re-scaling in $[0,1]^I$. In fact we shall show that every completely regular topometric space embeds in such a space, obtaining a Stone-Čech compactification.

Say that a family of functions $\mathcal{F} \subseteq \mathbf{C}^X$ *separates points from closed sets* if for every closed set $F \subseteq X$ and $x \in X \smallsetminus F$, there is a function $f \in \mathcal{F}$ which is constant on F and takes some different value at x.

Fact 2.1. *Let X be a Hausdorff topological space, $\mathcal{F} \subseteq \mathcal{C}(X)$ a family separating points from closed sets. Then the map $\theta\colon X \to \mathbf{C}^{\mathcal{F}}$ defined by $x \mapsto (f \mapsto f(x))$ is a topological embedding.*

Proof. This is fairly standard. First of all $\mathcal{F}$ separates points so θ is injective. To see that θ is continuous, it is enough to consider a sub-basic open set $U = \pi_f^{-1}(V) \subseteq \mathbf{C}^{\mathcal{F}}$, where $V \subseteq \mathbf{C}$ is open and π_f is the projection on the fth coördinate. Then $\theta^{-1}(U) = f^{-1}(V)$ is open. In order to show that θ is a homeomorphism with its image it will be enough to show that for $F \subseteq X$ closed and $x \notin F$ there is a closed set $F' \subseteq \mathbf{C}^{\mathcal{F}}$ such that $\theta(F) \subseteq F'$ and $\theta(x) \notin F'$. Since $\mathcal{F}$ separates points from closed sets there is $f \in \mathcal{F}$ such that $f{\restriction}_F = t$ and $f(x) \neq t$. Then $F' = \{\bar{y} \in \mathbf{C}^{\mathcal{F}} : y_f = t\}$ will do. $\blacksquare_{2.1}$

Definition 2.2. Let X be a topometric space. Say that a family of functions $\mathcal{F} \subseteq \mathcal{C}_{\mathcal{L}(1)}(X)$ is *sufficient* if

(i) It separates points and closed sets.
(ii) For $x, y \in X$ we have

$$d(x,y) = \sup\big\{|f(x) - f(y)| : f \in \mathcal{F}\big\}.$$

(Clearly, $\geq$ always holds.)

A topometric space X is *completely regular* if $\mathcal{C}_{\mathcal{L}(1)}(X)$ is sufficient. This is clearly equivalent to $\mathcal{C}_{\mathcal{L}(1)}(X, \mathbf{R}^+)$ being sufficient.

In view of Fact 2.1 we may say that a topometric space X is completely regular if $\mathcal{C}_{\mathcal{L}(1)}(X)$ captures both the topological structure and the metric structure of X.

Proposition 2.3. (i) *Every normal topometric space is completely regular.*
(ii) *Every subspace of a completely regular space is completely regular.*
(iii) *Let X be a maximal topometric space. Then it is topologically completely regular if and only if it is topometrically completely regular.*

Proof. The first item follows from Corollary 1.7, keeping in mind that since the metric of a topometric space X refines its topology, if $F \subseteq X$ is closed and $x \notin F$ then $d(x, F) > 0$.

For the second item, assume that X is completely regular, $Y \subseteq X$. If $F \subseteq Y$ is closed then $F = Y \cap \overline{F}$, where $\overline{F}$ is the closure in X. Thus if $x \in Y \smallsetminus F$ then $x \in X \smallsetminus \overline{F}$, so there is a 1-Lipschitz continuous function separating $\overline{F}$ from x, and its restriction to Y is continuous and 1-Lipschitz as well. The same argument works for witnessing distances.

The last item follows from the fact that every function from a maximal topometric space to $[0,1]$ is 1-Lipschitz. $\blacksquare_{2.3}$

Proposition 2.4. *Let X be a completely regular topometric space and let $\mathcal{F} = \mathcal{C}_{\mathcal{L}(1)}(X, \mathbf{R}^+)$. Then the map $\theta\colon X \to (\mathbf{R}^+)^{\mathcal{F}}$ from Fact 2.1 is a topometric embedding, i.e., an isometric homeomorphic embedding.*

Proof. Immediate from the definitions. $\blacksquare_{2.4}$

Corollary 2.5. *Every completely regular topometric space X (and thus in particular every normal or compact one) embeds in some power of $[0,\infty]$. If in addition X is bounded, say of diameter 1, then it embeds in a power of $[0,1]$.*

Proof. We just have to show the last part. Indeed let $\theta\colon X \to [0,\infty]^I$ be any embedding. Define $\theta'\colon X \to [0,\infty]^I$ by $\theta'(x)(i) = \theta(x)(i) - \inf\{\theta(y)(i) : y \in X\}$ (here $\infty - \infty = d(\infty,\infty) = 0$). Then θ' is an embedding as well, and $\bar{0} \in \theta(X)$. If X is bounded of diameter 1 then $\theta(X) \subseteq [0,1]^I$. $\blacksquare_{2.5}$

Theorem 2.6. *A topometric space admits a compactification if and only if it is completely regular.*

Proof. If X is completely regular then we can identify it with a subspace of $[0,\infty]^I$, and then its closure there is a compactification. Conversely, assume X admits a compactification $\bar{X}$. Then $\bar{X}$ is completely regular, whereby so is X. $\blacksquare_{2.6}$

Theorem 2.7. *Let X be completely regular. Then it admits a compactification βX satisfying the following universal property: Every 1-Lipschitz continuous function $f\colon X \to [0,\infty]$ can be extended to such a function on βX (and the extension is unique).*

Moreover, βX is unique up to a unique isomorphism (i.e., isometric homeomorphism) and satisfies the same universal property with any compact topometric space Y instead of $[0,\infty]$.

Proof. Let $\mathcal{F} = \mathcal{C}_{\mathcal{L}(1)}(X, \mathbf{R}^+)$ and let $\theta\colon X \to (\mathbf{R}^+)^{\mathcal{F}} \subseteq [0,\infty]^{\mathcal{F}}$ be as in Proposition 2.4. Identify X with $\theta(X)$ and let βX be its closure in $[0,\infty]^{\mathcal{F}}$.

For $f \in \mathcal{F}$, let $\pi_f\colon [0,\infty]^{\mathcal{F}} \to [0,\infty]$ be the projection on the fth coordinate. Then $\pi_f \circ \theta = f$, so $\pi_f\colon \beta X \to [0,\infty]$ is as required. Given $f \in \mathcal{C}_{\mathcal{L}(1)}(X,[0,\infty])$ and $n \in \mathbf{N}$, the truncation $f \wedge n\colon X \to [0,n]$ belongs to $\mathcal{F}$ and the sequence $\pi_{f\wedge n}$ is increasing, converging point-wise to some $g\colon \beta X \to [0,\infty]$. The collection of open subsets of $[0,\infty]$ which are either bounded or contain ∞ forms a base. For such an open set U there is n such that either $[n,\infty] \subseteq U$ or $U \subseteq [0,n]$, and in either case $g^{-1}(U) = (f\wedge n)^{-1}(U)$ is open. Thus g is continuous. (Of course we could have also let $\mathcal{F} = \mathcal{C}_{\mathcal{L}(1)}(X,[0,\infty])$ to begin with.)

Now let Y be any compact topometric space. Then Y embeds in $[0,\infty]^J$ for some J. If $f \in \mathcal{C}_{\mathcal{L}(1)}(X,Y)$ then $\pi_j \circ f \in \mathcal{C}_{\mathcal{L}(1)}(X,[0,\infty])$ for $j \in J$ and thus extends to $g_j \in \mathcal{C}_{\mathcal{L}(1)}(\beta X,[0,\infty])$. Let $g = (g_j)\colon \beta X \to [0,\infty]^J$, so $g{\restriction}_X = f$. Then $g(X) \subseteq Y$, X is dense in βX and Y is closed in $[0,\infty]^J$, so $g(\beta X) \subseteq Y$ as required.

The uniqueness of an object satisfying this universal property is now standard. $\blacksquare_{2.7}$

In other words, for every compact Y the restriction $\mathcal{C}_{\mathcal{L}(1)}(\beta X, Y) \to \mathcal{C}_{\mathcal{L}(1)}(X,Y)$ is bijective.

Definition 2.8. The compactification βX, if it exists (i.e., if X is completely regular) is called the *Stone-Čech compactification* of X.

Automorphism groups of metric structures probably form the most natural class of examples of non (locally) compact topometric spaces. They are easily checked to be completely regular.

Proposition 2.9. *Let $\mathcal{M}$ be a metric structure and let $G = \mathrm{Aut}(\mathcal{M})$, equipped with the topology $\mathscr{T}$ of point-wise convergence and with the distance d_u of uniform convergence. Then $(G,\mathscr{T},d_u)$ is a completely regular topometric space.*

Similarly, if $(G,\mathscr{T})$ is any metrisable topological group, with left-invariant compatible distance d_L, and $d_u(f,g) = \sup_h d_L(fh,gh)$, then $(G\mathscr{T},d_u)$ is a completely regular topometric space.

Proof. Since $d_u(f,g) = \sup_{a\in M} d(fa,ga)$, and for each a the function $(f,g) \mapsto d(fa,ga)$ is continuous, d_u is lower semi-continuous. Assume that $d_u(f,g) > r$. Then there exists $a \in M$ such that $d(fa,ga) > r$, and we may define $\theta(x) = d(fa,xa)$. Then θ is continuous and 1-Lipschitz (by definition of point-wise and uniform convergence). In addition, $\theta(f) = 0$ and $\theta(g) > r$. Thus continuous 1-Lipschitz functions witness distances, and it follows that d_u is lower semi-continuous. Now let U be a topological neighbourhood of f. Then there is a finite tuple $\bar a \in M^n$ and $\varepsilon > 0$ such that U contains the set

$$U_{\bar a, f\bar a, \varepsilon} = \{h\colon d(h\bar a, f\bar a) < \varepsilon\}.$$

Then the function $\rho(x) = d(f\bar a, x\bar a)$ separates f from $G \setminus U$.

A similar reasoning applies to the case of an abstract group (acting on itself on the left). In fact, when G is completely metrisable then this case can be shown to be a special case of the first, and every metrisable group can be embedded in a completely metrisable one. $\blacksquare_{2.9}$

Question 2.10. Are automorphism groups of metric structures topometrically normal? In other words, do continuous 1-Lipschitz functions witness distance between closed sets?

Most topometric spaces one would encounter, such as compact ones (e.g., type spaces) or automorphism groups, are (metrically) complete. If X is an incomplete topometric space then the metric structure carries obviously over to the completion $\hat X$, and it is legitimate to ask whether, or how, the topological structure carries there as well. Let us concentrate on the case where X is completely regular.

Definition 2.11. Let X be a completely regular topometric space. We equip its completion $\hat X$ with the least topology such that for every $f \in \mathcal{C}_{\mathcal{L}(1)}(X)$, the unique 1-Lipschitz extension of f to $\hat f\colon \hat X \to \mathbf{C}$ is continuous. In other words, we define it so that the restriction map $\mathcal{C}_{\mathcal{L}(1)}(\hat X) \to \mathcal{C}_{\mathcal{L}(1)}(X)$ is a bijection.

Lemma 2.12. *Let X be a completely regular topometric space. Then so is $\hat X$.*

Proof. The Stone-Čech compactification βX is compact and therefore complete, and the canonical identification of $\hat X$ with a subset of βX is homeomorphic. $\blacksquare_{2.12}$

The topometric structure we put on $\hat X$ is clearly the strongest possible regular one, and it is natural to ask whether it is unique. For a positive result in this direction, let us consider the following two conditions on a topometric space X:

($*$) For every open set $U \subseteq X$ and $r > 0$, the open metric neighbourhood $B(U, r)$ is (topologically) open.

($**$) For every open set $U \subseteq X$ and $r > 0$ we have $\overline{U}^d \subseteq B(U, r)^\circ$.

Clearly ($*$) implies ($**$).

Proposition 2.13. *Let X be a completely regular topometric space in which condition ($**$) holds, and let $X_0 \subseteq X$ be a metrically dense subspace. Then every $f \in \mathcal{C}_{\mathcal{L}(1)}(X_0)$ extends to $\hat{f} \in \mathcal{C}_{\mathcal{L}(1)}(X)$.*

Proof. Let $f \in \mathcal{C}_{\mathcal{L}(1)}(X_0)$. Then it extends uniquely to a 1-Lipschitz function $\hat{f}\colon X \to \mathbf{C}$, and all we need to show is that $\hat{f}$ is continuous at every $x \in X$. Assuming, as we may, that $\hat{f}(x) = 0$, let $U = \{y \in X_0 \colon |f(y)| < \varepsilon\}$ for some $\varepsilon > 0$. Then $U \subseteq X_0$ is open, so of the form $V \cap X_0$ for some open $V \subseteq X$. Since $x \in \overline{V}^d$, by ($**$) we have $x \in B(V, \varepsilon)^\circ$. Now let $w \in B(V, \varepsilon)$. Then there is $z \in V \cap B(w, \varepsilon)$, and for some $0 < \delta < \varepsilon$ we have $B(z, \delta) \subseteq V$. Since X is dense, there is $y \in B(z, \delta) \cap X_0 \subseteq U$. Thus $|\hat{f}(y)| = |f(y)| < \varepsilon$, so $|\hat{f}(w)| < 3\varepsilon$, which is enough. $\blacksquare_{2.13}$

Lemma 2.14. *Condition ($*$) holds in every topometric space of the form $\prod[s_i, r_i]$. More generally, it holds in every minimal or maximal topometric space, and if it holds in each X_i then it holds in $\prod X_i$. Similarly, if condition ($**$) holds in each X_i then it also holds in $\prod X_i$.*

Proof. Easy. $\blacksquare_{2.14}$

Lemma 2.15. *Condition ($*$) holds in every topometric group. In fact, while we usually require that the distance in a topometric group be biïnvariant, here it is enough that it be invariant on one side.*

Proof. Assume that the distance is left-invariant. Then one checks that $B(U, r) = \bigcup_{d(h,1)<r} Uh$. $\blacksquare_{2.15}$

On the other hands, it is not difficult to construct even compact topometric spaces where the properties discussed in this section fail.

Example 2.16. In [Ben08a, Example 3.11 & Theorem 3.15] an example was given somewhat indirectly of a compact topometric space in which condition ($*$) fails (in the terminology used there, in which the perturbation distance was not *open* or even weakly so).

Example 2.17. We give a more explicit example in which Proposition 2.13 fails (so in particular, so do ($**$) and ($*$)). Let X be the disjoint union of $[0, 1]$ with $\mathbf{N}$, where $[0, 1]$ is equipped with the usual minimal structure (i.e., usual topology and distance), $\mathbf{N}$ is equipped with the discrete topology and $0/1$ distance (which is curiously both maximal and minimal). The distance between any point of $[0, 1]$ and of $\mathbf{N}$ is one, and 0 (hereafter always referring to $0 \in [0, 1]$ and not to $0 \in \mathbf{N}$) is the limit of $\mathbf{N}$. Thus X is a compact topometric space, which can be naturally viewed as a subspace of $[0, 1]^{\mathbf{N}}$ by sending $t \in [0, 1]$ to $(t, 0, 0, \ldots)$, and sending $n \in \mathbf{N}$ to the sequence $(0, 0, \ldots, 0, 1, 1, \ldots)$ consisting of n initial zeroes. Let $U = (0, 1) \subseteq X$ and $0 < r < 1$. Then $\overline{U}^d = [0, 1] = B(U, r)$, while every neighbourhood of 0 must contains members of $\mathbf{N}$, so ($**$) fails. Now let $X_0 = (0, 1) \cup \mathbf{N}$. Then X_0 is metrically dense in X, and the function $\mathbf{1}_{(0,1)}$ is continuous and 1-Lipschitz on X_0, but its 1-Lipschitz extension to X fails to be continuous at 0, failing Proposition 2.13. The topometric structure defined earlier on $\hat{X}_0$ differs from that on X only in that 0 is no longer an accumulation point of $\mathbf{N}$.

3. An abstract characterisation of the set of (continuous) 1-Lipschitz functions

It is a classical fact that for a compact space X, $\mathcal{C}(X)$ is a commutative unital C^*-algebra, and that conversely, every such algebra is of the form $\mathcal{C}(X)$ for a compact X which is moreover unique up to a unique homeomorphism. Since a compact topometric space is completely regular, the distance is captured by the subset $\mathcal{C}_{\mathcal{L}(1)}(X) \subseteq \mathcal{C}(X)$. Here we ask the opposite question, namely, given commutative unital C^*-algebra, which we may already consider to be of the form $\mathcal{C}(X)$ for some compact space X, which subsets of the algebra can be of the form $\mathcal{C}_{\mathcal{L}(1)}(X)$ for some topometric structure on X.

Definition 3.1. Let X be a compact topological space. We say that a set $A \subseteq \mathcal{C}(X)$ is an $\mathcal{L}(1)$-*set* if

(i) It is convex, closed under multiplication by scalars $\alpha \in \mathbf{C}$, $|\alpha| \leq 1$ and under taking the absolute value.

(ii) It separates points in X.

(iii) $\mathbf{C} \subseteq A$.

(iv) If $f \notin A$ then there are two points $x, y \in X$ and some $\varepsilon > 0$ such that for all $g \in \mathcal{C}(X)$, if $|f(x) - g(x)|, |f(y) - g(y)| < \varepsilon$ then $g \notin A$ as well.

Lemma 3.2. *Let X be a compact topological space and $A \subseteq \mathcal{C}(X)$ an $\mathcal{L}(1)$-set. Then A is closed in the topology of point-wise convergence, separates points from closed sets and $A + \mathbf{C} = A$.*

Proof. That A is closed in point-wise convergence follows directly from the last condition of Definition 3.1. Now let $f \in A$ and $\alpha \in \mathbf{C}$. For $0 < \lambda < 1$ we have $\lambda f + (1-\lambda)\frac{\alpha}{1-\lambda} \in A$, and since this converges uniformly to $f + \alpha$ when $\lambda \to 1$ we have $f + \alpha \in \mathbf{C}$. Now let $x \in X$ disjoint from a closed set F. For $y \in F$ there is $f_y \in A$ such that $f_y(x) \neq f_y(y)$. Translating by a constant and taking the absolute value we may assume that $f_y \geq 0$, $f_y(x) = 0$ and $f_y(y) > 0$. By compactness there is a finite family $\{y_i\}_{i<k}$ such that for all $y \in F$ there is $i < k$ for which $f_{y_i}(y) > \frac{1}{2} f_{y_i}(y_i)$. Letting $f = \frac{1}{k}\sum f_{y_i} \in A$ we have $f(x) = 0$ and $f(y) \geq r > 0$ for all $y \in F$. $\blacksquare_{3.2}$

Remark 3.3. Modulo conditions (i)–(iii), condition (iv) of Definition 3.1 is equivalent to

(iv′) If $f \notin A$ then there are two points $x, y \in X$ and some $\varepsilon > 0$ such that for all $g \in \mathcal{C}(X)$, if $|f(x) - g(x) - f(y) + g(y)| < \varepsilon$ then $g \notin A$ as well.

Indeed, (iv′) clearly implies (vi). For the other direction we already know that A is translation invariant, so we may always assume that $f(x) = g(x)$, in which case (iv) and (iv′) are the same.

In pure C^*-algebraic terms, we can express $\mathcal{C}(X \times X)$ as the C^* tensor product $\mathcal{C}(X) \otimes \mathcal{C}(X)$, and define $\delta\colon \mathcal{C}(X) \to \mathcal{C}(X) \otimes \mathcal{C}(X)$ by $\delta f = f \otimes 1 - 1 \otimes f$, i.e., $\delta f(x,y) = f(x) - f(y)$. Since a point in $X \times X$ corresponds to a maximal ideal in $\mathcal{C}(X) \otimes \mathcal{C}(X)$, we obtain that (vi) is further equivalent to

(iv″) If $f \notin A$ then there exists $\varepsilon > 0$ such that the family of all $\varepsilon \dot{-} |\delta f - \delta g|$, in the sense of continuous functional calculus, as g varies over A, generates a proper ideal in $\mathcal{C}(X) \otimes \mathcal{C}(X)$.

Theorem 3.4. *Let X be a compact topological space, $A \subseteq \mathcal{C}(X)$. Then the following are equivalent:*

(i) *The set A is an $\mathcal{L}(1)$-set.*

(ii) *There is a topometric structure (X, d) on X such that $A = \mathcal{L}_{\mathcal{L}(1)}(X)$.*

In this case the metric d is unique and can be recovered by

$$d(x,y) = \sup_{f \in A} |f(x) - f(y)|. \tag{1}$$

Proof. Bottom to top is easy, and (1) follows from Urysohn's Lemma for normal topometric spaces and the fact that a compact topometric space is normal. Assume therefore that A is an $\mathcal{L}(1)$-set, and let us define d by (1).

Clearly d is a pseudo-distance, and is lower semi-continuous being the supremum of continuous functions. Since A separates points from closed sets, d refines the topology, and in particular is a distance (rather than a pseudo-distance). Thus (X, d) is a topometric space, and we view it henceforth as such. It is then immediate from the construction that $A \subseteq \mathcal{C}_{\mathcal{L}(1)}(X)$. Finally, assume that $f \notin A$, and let $x, y \in X$ and $\varepsilon > 0$ be such that if $|f(x) - g(x) - f(y) + g(y)| < \varepsilon$ then $g \notin A$. Since A is closed under multiplication by complex scalar of absolute value ≤ 1, this is only possible if $|f(x) - f(y)| \geq |g(x) - g(y)| + \varepsilon$ for all $g \in A$. It follows that $|f(x) - f(y)| \geq d(x,y) + \varepsilon$, so $f \notin \mathcal{C}_{\mathcal{L}(1)}(X)$, as desired. $\blacksquare_{3.4}$

This is quite different from [Wea99, Theorem 4.3.2], which still seems to be the most closely analogous result therein.

References

[BBM] Itaï Ben Yaacov, Alexander Berenstein, and Julien Melleray, *Polish topometric groups*, submitted, arXiv:1007.3367.

[Ben08a] Itaï Ben Yaacov, *On perturbations of continuous structures*, Journal of Mathematical Logic **8** (2008), no. 2, 225–249, doi:10.1142/S0219061308000762, arXiv:0802.4388.

[Ben08b] ———, *Topometric spaces and perturbations of metric structures*, Logic and Analysis **1** (2008), no. 3–4, 235–272, doi:10.1007/s11813-008-0009-x, arXiv:0802.4458.

[BU10] Itaï Ben Yaacov and Alexander Usvyatsov, *Continuous first order logic and local stability*, Transactions of the American Mathematical Society **362** (2010), no. 10, 5213–5259, doi:10.1090/S0002-9947-10-04837-3, arXiv:0801.4303.

[Mun75] James R. Munkres, *Topology: a first course*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.

[Wea99] Nik Weaver, *Lipschitz algebras*, World Scientific Publishing Co. Inc., River Edge, NJ, 1999.

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