TRANSFER OF PROPERTIES BETWEEN MEASURES AND RANDOM TYPES

ITAÏ BEN YAACOV

Abstract.

1. The case of a general theory

Let $\mathcal{M}$ be a classical structure (although everything we say can be generalised to the case of a continuous structure). Let also $\mathscr{A}$ be an atomless measure algebra. Let $(\mathcal{M} \otimes \mathscr{A})_0$ be the $\mathcal{L}^R$-pre-structure defined as follows. The domain consists of all formal finite sums $\sum_{i<m} m_i \otimes e_i$, also written $\bar{m} \otimes \bar{e}$ or simply $\bar{m}\bar{e}$, where $m_i \in M$ and $\bar{e} = (e_i)_{i<n} \subseteq \mathscr{A}$ is a partition of the identity. If e' is any other event then we identify $\bar{m} \otimes \bar{e}$ with $(\bar{m}, \bar{m}) \otimes (\bar{e} \wedge e', \bar{e} \setminus e')$. In other words, we identify members of $(\mathcal{M} \otimes \mathscr{A})_0$ with other members obtained by refinement of the partition. The reader will not find it difficult to check that the definitions that follow are compatible with this identification. We then define:

$$f\left(\bar{a} \otimes \bar{e}, \bar{b} \otimes \bar{e}, \ldots\right) = \left(f(a_i, b_i, \ldots)\right) \otimes \bar{e},$$
$$\left[\!\left[P\left(\bar{a} \otimes \bar{e}, \bar{b} \otimes \bar{e}, \ldots\right) \right]\!\right] = \bigvee \left\{ e_i \colon P(a_i, b_i, \ldots) \right\} \in \mathscr{A}.$$

We notice that the distance symbol interprets a metric on $(\mathcal{M} \otimes \mathscr{A})_0$, whose completion we call $\mathcal{M} \otimes \mathscr{A}$. We observe that if $\mathcal{M} \vDash T$ then $\mathcal{M} \otimes \mathscr{A} \vDash T^R$. In addition, the original structure $\mathcal{M}$ can be viewed as a subset, a sub-structure in fact, of $\mathcal{M} \otimes \mathscr{A}$ via $m \mapsto m \otimes 1$.

Let μ be an n-ary measure over $\mathcal{M}$. In the sense of $\mathcal{M} \otimes \mathscr{A}$ it is a type over the subset $\mathcal{M}$, but this is not a type over a model in the sense of T^R. Nonetheless, μ admits a *natural extension* to the model $\mathcal{M} \otimes \mathscr{A}$. This natural extension is denoted $\mu \otimes \mathscr{A}$, and is defined by letting, for every $\mathcal{L}$-formula φ:

$$\mathbb{P}\left[\varphi\left(\bar{x}, \sum m_i e_i\right)\right]^{\mu \otimes \mathscr{A}} = \sum \mathbb{P}[e_i] \mathbb{P}^{\mu}\left[\varphi(\bar{x}, m_i)\right].$$

This is only defined for formulae over the parameter set $(\mathcal{M} \otimes \mathscr{A})_0$ and the extended by continuity to the whole structure.

This note answers a few questions about the relation between measures and random types, raised (among others) by Pierre Simon.

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Assume now that μ is *Borel definable* over $A \subseteq M$. In other words, assume that for every $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, $|\bar{y}| = m$, there is a Borel function $D_\mu\varphi\colon \mathrm{S}_m(A) \to [0,1]$ such that $\mathbb{P}^\mu[\varphi(\bar{x}, \bar{b})] = D_\mu\varphi(\bar{b})$ (i.e., $= D_\mu\varphi\big(\mathrm{tp}(\bar{b}/A)\big)$.) If $\mathcal{M}$ realises every type over A then the Borel definition of μ over A is unique. We say that μ is *locally Borel definable* if every function $D_\mu\varphi$ factors via $\mathrm{S}_{\tilde{\varphi}}(A)$, i.e., if $D_\mu\varphi(\bar{b})$ depends only on the $\tilde{\varphi}$-type $\mathrm{tp}_{\tilde{\varphi}}(\bar{b}/A)$. In this case, it is enough to have $\mathcal{M}$ realise every local type over A for the definition of μ to be unique. If the defining functions are continuous, namely, are definable predicates, then they are uniquely determined by μ without any saturation hypothesis for $\mathcal{M}$, and we say that μ is *definable*. If $\boldsymbol{\mathcal{M}}$ is any model of T^R containing $\mathcal{M}$ (e.g., if $\boldsymbol{\mathcal{M}} \succeq \mathcal{M} \otimes \mathscr{A}$) then we can extend μ to a complete type over $\mathbf{M}$, denoted $\mu{\restriction}^{\mathbf{M}}$:

$$\mathbb{P}[\varphi(x, \mathbf{b})]^{\mu\restriction^{\mathbf{M}}} = \int D_\varphi(p)\, d\,\mathrm{tp}(\mathbf{b}/A), \tag{1}$$

where $\mathrm{tp}(\mathbf{b}/A)$ is viewed as a Borel probability measure on $\mathrm{S}_m(A)$. In the case of $\mathcal{M}\otimes\mathscr{A}$, the natural extension $\mu \otimes \mathscr{A}$ and the extension by definition $\mu{\restriction}^{\mathcal{M}\otimes\mathscr{A}}$ coincide.

Recall that a type $p(\bar{x})$ over a model $\mathcal{M}$ is finitely satisfiable in a sub-model $\mathcal{M}_0 \preceq \mathcal{M}$ if it lies in the topological closure in $\mathrm{S}_n(M)$ of the set of types realised in M_0. In continuous logic, this is equivalent to saying that for every formula $\varphi(\bar{x})$ with parameters in M_0, if $\varphi^p = 0$ then there is $\bar{a} \in M_0$ such that $\varphi(\bar{a}) < 1$. This is further equivalent to saying that for every formula $\varphi(\bar{x})$ with parameters in M_0 and every $\varepsilon > 0$ there is $\bar{a} \in M_0$ such that $|\varphi(\bar{a}) - \varphi^p| < \varepsilon$. In the case of randomised structures we need to be careful with this definition, since not every $\mathcal{L}^R$ formula is of the form $\mathbb{P}[\varphi]$ for an $\mathcal{L}$-formula φ: we only know that every $\mathcal{L}^R$ formula can be approximated arbitrarily well by continuous combinations of formulae of the form $\mathbb{P}[\varphi]$. Thus, a type $\mathbf{p}$ over $\boldsymbol{\mathcal{M}} \vDash T^R$ is finitely realised in a sub-model $\boldsymbol{\mathcal{M}}_0 \preceq \boldsymbol{\mathcal{M}}$ if and only if, for every finite family of $\mathcal{L}$-formulae $\varphi_i(x, y)$, $i < m$, every parameter $\mathbf{b} \in \mathbf{M}$ (or in some fixed dense subset of $\mathbf{M}$), and for every $\varepsilon > 0$, there is $\mathbf{a} \in \mathbf{M}_0$ such that

$$\big|\mathbb{P}[\varphi_i(\mathbf{a}, \mathbf{b})] - \mathbb{P}[\varphi_i(x, \mathbf{b})]^{\mathbf{p}}\big| < \varepsilon, \qquad \forall i < m.$$

Proposition 1.1. *Let $\mathcal{M}_0 \preceq \mathcal{M}$, $\mathscr{A}_0 \preceq \mathscr{A}$, and let μ be a measure over $\mathcal{M}$.*

(i) *We have $\mathcal{M}_0 \otimes \mathscr{A}_0 \preceq \mathcal{M} \otimes \mathscr{A}$.*
(ii) *The measure μ is definable over M_0 if and only if the type $\mu \otimes \mathscr{A}$ is definable over M_0.*
(iii) *The measure μ is finitely satisfied in $\mathcal{M}_0$ if and only if the type $\mu \otimes \mathscr{A}$ is finitely satisfied in $\mathcal{M}_0 \otimes \mathscr{A}_0$.*

Proof. The first item is by quantifier elimination for T^R.

For the second item, assume that μ is definable over M_0. Its φ-definition $I_\mu\varphi(\bar{y})$ is a continuous M_0-definable predicate in the sense of T. Therefore, $\mathbb{E}[I_\mu\varphi(\bar{y})]$ is an M_0-definable predicate in the sense of T^R and is the $\mathbb{P}[\varphi]$-definition of $\mu\otimes\mathscr{A}$. Since T^R admits quantifier elimination down to formulae of the form $\mathbb{P}[\varphi]$, we are done. The converse is easy.

For the third item we use the characterisation of finite satisfiability given earlier for $\boldsymbol{\mathcal{M}}_0 = \mathcal{M}_0 \otimes \mathscr{A}_0 \preceq \boldsymbol{\mathcal{M}} = \mathcal{M} \otimes \mathscr{A}$. Since we may restrict ourselves to a dense set of parameters, we may assume that $\mathbf{b} = \bar{b} \otimes \bar{e} = \sum_{j<k} b_j e_j$.

Let $s \in s^{mk}$. Define $\varphi_s(x, \bar{y}) = \bigwedge_{i<m,j<k} \varphi_i^{s_{ij}}(x, y_j)$, where $\varphi^0 = \varphi$, $\varphi^1 = \neg\varphi$. Then choose $a_s \in M_0$, if possible, such that $\varphi_s(a_s, \bar{b})$, i.e., such that $\varphi_i(a_s, b_j) \iff s_{ij} = 0$ for all i, j. Choose also a partition $\bar{f} = (f_s)_{s \in 2^{mk}} \subseteq \mathscr{A}_0$ such that $\mathbb{P}[f_s] = \mathbb{P}^{\mu}[\varphi_s(\bar{x}, \bar{b})]$.

We now have a little problem, since we wish $\bar{f}$ to be independent from $\bar{e}$, but it may well be that no such partition $\bar{f}$ (with the desired probabilities) exists in $\mathscr{A}_0$. Instead, we may construct in $\mathscr{A}_0$ an independent sequence of partitions $(\bar{f}^{\ell})_{\ell \in \mathbb{N}}$ (with the same probabilities as $\bar{f} = \bar{f}^0$). By a superstability argument, each e_i is arbitrarily close to being independent from $\bar{f}^{\ell}$ for ℓ sufficiently big. Since the tuple $\bar{e}$ is finite, there exists ℓ such that

$$|\mathbb{P}[e_i \wedge f_s^{\ell}] - \mathbb{P}[e_i]\mathbb{P}[f_s]| < \varepsilon \mathbb{P}[e_i]\mathbb{P}[f_s], \qquad \forall i, s.$$

We may then replace our original partition $\bar{f}$ with $\bar{f}^{\ell}$. In other words, we may assume that $\bar{f}$ and $\bar{e}$ are arbitrarily close to being independent.

Notice that by hypothesis, if $f_s \neq 0$ then a_s exists, so the expression $\mathbf{a} = \bar{a} \otimes \bar{f}$ makes sense an is a member of $\mathcal{M}_0 \otimes \mathscr{A}_0$. We then have

$$\mathbb{P}[\varphi_i(\mathbf{a}, \mathbf{b})] = \sum_s \mathbb{P}[f_s] - \mathbb{P}[\varphi(x, \mathbf{b})]^{\mathbf{p}} \Big| < \varepsilon, \qquad \forall i < m.$$

$$\begin{aligned} \left|\mathbb{P}[\varphi_i(\mathbf{a}, \mathbf{b})] - \mathbb{P}[\varphi_i(x, \mathbf{b})]^{\mu \otimes \mathscr{A}}\right| &= \left| \sum_{s,j \colon s_{ij}=1} \mathbb{P}[f_s \wedge e_j] - \sum_j \mathbb{P}[e_j]\mathbb{P}^{\mu}\big[\varphi_i(\bar{x}, b_j)\big] \right| \\ &= \left| \sum_{s,j \colon s_{ij}=1} \mathbb{P}[f_s \wedge e_j] - \sum_{s,j \colon s_{ij}=1} \mathbb{P}[e_j]\mathbb{P}[f_s] \right| \\ &\leq \sum_{s,j \colon s_{ij}=1} \big|\mathbb{P}[f_s \wedge e_j] - \mathbb{P}[e_j]\mathbb{P}[f_s]\big| < \varepsilon, \end{aligned}$$

as desired. $\blacksquare_{1.1}$

2. The case of a dependent theory

Proposition 2.1. *Assume $\mathcal{M}_0 \preceq \mathcal{M}$ are models of a dependent theory T, where $\mathcal{M}$ is weakly saturated over $\mathcal{M}_0$. Let also μ be a measure over $\mathcal{M}$, finitely satisfiable over $\mathcal{M}_0$. Finally, let $\mathscr{A}_0$ be any atomless probability algebra. Then*

(i) *The measure μ admits a unique Borel definition over M_0, which is moreover local.*

(ii) *If* $\boldsymbol{\mathcal{M}} \succeq \mathcal{M} \otimes \mathscr{A}_0$ *then the extension by definition* $\mu^{\mathbf{M}}$ *is approximately finitely satisfiable in* $\mathcal{M}_0 \otimes \mathscr{A}_0$*. Conversely, if* $\mu^{\mathcal{M} \otimes \mathscr{A}_0}$ *is approximately finitely satisfiable in* $\mathcal{M}_0 \otimes \mathscr{A}_0$ *then* μ *is finitely satisfiable in* $\mathcal{M}_0$*.*

Proof. Since μ is finitely satisfiable it is invariant over M_0, and by [HP] it admits a Borel definition. This definition is unique by the saturation assumption. In fact, finite satisfiability implies that μ is *locally invariant*, meaning that $\mathbb{P}^\mu[\varphi(\bar{x}, \bar{b})]$ depends only on $\operatorname{tp}_{\tilde{\varphi}}(\bar{b}/M_0)$. Locally of the Borel definition follows.

For the second item, fix a formula $\varphi(x, y)$ and a parameter $\mathbf{b} \in \mathbf{M}$. Let $\nu = \operatorname{tp}_{\tilde{\varphi}}(\mathbf{b}/M_0)$, viewed as a $\tilde{\varphi}$-measure over M_0. By [HP, Lemma 4.8], ν can be approximated up to arbitrary $\varepsilon > 0$ by an average $\frac{1}{k}\sum_{\ell<k} p_\ell$, where $p_\ell \in \mathrm{S}_{\tilde{\varphi}}(M_0)$ are actual types, identified with the corresponding Dirac measures. The same can be done simultaneously with a finite family of formulae $\varphi_i(x, y)$, $i < m$, namely fine k and $p_\ell \in \mathrm{S}_{\bar{y}}(M_0)$ such that $\operatorname{tp}_{\tilde{\varphi}_i}(\mathbf{b}/M_0)$ is approximated up to ε by $\frac{1}{k}\sum_{\ell<k} p_\ell{\restriction}_{\tilde{\varphi}_i}$.

Let $\mathscr{A} \succeq \mathscr{A}_0$ contain an equal k-partition $\bar{e}$, independent of $\mathscr{A}_0$. Let also $\bar{b}_\ell \in M$ realise p_ℓ for each $\ell < k$. Then

$$\left|\varphi_i(\bar{x}, \bar{b} \otimes \bar{e})^{\mu \otimes \mathscr{A}} - \varphi_i(\bar{x}, \mathbf{b})^{\mu{\restriction}^{\mathbf{M}}}\right| < \varepsilon, \qquad i < m.$$

Since we already know that $\mu \otimes \mathscr{A}$ is finitely realised in $\mathcal{M}_0 \otimes \mathscr{A}_0$, we conclude that so is $\mu{\restriction}^{\mathbf{M}}$. ■$_{2.1}$

Commutativity of the product of a definable measure with a finitely realised measure (originally proved by Pillay) follows.

References

[HP] Ehud Hrushovski and Anand Pillay, *On NIP and invariant measures*, .

Itaï Ben Yaacov Université Claude Bernard – Lyon 1, Institut Camille Jordan, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France

URL: `http://math.univ-lyon1.fr/~begnac/`