THE SMALL DENSITY PROPERTY FOR POLISH TOPOMETRIC GROUPS

ITAÏ BEN YAACOV AND JULIEN MELLERAY

ABSTRACT. We develop the basics of an analogue of descriptive set theory for functions on a Polish space X. We use this to define a version of the small index property in the context of Polish topometric groups, and show that Polish topometric groups with ample generics have this property.

1. INTRODUCTION

This paper is a follow-up to [BBM], in which we introduced the notion of a *Polish topometric group*, and defined a notion of *ample generics* in that context. We first recall some basic terminology: a *Polish metric structure* $\mathcal{M}$ is a complete, separable metric space (M,d), along with a family $(R_i)_{i\in I}$ such that each R_i is a uniformly continuous map from some M^{n_i} to $\mathbf{R}$. The automorphism group $\mathrm{Aut}(\mathcal{M})$ is made up of all the isometries of (M,d) which preserve all the relations R_i. When endowed with the pointwise convergence topology τ, $\mathrm{Aut}(\mathcal{M})$ is a Polish group. It is also natural to consider the metric of uniform convergence ∂ defined by

$$\partial(g,h) = \sup\big\{d(gm,hm) \colon m \in M\big\}.$$

This metric ∂ is complete, bi-invariant, and in general not separable. It is also τ-lower semi-continuous, i.e the sets $\{(g,h) \colon \partial(g,h) \le r\}$ are closed for all r. The automorphism groups of Polish metric structures, when endowed with the topology τ and the metric ∂, provide the paradigm for Polish topometric groups, and are ubiquitous in analysis. Of particular interest to us are the unitary group $\mathcal{U}(\ell_2)$ of a complex separable infinite-dimensional Hilbert space, and the isometry group of the Urysohn space $\mathbb{U}$. These are highly homogeneous structures, and this leads to them having "large" diagonal conjugacy classes. The exact notion of largeness we use involves an interplay of the topology and metric, since in both cases above it is known that each conjugacy class is topologically meagre.

Recall from [BBM] that a Polish topometric group (G,τ,∂) has *(topometric) ample generics* if for any $n \in \mathbb{N}$ there exists an uple $(g_1,\ldots,g_n) \in G^n$ such that the ∂-closure of $\{(kg_1k^{-1},\ldots,kg_nk^{-1}) \colon k \in G\}$ is co-meagre in G^n. This definition extends the usual notion of ample generics (obtained when ∂ above is the discrete metric). As mentioned above, natural examples of groups with topometric ample generics do not have ample generics as Polish groups, and it is in fact an open question whether there exist Polish groups with ample generics which are not isomorphic to a closed subgroups of $\mathfrak{S}_\infty$, the group of permutations of the integers, whence the need for this relaxed version of ample generics.

Many automorphism group of highly homogeneous Polish metric structures with meagre conjugacy classes turn out to have ample generics when considered as Polish topometric group when endowed with the topology of pointwise convergence and the metric of uniform convergence. For instance, $\mathcal{U}(\ell_2)$, $\mathrm{Iso}(\mathbb{U})$ and the automorphism group $Aut([0,1],\lambda)$ of the unit interval endowed with the Lebesgue measure all have ample metric generics when endowed with their natural topometric structure.

We are especially interested in the link of ample generics with the *automatic continuity property*; the following theorem ([BBM, Theorem 4.6]) extends to the topometric context an automatic continuity theorem proved by Kechris and Rosendal in [KR07].

Theorem 1.1. *Let (G,τ,∂) be a Polish topometric group with ample generics, H a topological group with uniform Suslin number strictly less than $2^{\aleph_0}$ (e.g a metrisable group of density character $< 2^{\aleph_0}$), and $\varphi \colon G \to H$ a homomorphism such that φ is continuous from (G,∂) to H. Then φ must be continuous from (G,τ) to H.*

2010 *Mathematics Subject Classification.* 22A05; 03E15.

Key words and phrases. topometric group; graded subset; graded subgroup; small index property; small density property.

First author supported by ANR chaire d'excellence junior THEMODMET (ANR-06-CEXC-007) and by the Institut Universitaire de France.

The authors wish to thank Alexander Berenstein for several helpful discussions.

Revision 1218 *of* 14th March 2011.

In the context of subgroups of $\mathfrak{S}_\infty$, the *small index property*, which says that subgroups of index $< 2^{\aleph_0}$ are open, has been extensively studied. Kechris and Rosendal showed that a Polish group G with ample generics always has the small index property; it is a consequence of the automatic continuity theorem above (when ∂ is discrete).

Our aim here is to establish the analogue of the small index property in the topometric context. Since most of the groups in which we are interested do not have proper open subgroups, an "analogous result" involving subgroups in the ordinary sense (which can be proved) would seem too weak. In order to state a stronger result, we replace subsets by functions with values in $[0, \infty]$, which we call *graded subsets*. Then, closed subsets correspond naturally to lower semi-continuous functions, open subsets to upper semi-continuous functions (hence clopen subsets correspond to continuous functions), and one is led to define the notion of a meagre graded subset. We then obtain extensions of some classical elementary results of descriptive set theory (the Kuratowski-Ulam theorem, the Pettis theorem, etc.) to the "graded context".

It turns out that the analogue of a subgroup in that context (i.e, a *graded subgroup*) is a *semi-norm* on G, i.e a map H such that

- $H(1) = 0$;
- $\forall g \in G\ H(g) = H(g^{-1})$;
- $\forall g, g' \in G\ H(gg') \le H(g) + H(g')$.

Such a function H naturally defines a left-invariant pseudo-metric d_H on G, defined by $d_H(g, h) = H(g^{-1}h)$; the *index* of H is simply the density character of the metric space obtained when identifying points g, h such that $d_H(g, h) = 0$. A "graded" analogue of the small index property, which at least on its face seems stronger than the classical version, would then be: whenever H is a left-invariant pseudo-metric on G with density character $< 2^{\aleph_0}$, H must be continuous with respect to τ. A topometric version thereof must take ∂ into account (or else be too strong), and one is led to the following statement, which is our main result.

Theorem 1.2. *Assume that (G, τ, ∂) is a Polish topometric group with ample generics, and that H is a semi-norm on G which is lower semi-continuous with respect to ∂. Then H is continuous with respect to τ.*

It is clear that this result extends the automatic continuity theorem 1.1 when the range group H is metrisable; we say that a topometric group satisfying the conclusion of the above theorem has the *small density property*. Our hope is that, for certain Polish groups (first among them the unitary group $\mathcal{U}(\ell_2)$) one can obtain an outright automatic continuity theorem (i.e remove the assumption of ∂-lower semi-continuity of H). In [BBM] we showed that this is possible for $Aut([0, 1], \lambda)$; we do not know if it is also possible for, say, $\mathcal{U}(\ell_2)$.

2. Preliminaries

We recall that the classical setting for descriptive set theory is that of *Polish spaces*, i.e separable metrisable topological spaces whose topology is induced by a complete metric. Our aim is to "do some topology", or descriptive set theory, where instead of considering subsets of a topological space X we consider functions on X, say valued in $[0, \infty]$. For all the basic facts and theorems of descriptive set theory we use below, we refer the reader to [Kec95]. We start with a few general definitions and facts.

Notation 2.1. For any two sets $A \subseteq X$ and two values a, b, we define $\langle A, a, b\rangle \colon X \to \{a, b\}$ by

$$\langle A, a, b\rangle(x) = \begin{cases} a & x \in A, \\ b & x \notin A. \end{cases}$$

Notation 2.2. Given a function $\varphi \colon X \to [-\infty, \infty]$ and $r \in \mathbf{R}$, we define $\varphi_{<r} = \{x \colon \varphi(x) < r\}$, and similarly for $\varphi_{\le r}$, $\varphi_{>r}$ and $\varphi_{\ge r}$.

Recall that a function $\varphi \colon X \to [-\infty, \infty]$ on a topological space X is *upper (lower) semi-continuous* if the set $\varphi_{<r}$ ($\varphi_{>r}$) is open for all $r \in \mathbf{R}$.

Lemma 2.3. *Let Φ be a family of upper semi-continuous functions on a topological space X. Then* $\inf \Phi \colon x \mapsto \inf\{f(x) \colon f \in \Phi\}$ *is upper semi-continuous as well.*

If X admits a countable base then there exists a countable sub-family $\Phi_0 \subseteq \Phi$ such that $\inf \Phi = \inf \Phi_0$.

Proof. The first assertion is quite standard. For the second, let $\mathcal{B}$ be a countable base for X. We let Ψ consist of all functions of the form $\langle U, q, \infty\rangle$, where $q \in \mathbf{Q}$ and $U \in \mathcal{B}$. Then Ψ is a countable family of upper semi-continuous functions on X, and every upper semi-continuous function φ is the infimum of those $\psi \in \Psi$ greater than φ.

Now, choose $\Phi_0 \subseteq \Phi$ countable such that for each $\psi \in \Psi$, if there is $\varphi \in \Phi$ such that $\varphi \leq \psi$ then there is such φ in Φ_0 as well. Then $\inf \Phi = \inf \Phi_0$. $\blacksquare_{2.3}$

Whenever $A, B \subseteq X$ and $A \setminus B$ is meagre in X we write $A \subseteq^* B$, and if $A \subseteq^* B \subseteq^* A$ then we write $A =^* B$. A subset $A \subseteq X$ is called *Baire measurable* if there exists an open set U such that $A =^* U$. The family of all Baire measurable sets forms a σ-algebra which contains all open sets and therefore all Borel sets.

When X is Polish (or more generally, when it admits a countable base) we define

$$U(A) = \bigcup \{U \text{ open in } X \colon A \supseteq^* U\}.$$

The union is then equal to a countable sub-union, so $A \supseteq^* U(A)$, and $U(A)$ is the largest open set with this property. The set A is then Baire measurable if and only if $A =^* U(A)$, if and only if $A \subseteq^* U(A)$.

Definition 2.4. Let X be a Polish space, $\varphi \colon X \to [-\infty, \infty]$ any function. We define

$$\inf_{x\in X}{}^{*} \varphi(x) = \max\{r \colon \forall^* x\, \varphi(x) \geq r\}, \qquad \sup_{x\in X}{}^{*} \varphi(x) = \min\{r \colon \forall^* x\, \varphi(x) \leq r\},$$

observing that the maximum and minimum are indeed attained.

Proposition 2.5 (Kuratowski-Ulam Theorem for functions). *Let X and Y be Polish spaces, $\varphi \colon X \times Y \to [-\infty, \infty]$ Baire measurable. Then*

$$\inf_{x,y}{}^{*} \varphi(x, y) = \inf_{x}{}^{*} \inf_{y}{}^{*} \varphi(x, y) = \inf_{y}{}^{*} \inf_{x}{}^{*} \varphi(x, y),$$

and the sets

$$\{x \colon y \mapsto \varphi(x, y) \text{ is Baire measurable}\}, \qquad \{y \colon x \mapsto \varphi(x, y) \text{ is Baire measurable}\}$$

are co-meagre in X and Y, respectively.

Proof. The first assertion follows immediately from the set version of the Kuratowski-Ulam Theorem, applied to the Baire measurable set $\{(x, y) \colon \varphi(x, y) \geq r\}$. For the second assertion, we know by the classical version that $A_{r,s} = \big\{x \colon \{y \colon \varphi(x, y) \in (r, s)\} \text{ is Baire measurable}\big\}$ is co-meagre in X for every interval (r, s). The intersection of all such sets, as (r, s) varies over all open intervals with rational endpoints, is co-meagre as well, and is exactly the set $\{x \colon y \mapsto \varphi(x, y) \text{ is Baire measurable}\}$. $\blacksquare_{2.5}$

3. Graded sets

Throughout, let X denote a Polish space.

Definition 3.1. By a *graded subset* of X, denoted $\varphi \sqsubseteq X$, we mean a function $\varphi \colon X \to [0, \infty]$. We say that a graded set φ is *open* ($\varphi \sqsubseteq_o X$) if it is upper semi-continuous as a function. Similarly, it is *closed* ($\varphi \sqsubseteq_c X$) if it is lower semi-continuous, and *clopen* if it is both, namely continuous.

If φ and ψ are two graded subsets of X then we say that $\varphi \sqsubseteq \psi$ if $\varphi \geq \psi$.

For example, one may identify an ordinary subset $A \subseteq X$ with its *zero-indicator function* $\mathbf{0}_A = \langle A, 0, \infty\rangle$. Notice that $\mathbf{0}_A$ is open, i.e., upper semi-continuous (respectively, closed, i.e., lower semi-continuous) if and only if A is open (respectively, closed). Also, we have $\varphi \sqsubseteq X$ if and only if $\varphi \sqsubseteq \mathbf{0}_X$.

One may further restrict graded subsets to values in $[0, 1]$ (in which case we define $\mathbf{0}_A = \langle A, 0, 1\rangle$), or extend to $[-\infty, \infty]$, since these are isomorphic ordered sets, equipped with the order topology, choosing one or the other has essentially no effect on most of our results.

Lemma 3.2. *Let φ be a graded subset of X, and define*

$$U(\varphi) = \inf\{\psi \sqsubseteq_o X \colon \varphi \leq^* \psi\}.$$

Then $\varphi \leq^* U(\varphi)$, $U(\varphi)$ is least u.s.c. with this property, and the following are equivalent:

(i) *We have $\varphi \geq^* U(\varphi)$.*
(ii) *We have $\varphi =^* U(\varphi)$.*
(iii) *There exists a u.s.c. function ψ such that $\varphi =^* \psi$.*
(iv) *As a function, φ is Baire measurable.*

Proof. The first assertion is by Lemma 2.3 and the fact that a Polish space admits a countable base. Then, implication from top to bottom is clear (since every upper semi-continuous function is Borel and therefore Baire measurable).

Assume now that φ is Baire measurable, but for some $\varepsilon > 0$ the set $A = \{x \colon \varphi(x) < U(\varphi)(x) - \varepsilon\}$ is non meagre. Then A is Baire measurable, so $A =^* U(A)$, and since X is a Baire space, $U(A) \neq \varnothing$. Let $\psi = U(\varphi) - \varepsilon \mathbf{1}_{U(A)}$. Then ψ is also u.s.c. and $\varphi \leq^* \psi$, contradicting the minimality of $U(\varphi)$. This contradiction implies that $\varphi \geq^* U(\varphi)$ concluding the proof. $\blacksquare_{3.2}$

In other words, if we defined Baire measurable graded subsets as we did for ordinary subsets, this would coincide with the usual definition of Baire measurable functions. Accordingly, we will call a graded subset satisfying the equivalent conditions above a *Baire-measurable graded subset* of X.

Throughout, let G denote a Polish group, i.e a topological group whose topology is Polish. We now introduce two operations on graded subsets; the operations $\diamond$ reminds one of convolution.

Definition 3.3. For two graded subsets $\varphi, \psi \sqsubseteq G$ we define $\varphi^{\diamond -1}, \varphi \diamond \psi \sqsubseteq G$ by

$$\varphi^{\diamond -1}(x) = \varphi(x^{-1}), \qquad \varphi \diamond \psi(x) = \inf_{h \in G} \varphi(h) + \psi(h^{-1}x).$$

Note that $\diamond$ is associative and has $\mathbf{0}_{\{1_G\}}$ as a neutral element. We observe that for $A, B \subseteq G$, $\mathbf{0}_A^{\diamond -1} = \mathbf{0}_{A^{-1}}$ and $\mathbf{0}_A \diamond \mathbf{0}_B = \mathbf{0}_{A \cdot B}$. Thus, ${}^{\diamond -1}$ and $\diamond$ extend the group operations of G, applied to subsets, to graded subsets (and should be thought of as operations on subsets, rather than as operations on group elements). As expected, we have, for all graded subsets φ, ψ of G, that

$$(\varphi \diamond \psi)^{\diamond -1} = \psi^{\diamond -1} \diamond \varphi^{\diamond -1}.$$

By extension, for $g \in G$ we define $g\varphi = \mathbf{0}_{\{g\}} \diamond \varphi$, namely $(g\varphi)(x) = \varphi(g^{-1}x)$, so $g\mathbf{0}_A = \mathbf{0}_{g \cdot A}$. We then obtain

$$\varphi \diamond \psi(g) = \inf \varphi + g\psi^{\diamond -1}.$$

Proposition 3.4 (Pettis Theorem for graded subsets)**.** *Let G be a Polish group, $\varphi, \psi \sqsubseteq G$ graded subsets. Then $U(\varphi) \diamond U(\psi) \sqsubseteq \varphi \diamond \psi$.*

Proof. It will be enough to show that if $U(\varphi) \diamond U(\psi)(g) < r$ for some $g \in G$ and $r \in \mathbf{R}$ then $\varphi \diamond \psi(g) < r$. Indeed, in this case the set $\{x \colon \big(U(\varphi) + gU(\psi)^{\diamond -1}\big)(x) < r\}$ is a non empty open subset of G, and in particular is not meagre. Since $\varphi + g\psi^{\diamond -1} \leq^* U(\varphi) + gU(\psi)^{\diamond -1}$, we obtain that $\{x \colon (\varphi + g\psi^{\diamond -1})(x) < r\}$ is non meagre, and in particular non empty, so $\varphi \diamond \psi(g) = \inf(\varphi + g\psi^{\diamond -1}) < r$, as desired. $\blacksquare_{3.4}$

Definition 3.5. Let X be a Polish space, $\varphi \sqsubseteq X$ a graded subset.

(i) We say that φ is *meagre* if there exists $r > 0$ such that $\varphi_{<r}$ is meagre, i.e., such that $\forall^* x\, \varphi(x) \geq r$. This implies $U(\varphi) \geq r$, and for Baire measurable φ the two are equivalent.
(ii) We define φ°, the *interior* of φ to be the least u.s.c. ψ greater than φ.
(iii) We define $\overline{\varphi}$, the *closure* of φ to be the greatest l.s.c. ψ less than φ.

One can check that

$$\varphi^\circ(x) = \limsup_{y \to x} \varphi(y), \qquad \overline{\varphi}(x) = \liminf_{y \to x} \varphi(y).$$

Lemma 3.6. *Let $\varphi \sqsubseteq G$ be a non meagre Baire measurable graded subset. Then $(\varphi \diamond \varphi^{\diamond -1})^\circ(1) = 0$*

Proof. Let $\psi = U(\varphi)$. Since φ is Baire measurable and non meagre, we have $\inf \psi = 0$, and thus $\psi \diamond \psi^{\diamond -1}(1) = 2 \inf \psi = 0$. By Proposition 3.4 we have $\psi \diamond \psi^{\diamond -1} \sqsubseteq \varphi \diamond \varphi^{\diamond -1}$, and since $\psi \diamond \psi^{\diamond -1} \sqsubseteq_o G$, we obtain $\psi \diamond \psi^{\diamond -1} \sqsubseteq (\varphi \diamond \varphi^{\diamond -1})^\circ$. Thus $(\varphi \diamond \varphi^{\diamond -1})^\circ(1) = 0$. $\blacksquare_{3.6}$

We said that graded subsets of X are a natural analogue of subsets of X; we pursue this analogy further by introducing the analogue of subgroups in this context.

Definition 3.7. A *graded subgroup* of a Polish group G is a graded subset $H \sqsubseteq G$ satisfying the following properties:

- $H(1) = 0$,
- $H^{\diamond -1} = H$ (i.e., $H(x) = H(x^{-1})$), and
- $H \diamond H \sqsubseteq H$ (i.e., $H \diamond H \geq H$, or equivalently, $H(h) + H(g) \geq H(hg)$).

Notice that even if we did allow negative values for graded subsets, a graded subgroup would still always be non negative. Also, $H(1) = 0$ implies that $H \diamond H \sqsupseteq H$, so in fact $H \diamond H = H$. Finally, if $H \sqsubseteq G$ is a graded subgroup then so is $\overline{H}$, since

$$\overline{H}(g) + \overline{H}(h) = \liminf_{g' \to g, h' \to h} H(g') + H(h') \geq \liminf_{g' \to g, h' \to h} H(g'h') = \overline{H}(gh).$$

Remark 3.8. What we call a graded subgroup of G is usually called a *semi-norm* on G. These correpond to left-invariant pseudo-metrics on G as follows: for a graded subgroup H of G and left-invariant pseudo-metric d on G define

$$d_H(g, h) = H(g^{-1}h), \qquad H_d(g) = d(1, g).$$

Then $d \mapsto H_d$ and $H \mapsto d_H$ are inverses, yielding a natural bijection between graded subgroups of a group G and left-invariant pseudo-distances on G (allowing $+\infty$ as a value for the pseudo-distance, or excluding it for the graded subgroup).

Notice also that $H \sqsubseteq G$ is closed (open) if and only if d_H is, and more generally, $\overline{d_H} = d_{\overline{H}}$ and $d_H^\circ = d_{H^\circ}$.

Lemma 3.9. *Let $H \sqsubseteq G$ be a graded subgroup. Then the following are equivalent*

(i) $\inf H^\circ = 0$.
(ii) *H is clopen in G.*
(iii) *H is open in G.*

Proof. It is enough to show that if $\inf H^\circ = 0$ then H is clopen. First, we observe that this implies that $H^\circ(1) = 0$. Indeed, if $H^\circ(y) < \varepsilon$ then one has $H(y^{-1}) = H(y) < \varepsilon$, and then the fact that H is a graded subgroup gives

$$H^\circ(1) = \limsup_{x \to 1} H(x) \leq \limsup_{x \to 1} H(xy) + H(y^{-1}) = H^\circ(y) + H(y^{-1}) < 2\varepsilon\ .$$

Let us show that H is open, i.e., that $H_{<r}$ is open for all r. Indeed, assume that $H(g) = r - \delta$ for some $\delta > 0$. Let $U = H^\circ_{<\delta}$, which is open and contains the identity by assumption, so gU is a neighbourhood of g. In addition, $H^\circ \sqsubseteq H$ implies that $H(h) < \delta$ for all $h \in U$. Thus, if $h \in U$ then $H(gh) \leq H(g) + H(h) < H(g) + \delta = r$, so $H(x) < r$ for all $x \in gU$.

Now let us show that H is closed, i.e., that $H_{>r}$ is open. Assume this time that $H(g) = r + \delta$ for some $\delta > 0$, and let U be as before. If $h \in U$ then $H(g) \leq H(gh) + H(h^{-1}) < H(gh) + \delta$, so $H(x) > H(g) - \delta \geq r$ for all $x \in gU$, and the proof is complete. $\blacksquare_{3.9}$

Lemma 3.10. *If $\varphi \sqsubseteq_o G$ and $\psi \sqsubseteq G$ then $\varphi^{\diamond -1} \sqsubseteq_o G$ and $\varphi \diamond \psi \sqsubseteq_o G$.*

Proof. The first assertion is obvious. The second assertion is also immediate when one goes back to the definition of $\varphi \diamond \psi$: if $\varphi \diamond \psi(g) < r$ then there is some x such that $\varphi(x) + \psi(x^{-1}g) = r - \delta < r$; the set $V = \{y \colon \psi(y) < \psi(x^{-1}g) + \delta\}$ is open and contains $x^{-1}g$, so xV is open, contains g, and for $h \in xV$ one has $\varphi(x) + \psi(x^{-1}h) < r$. This implies in particular that $\varphi \diamond \psi(h) < r$ for all $h \in xV$. $\blacksquare_{3.10}$

Lemma 3.11. *Let $H \sqsubseteq G$ be a non meagre Baire measurable graded subgroup. Then H is open, and thus clopen.*

Proof. By definition, $H \diamond H^{\diamond -1} = H \diamond H \sqsubseteq H$, implying $(H \diamond H^{\diamond -1})^\circ \geq H^\circ$. By Lemma 3.6 this yields $H^\circ(1) = 0$, so H is clopen by Lemma 3.9. $\blacksquare_{3.11}$

In usual terminology, this lemma says that if a semi-norm H on a Polish group G is Baire-measurable and for any $\varepsilon > 0$ the set of g such that $H(g) \leq \varepsilon$ is non meagre, then H must be continuous.

4. The small density property

Definition 4.1. Let G be a Polish group, $H \sqsubseteq G$ a graded subgroup. Then we define the *index* $[G : H]$ to be the density character of d_H, namely the least cardinal of a d_H-dense subset of G.

We observe that if H is an ordinary subgroup then this agrees with the usual definition of index. Here we are going to be interested in the condition $[G : H] < 2^{\aleph_0}$ (i.e., "H has small index"). Since the cofinality of $2^{\aleph_0}$ is uncountable, the following are equivalent:

(i) $[G : H] < 2^{\aleph_0}$.
(ii) For all $\varepsilon > 0$, G can be covered by fewer than continuum many left translates of $H_{<\varepsilon}$.
(iii) For all $\varepsilon > 0$, a family of disjoint left translates of $H_{<\varepsilon}$ has cardinal smaller than the continuum.

Recall from [BBM] that a *Polish topometric group* is a triple (G, τ, ∂), where (G, τ) is a Polish group, ∂ is a complete bi-invariant metric refining τ and τ-lower semi-continuous, i.e such that for any r the sets $\{(g,h)\colon \partial(g,h) \leq r\}$ are closed in $G \times G$. The canonical example one should have in mind when thinking of this is the isometry group of some Polish metric space (X, d) (or, more generally, the automorphism group of some Polish metric structure), endowed with the topology of pointwise convergence and the supremum metric $\partial(g,h) = \sup\{d(gx, hx)\colon x \in X\}$.

Definition 4.2. We say that a Polish topometric group (G, τ, ∂) has the *small density property* if whenever $H \sqsubseteq G$ is a ∂-closed graded subgroup of index $< 2^{\aleph_0}$ then H is open.

Remark 4.3. We do not call this property the small index property, because even when ∂ is the discrete metric the small density property as defined above is stronger than the usual small index property (which corresponds to left-invariant *ultrametrics* rather than left-invariant metrics).

Theorem 4.4. *Let (G, τ, ∂) be a Polish topometric group admitting ample generics. Then G has the small density property.*

Proof. If A is a subset of G, we will denote by $(A)_{<\varepsilon}$ the et $\{g\colon \exists a \in A \partial(g,a) < \varepsilon\}$.

Let $H \sqsubseteq G$ be ∂-closed graded subgroup of small index. For convenience, let us allow ourselves to consider ∂ as a unary function, $\partial(x) = \partial(x,1) = \partial(1,x)$. For $n \in \mathbf{N}$ define $H^{n\partial} = H \diamond n\partial$, namely (by invariance)

$$H^{n\partial}(g) = \inf_h H(h) + n\partial(h,g).$$

We observe that by symmetry, $H^{n\partial} = n\partial \diamond H$ as well. In addition, $H \diamond H^{n\partial} = H \diamond H \diamond n\partial = H \diamond n\partial = H^{n\partial}$. Clearly $H^{n\partial} \leq H$; it is also easy to check that $\sup H^{n\partial} = \overline{H}^{\partial}$ (that $\sup_n H^{n\partial} \leq \overline{H}^{\partial}$ comes from the existence of a sequence (h_n) such that $\partial(g,h_n) < 1/n^2$ and $H(h_n) \to \overline{H}^{\partial}(g)$; the other direction is similar, using the fact that $\overline{H}^{\partial}(g)$ is the $\liminf$ of $H(h)$ as h ∂-converges to g). From this fact and our assumption on H, we obtain $H = \sup H^{n\partial}$.

By the Kuratowski-Mycielski Theorem, H cannot be meagre, that is, $H_{<\varepsilon}$ is not meagre for any $\varepsilon > 0$. Assume, for a contradiction, that for some $\varepsilon > 0$ the set $G \smallsetminus (H_{<\varepsilon})_\varepsilon$ is non meagre in every open subset of G. Below, for $h \in G$ and $A \subseteq G$ we denote by A^h the set $h^{-1}Ah$. By [BBM, Lemma 3.4] we can find a mapping $a \in 2^\omega \mapsto h_a \in G$ such that if $a, b \in 2^\omega$ are distinct then $\partial\big(H_{<\varepsilon/3}^{h_a}, [G \smallsetminus (H_{<\varepsilon})_\varepsilon]^{h_b}\big) < \varepsilon$, i.e., $\partial\big(H_{<\varepsilon/3}^{h_a h_b^{-1}}, G \smallsetminus (H_{<\varepsilon})_\varepsilon\big) < \varepsilon$. It follows that $h_a h_b^{-1} \notin H_{<\varepsilon/3}$ for all $a \neq b$, implying that $[G : H] = 2^{\aleph_0}$, a contradiction.

Thus, for all $\varepsilon > 0$, the set $(H_{<\varepsilon})_\varepsilon$ is co-meagre in some non empty open set. Applying Pettis' Theorem to $\varepsilon/2$ we see that $(H_{<\varepsilon})_\varepsilon$ contains a neighbourhood of the identity. We remark that this last observation implies (in fact, is equivalent to) $(H^{n\partial})^\circ(1) = 0$ for all n. Indeed, if V is an open set contained in $(H_{<\varepsilon/2n})_{\varepsilon/2n}$ then for all $v \in V$ there is h such that $H(h) < \varepsilon/2n$ and $\partial(h,v) < \varepsilon/2n$, which implies that $H^{n\partial}(v) \leq H(h) + n\partial(h,v) < \varepsilon$.

Now fix n, $\varepsilon > 0$ and a non empty open set $U \subseteq G$. Then there exists a neighbourhood $1 \in V$ such that $H^{n\partial}(V) < \varepsilon/2$. If $g \in G$ then $H^{n\partial}(gV) = (H \diamond H^{n\partial})(gV) < H(g) + \varepsilon/2$. By definition of $\overline{H}$, the set $(H - \overline{H})_{<\varepsilon/2}$ is dense, so we may choose $g \in (H - \overline{H})_{<\varepsilon/2} \cap U$. In particular, $g \in \overline{H}_{>H(g)-\varepsilon/2}$, and this last set is open. Thus $W = gV \cap \overline{H}_{>H(g)-\varepsilon/2} \cap U$ is a non empty open set, and $H^{n\partial} < \overline{H} + \varepsilon$ on W.

Thus, for any open set U there is some nonempty open subset $W \subseteq U$ such that $H^{n\partial} < \overline{H} + \varepsilon$ on W. Hence $H^{n\partial} < \overline{H} + \varepsilon$ on a dense open set for all $\varepsilon > 0$. It follows that $H^{n\partial} \leq^* \overline{H}$ and therefore $H = \sup_n H^{n\partial} \leq^* \overline{H}$. In particular, $U(H) = U(\overline{H})$.

Now, $\overline{H}$ is closed and non meagre (since H is non meagre), so $\overline{H}_{\leq\varepsilon}$ is closed and non meagre, and therefore of non empty interior, for all $\varepsilon > 0$. It follows that $\inf \overline{H}^\circ = 0$ (recall that $\overline{H}^\circ(g) = \limsup_{h \to g} \overline{H}(h)$), so by Lemma 3.9 $\overline{H}$ is clopen. In particular $U(\overline{H}) = \overline{H}$. Thus $U(H) = \overline{H}$, and by Pettis' Theorem $\overline{H} = \overline{H} \diamond \overline{H} = U(H) \diamond U(H) \geq H \diamond H = H$. Thus $H = \overline{H}$ is clopen. $\blacksquare_{4.4}$

It is probably worthwhile to translate this result in the more common language of left-invariant metrics. Assume that (G, τ, ∂) is a Polish topometric group with ample generics, and let ρ be a left-invariant pseudo-metric on G. Let G_ρ denote the metric space obtained by identifying points g, g' such that $\rho(g, g') = 0$, and assume that the density character of G_ρ is strictly less than the continuum. The theorem above says that, under these assumptions, ρ must be continuous (with regard to the topology τ of G) as soon as each set $\{g\colon \rho(g,1) \leq r\}$ is ∂-closed. Even without this assumption on the level sets of

ρ, one can obtain some information: indeed, the function ρ' defined by

$$\rho'(g,h) = \liminf_{g' \xrightarrow[\partial]{} g, h' \xrightarrow[\partial]{} h} \rho(g',h')$$

is still a left-invariant pseudo metric on G, and the density character of the associated metric space is still less than the continuum. The lim inf ensures that the associated graded subgroup is ∂-closed, so our theorem says that ρ' must be continuous with respect to the topology τ.

The result above is insufficient to obtain a full automatic continuity theorem for, say, $\mathcal{U}(\ell_2)$, the unitary group of separable Hilbert space. This is a Polish topometric group with ample generics when endowed with the strong operator topology and the norm-operator metric, i.e

$$\partial(S,T) = \|S-T\|.$$

Actually, we do not even know whether $\mathcal{U}(\ell_2)$ admits a non-trivial homomorphism into $\mathfrak{S}_\infty$, the permutation group of the integers. Such a homomorphism would necessarily be discontinuous; it can exist if and only if $\mathcal{U}(\ell_2)$ admits a non-trivial subgroup of countable index. Note that the results of [FS85] imply that there are no normal subgroups of *finite* index in $\mathcal{U}(\ell_2)$, so there are no nontrivial subgroups of finite index. This says nothing, however, about the existence of subgroups with countable index. Assume that Γ is such a subgroup, and let H_Γ denote the graded subgroup that takes the value 0 on Γ and 1 everywhere else. Then, using the remark following Theorem 4.4, and the fact that the strong operator topology is connected, we see that for all $U \in \mathcal{U}(\ell_2)$ one has

$$\liminf_{T \xrightarrow[\|\cdot\|]{} U} H_\Gamma(T) = 0.$$

In other words, Γ must be norm-dense in $\mathcal{U}(\ell_2)$. We do not know how to obtain more information about Γ; using a standard diagonal argument one may show that there must exist an infinite-dimensional subspace $\mathcal{H}_0$ of ℓ_2 such that Γ contains all unitary operators with support contained in $\mathcal{H}_0$, but this does not seem to be of much help either. So we are left with the following section to ponder:

Does there exist a nontrivial subgroup of $\mathcal{U}(\ell_2)$ which has a countable index?

All we said above would make sense also for the isometry group of the Urysohn space: any countable index subgroup must be dense for the uniform metric, and we do not know whether a nontrivial such subgroup exists.

We should also note here that R. Kallman has proved in [Kal00] that there are plenty of nontrivial homomorphisms from a *finite-dimensional* unitary group into $\mathfrak{S}_\infty$; actually, such homomorphisms separate points for any finite-dimensional linear group.

5. A remark concerning graded subgroups of automorphism groups

We conclude with a natural situation in which open graded subgroups arise. We assume that the reader of this section is familiar with the formalism of continuous logic ([BU10, BBHU08]).

Let $\mathcal{M}$ be a metric structure, $G = \mathrm{Aut}(\mathcal{M})$. If $\mathcal{M}$ is a classical structure then for every member $a \in M^{eq}$, the stabiliser G_a is an open subgroup, and if $\mathcal{M}$ is $\aleph_0$-categorical then every open subgroup is the stabiliser of some (real or imaginary) element. In the metric case, on the other hand, the stabiliser G_a is usually *not* open in G, and again we encounter the need to consider graded subgroups.

Definition 5.1. Let $\mathcal{M}$ be a metric structure, $G = \mathrm{Aut}(\mathcal{M})$. The *(graded) stabiliser* of $a \in M$, still denoted G_a, is defined by $G_a(g) = d(a, ga)$.

The graded stabiliser is, by definition, an open graded subgroup, and the exact stabiliser is $G_{a,\leq 0}$. As in classical logic, the converse holds for $\aleph_0$-categorical structures.

Proposition 5.2. *Let $\mathcal{M}$ be an $\aleph_0$-categorical separable structure, $G = \mathrm{Aut}(\mathcal{M})$, and let $H \sqsubseteq_o G$ be a real-valued open graded subgroup (namely, we exclude $+\infty$ from the range). Then there exists an imaginary $a \in M^{eq}$ such that $H = G_a$.*

Proof. We may consider $M^{\mathbf{N}}$ as a sort, equipped with the distance

$$d(b,c) = \bigvee_n 2^{-n} \wedge d(b_n, c_n).$$

We fix a in this sort which enumerates a dense subset of $\mathcal{M}$. By homogeneity, the set of realisations of $p = \mathrm{tp}(a)$ in $\mathcal{M}$ is exactly $\overline{Ga}$. We observe that $\{G_{a,<\varepsilon}\}_{\varepsilon>0}$ is a base of neighbourhoods of the identity, so for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $G_{a,<\delta(\varepsilon)} < H_{<\varepsilon}$.

Let $\varepsilon > 0$ and $g, g', h, h' \in G$, and assume that $d(ga, g'a)$ and $d(ha, h'a)$ are smaller than $\delta = \delta(\varepsilon/2)$. Then $g^{-1}g', h^{-1}h' \in G_{a,<\delta} \subseteq H_{<\varepsilon/2}$, whereby $|d_H(g,h) - d_H(g',h')| < \varepsilon$. Thus the map $\varphi\colon Ga \times Ga \to \mathbf{R}$ sending $(ga, ha) \mapsto d_H(g,h)$ is well defined and uniformly continuous, and thus extends uniquely to a uniformly continuous function $\varphi\colon \overline{Ga} \times \overline{Ga} \to [0, \infty]$. By the Ryll-Nardzewski Theorem [BU07, Fact 1.14], since φ is uniformly continuous and invariant by automorphism, it is a definable pseudo-metric on the set defined by p (and therefore in particular bounded). By [Ben10], and since by $\aleph_0$-categoricity the set defined by p is definable, φ extends to a definable pseudo-metric on all of $M^{\mathbf{N}}$. (To recall the argument, by the Tietze extension theorem we may extend φ to something definable on all of $M^{\mathbf{N}} \times M^{\mathbf{N}}$, call it $\varphi_0(x, y)$, and then $\varphi_1(x, y) = \sup_{z \models p} |\varphi_0(x, z) - \varphi_0(y, z)|$ is a definable pseudo-metric which agrees with φ on p.) Finally, with $[b]$ denoting the canonical parameter for $\varphi(x, b)$,

$$H(g) = d_H(g, 1) = \varphi(ga, a) = d\big([a], [ga]\big).$$

Thus H is precisely the graded stabiliser of $[a]$. $\blacksquare_{5.2}$

References

[BBHU08] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with Applications to Algebra and Analysis, volume 2 (Zoé Chatzidakis, Dugald Macpherson, Anand Pillay, and Alex Wilkie, eds.), London Math Society Lecture Note Series, vol. 350, Cambridge University Press, 2008, pp. 315–427.

[BBM] Itaï Ben Yaacov, Alexander Berenstein, and Julien Melleray, *Polish topometric groups*, submitted, arXiv:1007.3367.

[Ben10] Itaï Ben Yaacov, *Definability of groups in $\aleph_0$-stable metric structures*, Journal of Symbolic Logic **75** (2010), no. 3, 817–840, doi:10.2178/jsl/1278682202, arXiv:0802.4286.

[BU07] Itaï Ben Yaacov and Alexander Usvyatsov, *On d-finiteness in continuous structures*, Fundamenta Mathematicæ **194** (2007), 67–88, doi:10.4064/fm194-1-4.

[BU10] ———, *Continuous first order logic and local stability*, Transactions of the American Mathematical Society **362** (2010), no. 10, 5213–5259, doi:10.1090/S0002-9947-10-04837-3, arXiv:0801.4303.

[FS85] Che Kao Fong and Ahmed Ramzi Sourour, *Normal subgroups of infinite dimensional linear groups*, Technical Reports (Mathematics and Statistics), University of Victoria, 1985.

[Kal00] Robert R. Kallman, *Every reasonably sized matrix group is a subgroup of S_∞*, Fundamenta Mathematicae **164** (2000), no. 1, 35–40.

[Kec95] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995.

[KR07] Alexander S. Kechris and Christian Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*, Proceedings of the London Mathematical Society. Third Series **94** (2007), no. 2, 302–350, doi:10.1112/plms/pdl007, arXiv:math/0409567.

Itaï Ben Yaacov, Université Claude Bernard – Lyon 1, Institut Camille Jordan, CNRS UMR 5208, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France

URL: http://math.univ-lyon1.fr/~begnac/

Julien Melleray, Université Claude Bernard – Lyon 1, Institut Camille Jordan, CNRS UMR 5208, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France

URL: http://math.univ-lyon1.fr/~melleray/