CONTINUOUS FIRST ORDER LOGIC FOR UNBOUNDED METRIC STRUCTURES

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Abstract. We present an adaptation of continuous first order logic to unbounded metric structures. This has the advantage of being closer in spirit to C. Ward Henson's logic for Banach space structures than the unit ball approach (which has been the common approach so far to Banach space structures in continuous logic), as well as of applying in situations where the unit ball approach does not apply (i.e., when the unit ball is not a definable set).

We also introduce the process of single point *emboundment* (closely related to the topological single point compactification), allowing to bring unbounded structures back into the setting of bounded continuous first order logic.

Together with results from [Benc] regarding perturbations of bounded metric structures, we prove a Ryll-Nardzewski style characterisation of theories of Banach spaces which are separably categorical up to small perturbation of the norm. This last result is motivated by an unpublished result of Henson.

Introduction

Continuous first order logic is an extension of classical first order logic, introduced in [BU] as a model theoretic formalism for metric structures. It is convenient to consider that continuous logic also extends C. Ward Henson's logic for Banach space structures (see for example [HI02]), even though this statement is obviously false: continuous first order logic deals exclusively with *bounded* metric structures, immediately excluding Banach spaces from the picture. This is a technical hurdle which is relatively easy to overcome. What one usually does (e.g., in [BU, Example 4.5] and the discussion that follows it) is decompose a Banach space into a multi-sorted structure, with one sort for, say, each closed ball of radius $n \in \mathbb{N}$. One may further rescale all such sorts into the sort of the unit ball, which therefore suffices as a single sorted structure. The passage between Banach space structures in Henson's logic and unit ball structures in continuous logic preserves such notions as elementary classes, elementary extensions, type-definability of subsets of the unit ball, etc. This approach has allowed so far to translate almost every model theoretic question regarding Banach space structures to continuous logic.

The unit ball approach suffers nonetheless from several drawbacks. One drawback, which served as our original motivation, comes to light in the context of perturbations of metric structures introduced in [Benc]. Specifically, we wish to consider the notion of perturbation of the norm of a Banach space arising from the Banach-Mazur distance. However, any linear isomorphism of Banach spaces which respects the unit ball is necessarily isometric, precluding any possibility of a non trivial Banach-Mazur perturbation. Another drawback of the unit ball approach, also remedied by the tools introduced in the present paper, is that in some unbounded metric structures the unit ball is not a definable set (even though it is always type-definable), so naming it as a sort (and quantifying over it) adds undesired structure. For example,

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this is the case with complete metric valued fields (i.e., of fields equipped with a complete non trivial multiplicative valuation in $\mathbb{R}$), considered in detail in [Benb].

In the present paper we replace the unit ball approach with the formalism of *unbounded continuous first order logic*, directly applicable to unbounded metric structures and in particular to Banach space structures. Using some technical definitions introduced in Section 1, the syntax and semantics of unbounded logic are defined in Section 2. In Section 3 we prove Łoś's Theorem for unbounded logic, and deduce from it a Compactness Theorem inside bounded sets. It follows that the type space of an unbounded theory is locally compact. In Section 4 we show that unbounded continuous first order logic has the same expressive power as Henson's logic of positive bounded formulae.

In order to be able to apply to unbounded structures tools which are already developed in the context of standard (i.e., bounded) continuous logic, we introduce in Section 5 the process of *emboundment*. Trough the addition of a single point at infinity, to each unbounded metric structure we associate a bounded one, to which established tools apply. This method is used in Section 6 to adapt the framework of perturbations, developed in [Benc] for bounded structures, to unbounded ones. In particular, Theorem 6.9 asserts that the Ryll-Nardzewski style characterisation of $\aleph_0$-categoricity up to perturbation [Benc, Theorem 3.5] holds for unbounded metric structures as well.

As an application, we prove in Section 7 a Ryll-Nardzewski style characterisation of theories of Banach spaces which are $\aleph_0$-categorical up to arbitrarily small perturbation of the norm. This result is motivated by an unpublished result of Henson, whom we thank for the permission to include it in the present paper.

Notation is mostly standard. We use a, b, c, ... to denote members of structures, and use x, y, z, ... to denote variables. Bar notation is used for (usually finite) tuples, and uppercase letters are used for sets. We also write $\bar{a} \in A$ to say that $\bar{a}$ is a tuple consisting of members of A, i.e., $\bar{a} \in A^n$ where $n = |\bar{a}|$. When T is an $\mathcal{L}$-theory (whether bounded or unbounded) we always assume that T is closed under logical consequences. In particular, $|T| = |\mathcal{L}| + \aleph_0$ and T is countable if and only if $\mathcal{L}$ is. We shall assume familiarity with (bounded) continuous first order logic, as developed in [BU]. For the parts dealing with perturbations, familiarity with [Benc] is assumed as well. For a general survey of the model theory of metric structures we refer the reader to [BBHU08].

1. Gauged spaces

We would like to allow unbounded structures, while at the same time keeping some control over the behaviour of bounded parts thereof. The "bounded parts" of a structure are given by means of a *gauge*.

Definition 1.1. Let (X, d) be a metric space, $\nu\colon X \to \mathbb{R}$ any function. We define $X^{\nu \leq r} = \{x \in X : \nu(x) \leq r\}$ and similarly $X^{\nu \geq r}$, $X^{\nu < r}$, etc.

(i) We call $X^{\nu \leq r}$ and $X^{\nu < r}$ the *closed* and *open ν-balls* of radius r in X, respectively.
(ii) We say that ν is a *gauge* on (X, d), and call the triplet (X, d, ν) a *(ν-)gauged space* if ν is 1-Lipschitz in d and every ν-ball (of finite radius) is bounded in d.

Note that this implies that the bounded subsets of (X, d) are precisely those contained in some ν-ball.

Remark 1.2. We could have given a somewhat more general definition, replacing the 1-Lipschitz condition with the weaker condition that the gauge ν should be bounded and uniformly continuous on every bounded set. This does not cause any real loss of generality, since in that case we could define

$$d'(x, y) = d(x, y) + |\nu(x) - \nu(y)|.$$

Then ν is 1-Lipschitz with respect to d', and the two metrics d and d' are uniformly equivalent and induce the same notion of a bounded set.

Definition 1.3. Recall that a *(uniform) continuity modulus* is a left-continuous increasing function $\delta\colon (0, \infty) \to (0, \infty)$ (i.e., $\delta(\varepsilon) = \sup_{\varepsilon < \varepsilon'} \delta(\varepsilon'))$.

We say that a mapping $f\colon (X, d_X, \nu_X) \to (Y, d_Y, \nu_Y)$ between two gauged spaces *respects* δ *under* ν if for all $\varepsilon > 0$:

$$\text{(UC}_\nu\text{)} \qquad \begin{array}{l} \nu_X(x), \nu_X(y) < \frac{1}{\varepsilon}, \\ d_X(x,y) < \delta(\varepsilon) \end{array} \quad \Longrightarrow \quad \begin{array}{l} d_Y(f(x), f(y)) \leq \varepsilon, \\ \nu_Y(x) \leq \frac{1}{\delta(\varepsilon)}. \end{array}$$

We say that f is *uniformly continuous under* ν if it respects some δ under ν.

While respecting a given δ under ν depends on the choice of ν, the fact that some δ is respected under ν does not.

Lemma 1.4. *Let X and Y be gauged spaces, $f\colon X \to Y$ a mapping. Then*

(i) *Let $\delta\colon (0, \infty) \to (0, \infty)$ be any mapping, and assume that f respects δ under ν in the sense of* (UC$_\nu$)*. Define $\delta'(\varepsilon) = \varepsilon \wedge \sup_{0<\varepsilon'<\varepsilon} \delta(\varepsilon')$. Then $\delta' \leq \mathrm{id}$ is a continuity modulus and f respects δ' under ν as well. (If we used* sup *alone we could obtain infinite values, whence the need for truncation at ε.)*

(ii) *A mapping $f\colon X \to Y$ between gauged spaces is uniformly continuous under ν if and only if it restriction to every bounded set is uniformly continuous and bounded.*

Proof. Easy. $\blacksquare_{1.4}$

Definition 1.5. A Cartesian product of gauged metric spaces $X = \prod_{i<n} X_i$ is equipped with a gauged metric structure as follows:

$$d(\bar{x}, \bar{y}) = \bigvee_{i<n} d(x_i, y_i), \qquad \nu(\bar{x}) = \bigvee_{i<n} \nu(x_i). \tag{1}$$

In particular, if $n = 0$ then $X = \{*\}$ and $d(*, *) = \nu(*) = 0$.

We also identify $\mathbb{R}^+$ with the gauged space $(\mathbb{R}^+, |x - y|, |x|)$.

Lemma 1.6. *Let X, Y, and so on, denote gauged spaces.*

(i) *The projection mapping $X \times Y \to X$ respects the identity uniformly under ν.*

(ii) *Let $f_i\colon X \to Y_i$, $i < n$, be mappings between gauged spaces, each respecting δ_{f_i} under ν. Then $\bar{f}\colon X \to \prod Y_i$ respects the continuity modulus $\delta_{\bar{f}} = \bigwedge_{i<n} \delta_{f_i}$ under ν. In addition, if $\delta_{f_i} \leq \mathrm{id}$ for all (indeed, for some) $i < n$ then $\delta_{\bar{f}} \leq \mathrm{id}$ as well.*

(iii) *Let X, Y and Z be gauged spaces. Assume that $f\colon X \to Y$ and $g\colon Y \to Z$ respect continuity moduli δ_f and δ_g, respectively, under ν. Assume moreover that $\delta_f, \delta_g \leq \mathrm{id}$. Then $h = g \circ f\colon X \to Z$ respects the continuity modulus $\delta_h = \delta_f \circ \delta_g \circ \delta_f$ under ν. In particular, $\delta_h \leq \mathrm{id}$ is a continuity modulus.*

(iv) *Let X and Y be gauged spaces, and let $f\colon X \times Y \to \mathbb{R}^+$ and $g\colon Y \to \mathbb{R}^+$ mappings which respect δ_f and δ_g under ν, respectively. Assume also that f is* eventually equal *to g, namely that there exists a constant C such that $f(x,y) = g(y)$ whenever $\nu(x) \geq C$. Define*

$$h_1(y) = \sup_{x \in X} f(x,y), \qquad h_1'(y) = g(y) \vee \sup_{x \in X} f(x,y),$$
$$h_2(y) = \inf_{x \in X} f(x,y), \qquad h_2'(y) = g(y) \wedge \inf_{x \in X} f(x,y),$$
$$\delta_h(\varepsilon) = \delta_g(\varepsilon) \wedge \delta_f(\varepsilon \wedge \tfrac{1}{C}).$$

Then $h_i, h_i'\colon Y \to \mathbb{R}^+$ are well defined (i.e., the supremum is always finite) and respect δ_h under ν. Moreover, if either $\delta_f \leq \mathrm{id}$ or $\delta_g \leq \mathrm{id}$ then $\delta_h \leq \mathrm{id}$.

Proof. The first two items are easy.

For the third item we only prove that δ_h is respected under ν. Indeed, let $\varepsilon > 0$, $x, y \in X$, and assume that $\nu(x), \nu(y) < \frac{1}{\varepsilon}$ and $d(x,y) < \delta_h(\varepsilon)$. By the left continuity assumption there are s, t such that: $d(x,y) < \delta_f(s)$, $s < \delta_g(t)$, $t < \delta_f(\varepsilon)$. In particular $s < t < \varepsilon$. Using our hypotheses we obtain from top to bottom:

$$\begin{aligned} &d(x,y) < \delta_f(s), && \nu(x), \nu(y) < \tfrac{1}{\varepsilon} < \tfrac{1}{s}, \\ &d\big(f(x), f(y)\big) \leq s < \delta_g(t), && \nu \circ f(x), \nu \circ f(y) \leq \tfrac{1}{\delta_f(\varepsilon)} < \tfrac{1}{t}, \\ &d\big(h(x), h(y)\big) \leq t < \varepsilon, && \nu \circ h(x), \nu \circ h(y) \leq \tfrac{1}{\delta_g(t)} < \tfrac{1}{s}. \end{aligned}$$

In addition, s could have been chosen arbitrarily close to $\delta_g \circ \delta_f(\varepsilon)$ whereby $\nu \circ h(x) \leq \frac{1}{\delta_g \circ \delta_f(\varepsilon)} \leq \frac{1}{\delta_f \circ \delta_g \circ \delta_f(\varepsilon)} = \frac{1}{\delta_h(\varepsilon)}$, as desired.

For the fourth item, the existence of h_1 and h'_1 follows from the fact that for a fixed y, the function $x \mapsto f(x,y)$ is bounded on bounded sets and eventually constant. We show that h_1 respects δ_h under ν, a similar argument applies to the other functions. Let $y_1, y_2 \in Y$, and assume that $\nu(y_i) < \frac{1}{\varepsilon}$, $d(y_1, y_2) < \delta_h(\varepsilon)$. Let $r = \varepsilon \wedge \frac{1}{C}$, so $\nu(y_i) < \frac{1}{r}$ and $d(y_1, y_2) < \delta_g(\varepsilon) \wedge \delta_f(r)$. We may choose a point $x \in X$ such that $f(x, y_1)$ is arbitrarily close to $h_1(y_1)$. There are two cases to consider:

I. $\nu(x) \geq C$
$$f(x, y_1) = g(y_1) \leq \tfrac{1}{\delta_g(\varepsilon)} \leq \tfrac{1}{\delta_h(\varepsilon)}, \qquad |f(x,y_1) - f(x,y_2)| = |g(y_1) - g(y_2)| \leq \varepsilon,$$

II. $\nu(x) < C \leq \frac{1}{r}$
$$f(x, y_1) \leq \tfrac{1}{\delta_f(r)} \leq \tfrac{1}{\delta_h(\varepsilon)}, \qquad |f(x,y_1) - f(x,y_2)| \leq r \leq \varepsilon.$$

Either way we obtain that $h_1(y_1) \leq \frac{1}{\delta_h(\varepsilon)}$ and that $h_1(y_1) \leq h_1(y_2) + \varepsilon$, which is enough.

If there exists $x \in X$ such that $\nu(x) \geq C$ then $h_1 = h'_1$. If not then when dealing with h'_1 we need to consider the possibility that $h'_1(y_1) = g(y_1)$, which is treated identically to case **I**. The functions h_2 and h'_2 are treated analogously. $\blacksquare_{1.6}$

2. Unbounded continuous logic

We turn to define a $\mathbb{R}^+$-valued variant of continuous logic which can accommodate unbounded metric structures. We shall refer to this logic as *unbounded continuous logic*. The $[0,1]$-valued (or, more generally, bounded) continuous logic defined in [BU] will be referred to here as *standard* or *bounded*.

Definition 2.1. An *unbounded continuous signature* $\mathcal{L}$ consists of the following data:

(i) A set of *relation (or predicate) symbols* and of *function symbols*, each equipped with its *arity* (zero-ary function symbols are also called *constant symbols*).
(ii) For each n-ary symbol s, a continuity modulus $\delta_s \colon (0,\infty) \to (0,\infty)$.
(iii) For each sort S, a distinguished binary predicate symbol d_S called the *distance symbol*, as well as a distinguished unary predicate symbol ν_S called the *gauge symbol*. The subscript S is usually omitted.

We usually write down a signature merely by listing its non distinguished symbols.

Definition 2.2. Let $\mathcal{L}$ be an unbounded signature, and for the sake of simplicity let us assume it is single-sorted. An *(unbounded) $\mathcal{L}$-structure* is a complete metric gauged space $(M, d, \nu) = (M, d^M, \nu^M)$, possibly empty, equipped with interpretation of the symbols:

(i) The interpretation of an n-ary function symbol f is a mapping $f^M \colon M^n \to M$ which respects δ_f under ν.
(ii) The interpretation of an n-ary predicate symbol P is a mapping $P^M \colon M^n \to \mathbb{R}^+$ which respects δ_P under ν.

For this purpose we view M^n with a gauged space (M^n, d, ν) as per Definition 1.5. Similarly, $\mathbb{R}^+$ admits a standard gauge structure $(\mathbb{R}^+, d, \mathrm{id})$.

Thus, restricted to a ν-ball, everything is bounded and uniformly continuous as in bounded continuous logic, and closed ν-balls are metrically closed and therefore complete.

Remark 2.3. If the language contains a constant symbol 0 then the formula $\nu'(x) = d(x, 0)$ can act as an alternative gauge. Indeed, if $r \in \mathbb{R}^+$ then $M^{\nu' \leq r} \subseteq M^{\nu \leq r+\nu(0)}$, since ν is 1-Lipschitz, and conversely $M^{\nu \leq r} \subseteq M^{\nu' \leq \beta_d(r,\nu(0))}$ by definition of an unbounded structure. Thus we can pass between ν-balls and ν'-balls in a way which depends only on $\mathcal{L}$.

In most cases, ν will indeed be equal to $d(x, 0)$.

A *standard* continuity modulus for an n-ary symbol, when $n > 0$, is the function $x \mapsto \frac{x}{n}$. If a symbol s is 1-Lipschitz is each argument and $\nu(x) = d(x, 0)$ then s indeed respects the standard continuity modulus under ν. For a zero-ary symbol the standard continuity modulus is the identity.

Example 2.4. Let $\mathcal{L}$ be a standard (i.e., $[0,1]$-valued) continuous signature as defined in [BU]. In that case we chose to equip each n-ary symbol s with individual continuity moduli $\delta_{s,i}$, $i < n$, one for each argument. Let $\mathcal{L}'$ be the unbounded signature obtained from $\mathcal{L}$ by adding a gauge symbol ν_S for each sort S, and by setting $\delta_s(x) = 1 \wedge \bigwedge_i \delta_{s,i}(\frac{x}{n})$ (for zero-ary s let $\delta_s = 1$). Then every $\mathcal{L}$-structure M can be naturally viewed as an unbounded $\mathcal{L}'$-structure by interpreting all gauges as the constant 0. If $\mathcal{L}$ admits a constant symbol 0 then interpreting $\nu(x) = d(x, 0)$ works as well.

Example 2.5 (Banach spaces). We would like to view Banach spaces as unbounded structures. Let $\mathcal{L} = \{0, +, m_r : r \in \mathbb{Q}\}$, where m_r is unary scalar multiplication by r. We view $\|x\|$ as shorthand for $d(x, 0)$, and take it to be the gauge. Let $\delta_{m_r}(x) = |r|x$ and let all other continuity moduli be standard. Then every real Banach space is naturally an (unbounded) $\mathcal{L}$-structure.

This can be extended to additional structure on the Banach space. For example a complex Banach space also has a function symbol for multiplication by i, while a Banach lattice is given by binary function symbols $\vee$, $\wedge$ (again with standard continuity moduli).

Example 2.6 (Naming constants). Let $\mathcal{L}$ be an unbounded signature, M and $\mathcal{L}$-structure. Let $A \subseteq M$. We define $\mathcal{L}(A)$ as $\mathcal{L} \cup A$, where each $a \in A$ is viewed as a new constant symbol. We equip each symbol a with the uniform continuity modulus $\delta_a = \mathrm{id} \wedge \frac{1}{\nu(a)}$. Then for every $\varepsilon > 0$ we have $\nu(a) \leq \frac{1}{\delta_a(\varepsilon)}$, and we may render M an $\mathcal{L}(A)$-structure by interpreting $a^M = a$.

We now define the syntax of continuous logic. A term is defined, as usual, as either being a variable or a composition of a function symbol with simpler terms. Similarly, an atomic formula is a composition of a predicate symbol with terms. Connectives are continuous functions from $(\mathbb{R}^+)^n$ to $\mathbb{R}^+$, or any convenient family of such functions which is dense in the compact-open topology, i.e., in the topology of uniform convergence on every compact set. We shall use the system $\{1, x \dot{-} y, x+y, x/2\}$ which generates such a dense set through composition. While alternative systems may be legitimate, we shall always require the presence of 1 and $\dot{-}$ in what follows. We point out that as functions from $(\mathbb{R}^+)^n \to \mathbb{R}^+$, all the basic connectives we chose respect their respective standard continuity moduli (according to their arity). As one may expect, every combination of formulae by connectives is a formula.

On the other hand, care is needed when defining quantified formulae. First, $\sup_x \varphi$ could be infinite. Second, even if φ is bounded, we still need a uniform rate of convergence for $\sup_{\nu(x)<C} \varphi \to \sup_x \varphi$ as $C \to \infty$, or else we may run into trouble with compactness as well as with uniform continuity under ν. In Henson's logic of positive bounded formulae [HI02], where the truth values are True/False, one gets around this by restricting quantifiers to bounded balls (and then again, one needs to play around with the radii of the balls when considering approximations). If we tried to do the same thing with continuous quantifiers we could again run into trouble if, say, $\sup_{\nu(x)<r} \varphi < \sup_{\nu(x)\leq r} \varphi$. We shall follow a different path, looking for the simplest *syntactic* conditions on a formula φ that ensure that $\inf_x \varphi$ and $\sup_x \varphi$ are *semantically* legitimate. This approach will allow us nonetheless to recover approximate versions of bounded quantifiers later on.

Definition 2.7. We define formulae by induction, and at the same time we define whether a formula is syntactically eventually constant in a variable x and/or bounded.

- Atomic formulae are defined as above.
  – If φ is atomic and x does not appear in φ, then φ is eventually constant in x.
  – No atomic formula is bounded.
- A combination of formulae by connectives is a formula.
  – If all the components are bounded (respectively, eventually constant in x) then so is the compound formula.
  – If φ is bounded then $\varphi \dot- \psi$ is bounded for any ψ and $\varphi \dot- \nu(x)$ is eventually constant in x.
- If φ is eventually constant in x then $\inf_x \varphi$ and $\sup_x \varphi$ are formulae (but not otherwise).
  – If φ is bounded (respectively, eventually constant in a variable y) then so are $\sup_x \varphi$ and $\inf_x \varphi$. In particular, $\sup_x \varphi$ and $\inf_x \varphi$ are eventually constant in x.

Notice that the formula 1, being a combination of no formulae, is bounded and eventually constant in every variable. Similarly, every dyadic number $r = \frac{k}{2^m} = (\frac{1}{2})^m(1 + \cdots + 1)$ can be viewed as a formula, and is syntactically bounded and constant as such. It follows that for every formula φ, the formula $\varphi \wedge r = r \dot- (r \dot- \varphi)$ is syntactically bounded.

The qualitative syntactic properties of boundedness and eventual constancy can be translated to quantitative information.

Definition 2.8. For every syntactically bounded formula φ we extract a syntactic bound B_φ as follows:

$$\begin{aligned} &\varphi = \theta(\bar\psi),\ \psi_i \text{ bounded:} && B_\varphi = \sup_{\bar x \in \prod [0, B_{\psi_i}]} \theta(\bar x), \\ &\varphi = \psi \dot- \chi, \text{ or } \sup_x \psi, \text{ or } \inf_x \psi,\ \psi \text{ bounded:} && B_\varphi = B_\psi. \end{aligned}$$

Notice that no ambiguity arises for $\psi \dot- \chi$ when both ψ and χ are syntactically bounded.

Similarly, for a formula $\varphi(x, \bar y)$ which is syntactically eventually constant in x we extract a syntactic constancy threshold $C_{\varphi,x} \in \mathbb{R}^+$ and a formula $\varphi(\infty, \bar y)$, whose free variables lie among $\bar y$, and which is intended to agree with $\varphi(x, \bar y)$ once $\nu(x) \geq C_{\varphi,x}$.

$$\begin{aligned} &x \text{ not free in } \varphi: && \varphi(\infty, \bar y) = \varphi, && C_{\varphi,x} = 0, \\ &\varphi = \theta(\bar\psi),\ \psi_i \text{ e.c. in } x: && \varphi(\infty, \bar y) = \theta\big(\psi(\infty, \bar y)\big), && C_{\varphi,x} = \bigvee C_{\psi_i,x}, \\ &\varphi = \psi \dot- \nu(x),\ \psi \text{ bounded:} && \varphi(\infty, \bar y) = 0, && C_{\varphi,x} = B_\psi, \\ &\varphi = \sup_z \psi,\ \psi \text{ e.c. in } x \neq z: && \varphi(\infty, \bar y) = \sup_z \psi(\infty, z, \bar y), && C_{\varphi,x} = C_{\psi,x}. \end{aligned}$$

Again, when cases overlap the definitions agree.

The definition of the semantics can be somewhat delicate. The model for the definition is an unbounded structure M in which elements of arbitrarily high gauge exist (e.g., a non trivial Banach space). In this case the definition is entirely straightforward, namely

$$\begin{aligned} &(f\bar\tau)^M(\bar a) = f^M \circ \bar\tau^M(\bar a) \in M, && (P\bar\tau)^M(\bar a) = P^M \circ \bar\tau^M(\bar a) \in \mathbb{R}^+, \\ &\theta(\bar\varphi)^M(\bar a) = \theta\big(\bar\varphi^M(\bar a)\big) && \text{(where } \theta \text{ is a connective),} \end{aligned}$$

$$\big(\inf_x \varphi(x, \bar a)\big)^M = \inf_{b \in M} \varphi^M(b, \bar a), \qquad \text{idem for } \sup. \tag{Q}$$

Let us state some properties of this model situation, for the time being without proof. First, the interpretation of every term and formula is uniformly continuous under ν (essentially by Lemma 1.6). Second, if φ is syntactically bounded then it is bounded by B_φ. Third, if $\varphi(x,\bar{y})$ is syntactically eventually constant in x then $\varphi(x,\bar{y}) = \varphi(\infty,\bar{y})$ whenever $\nu(x) \geq C_{\varphi,x}$. In this case, $\varphi(x,\bar{a})$ is bounded for every $\bar{a}$, so the interpretation of the quantifiers makes sense and the following holds:

$$(\mathrm{Q}_\infty) \qquad \big(\inf_x \varphi(x,\bar{a})\big)^M = \inf_{b\in M\cup\{\infty\}} \varphi^M(b,\bar{a}), \qquad \text{idem for } \sup.$$

However, we must also take into account structures in which elements of arbitrarily high gauge need not exist. In order for ultra-products to behave reasonably, i.e., in order for Łoś's Theorem to hold, the definition of quantifier semantics in the general case must follow (Q_∞) and *not* (Q). This is illustrated in Remark 3.2 below.

Definition 2.9. Let M be an $\mathcal{L}$ structure. Then terms, atomic formulae and connectives are interpreted naturally, by composition. Quantifiers are interpreted according to (Q_∞), where $\varphi(\infty,\bar{a})$ is understood as per Definition 2.8.

Theorem 2.10. *Let M be an $\mathcal{L}$-structure. Then:*

(i) *All formulae are interpreted as $\mathbb{R}^+$-valued functions on Cartesian powers of M. In particular, in the interpretation of quantified formulae in M all the suprema are finite.*

(ii) *Every term τ and every formula φ are uniformly continuous under ν.*

(iii) *If a formula φ is syntactically constant then $\varphi^M(\bar{a}) \leq B_\varphi$ for all $\bar{a} \in M$.*

(iv) *If a formula $\varphi(x,\bar{y})$ is syntactically eventually constant in x then $\varphi(b,\bar{a}) = \varphi(\infty,\bar{a})$ whenever $\nu(b) \geq C_{\varphi,x}$.*

Proof. We prove this by induction on the complexity of terms and formulae. We observe that if $\varphi(x,\bar{y})$ is syntactically eventually constant in x then $\varphi(\infty,\bar{y})$ is of lesser or equal complexity. Thus, when treating $\sup_x \varphi$ and $\inf_x \varphi$, we may use the induction hypotheses both for $\varphi(x,\bar{y})$ and for $\varphi(\infty,\bar{y})$. We may assume that all the continuity moduli of symbols lie below the identity, and construct as we go continuity moduli below the identity for each term and formula.

The induction step itself now follows immediately from the definitions, the induction hypotheses and Lemma 1.6. $\blacksquare_{2.10}$

We leave it as an exercise to the reader to check that with our choice of connectives, every formula is equivalent to one in prenex normal form (one needs to make sure in particular that the natural transformations towards a prenex form do not violate the restrictions on quantification imposed in Definition 2.7).

It will be convenient later on to have some analogue of the restricted quantifier $\sup_{\nu(x)\leq r} \varphi$ (which is not part of our language). Let us assume that φ is syntactically bounded and let $k = \lceil B_\varphi \rceil$, namely the least integer syntactic bound for φ. We observe that for a dyadic r, the formula $\varphi \dot{-} (\nu(x) \dot{-} r)$ is equivalent to $(\varphi + (\nu(x) \wedge r)) \dot{-} \nu(x)$ which is syntactically bounded and eventually constant in x. It follows that for every natural $m > 0$ the formula $\varphi \dot{-} m(\nu(x) \dot{-} r)$ is equivalent to one which is syntactically bounded and eventually constant in x. Let $0 < r < r'$, and find the least m such that we can write $r \leq s = \ell 2^{-m} < (\ell+1)2^{-m} \leq r'$, and choose the least possible s. Define:

$$\varphi{\downarrow}^{x\leq r,r'} = \varphi \dot{-} k2^m(\nu(x) \dot{-} s), \qquad \sup_x{}^{r,r'} \varphi = \sup_x \varphi{\downarrow}^{x\leq r,r'},$$

$$\varphi{\uparrow}^{x\leq r,r'} = k \dot{-} (k \dot{-} \varphi){\downarrow}^{x\leq r,r'}, \qquad \inf_x{}^{r,r'} \varphi = \inf_x \varphi{\uparrow}^{x\leq r,r'}.$$

Both formulae on the left are syntactically bounded and eventually constant in x, so the expressions on the right are indeed formulae. By construction we always have $\varphi{\downarrow}^{x\leq r,r'} \leq \varphi$, and in addition $\varphi{\downarrow}^{x\leq r,r'} = \varphi$

when $\nu(x) \leq r$ and $\varphi\downarrow^{x\leq r,r'} = 0$ when $\nu(x) \geq r'$. Thus $\sup_{\nu(x)\leq r} \varphi \leq sup_x^{r,r'} \varphi \leq \sup_{\nu(x)<r'} \varphi$. Similarly, $\inf_{\nu(x)\leq r} \varphi \geq \inf_x^{r,r'} \varphi \geq \inf_{\nu(x)<r'} \varphi$.

We may further extend these abbreviations to the case where φ is not syntactically bounded by truncating it at 1, defining $\varphi\downarrow^{x\leq r,r'} = (\varphi \wedge 1)\downarrow^{x\leq r,r'}$ (and proceeding as above). This will only be used in conditions of the form $\sup_x^{r,r'} \inf_y^{s,s'} \ldots \varphi = 0$, whose satisfaction does not depend on our particular choice of constant at which we truncate.

3. Łoś's Theorem, compactness and theories

Let $\mathcal{L}$ be an unbounded signature, $\{M_i : i \in I\}$ a family of $\mathcal{L}$-structures and $\mathscr{U}$ an ultra-filter on I. Let $I_0 = \{i : M_i \neq \varnothing\}$. If $I_0 \in \mathscr{U}$ define

$$N_0 = \left\{ (a_i) \in \prod_{i\in I_0} M_i : \lim_{\mathscr{U}} \nu^{M_i}(a_i) < \infty \right\},$$

otherwise $N_0 = \varnothing$. Alternatively, one may introduce a new formal element ∞ with $\nu(\infty) = +\infty$, and define

$$N_0 = \left\{ (a_i) \in \prod_{i\in I} \bigl(M_i \cup \{\infty\}\bigr) : \lim_{\mathscr{U}} \nu^{M_i}(a_i) < \infty \right\}.$$

Under this definition a member $(a_i) \in N_0$ can have few (according to $\mathscr{U}$) coordinates which are equal to ∞ and which may be ignored in the definitions that follow. Either approach leads to the same construction.

For a function symbol f or predicate symbol P, and arguments $(a_i), (b_i), \ldots \in N_0$, define:

$$f^{N_0}\bigl((a_i), (b_i), \ldots\bigr) = (f^{M_i}(a_i, b_i, \ldots)),$$
$$P^{N_0}\bigl((a_i), (b_i), \ldots\bigr) = \lim_{\mathscr{U}} P^{M_i}(a_i, b_i, \ldots).$$

Note that by definition of N_0, the values of $P^{M_i}(a_i, b_i, \ldots)$ are bounded on a large set of indexes, so $\lim_{\mathscr{U}} P^{M_i}(a_i, b_i, \ldots) \in \mathbb{R}^+$. It is now straightforward verification that N_0 is an $\mathcal{L}$-pre-structure, i.e., that it verifies all the properties of a structure with the exception that d^{N_0} might be a pseudo-metric and needs not be complete. Let $N = \widehat{N}_0$ be the associated $\mathcal{L}$-structure, obtained by dividing by the zero distance equivalence relation and passing to the metric completion. We call N the *ultra-product of* $\{M_i : i \in I\}$ *modulo* $\mathscr{U}$, denoted $\prod M_i/\mathscr{U}$. The image in N of $(a_i) \in N_0$ will be denoted $[a_i]$. (Compare with the construction of ultra-products of Banach spaces in [HI02] and of bounded continuous structures in [BU].)

Theorem 3.1 (Łoś's Theorem)**.** *For every formula $\varphi(\bar{x})$ and $[a_i], [b_i], \ldots \in \prod M_i/\mathscr{U}$:*

$$\varphi([a_i], [b_i], \ldots)^{\prod M_i/\mathscr{U}} = \lim_{\mathscr{U}} \varphi(a_i, b_i, \ldots)^{M_i}.$$

Proof. Mostly as for bounded logic. The only significant difference is in the treatment of quantifiers, which we sketch below.

If $\lim_{\mathscr{U}} \inf_x \varphi(x, a_i, \ldots)^{M_i} < r$ then there is a large set on which $\inf_x \varphi(x, a_i, \ldots)^{M_i} < r$ and we can find witnesses b_i there (possibly the formal infinity) such that $\varphi(b_i, a_i, \ldots)^{M_i} < r$. If $\nu(b_i) \leq C_{\varphi,x}$ on a large set then $\nu([b_i]) \leq C_{\varphi,x}$, so in particular $[b_i]$ belongs to the ultra-product and

$$\inf_x \varphi(x, [a_i], \ldots) \leq \varphi([b_i], [a_i], \ldots) = \lim_{\mathscr{U}} \varphi(b_i, a_i, \ldots) \leq r.$$

If, on the other hand, $b_i = \infty$ or $\nu(b_i) \geq C_{\varphi,x}$ on a large set then

$$\inf_x \varphi(x, [a_i], \ldots) \leq \varphi(\infty, [a_i], \ldots) = \lim_{\mathscr{U}} \varphi(\infty, a_i, \ldots) = \lim_{\mathscr{U}} \varphi(b_i, a_i, \ldots) \leq r.$$

Conversely, assume that $\inf_x \varphi(x, [a_i], \ldots) < r$. Then again, either there is $[b_i]$ such that $\varphi([b_i], [a_i], \ldots) < r$ or $\varphi(\infty, [a_i], \ldots) < r$, and in either case $\lim_{\mathscr{U}} \inf_x \varphi(x, a_i, \ldots) \leq r$. $\blacksquare_{3.1}$

Remark 3.2. Łoś's Theorem might fail if our semantic interpretation did not take the value at infinity into account. For example, consider the sentence $\varphi = \inf_x(1 \dot{-} \nu(x))$. Let M_n be the structure consisting of two points, $\nu(a_{n,0}) = 0$, $\nu(a_{n,1}) = n$. Then the ultra-product contains a single point $a_0 = [a_n, 0]$, $\nu(a_0) = 0$, and we would have $\varphi^{M_n} = 0$ for all $n \geq 1$ and yet $\varphi^{\prod M_n/\mathscr{U}} = 1$.

Worse still, if M_n consisted only of $a_{n,1}$ then $\prod M_i/\mathscr{U}$ would be empty, making the naïve interpretation of quantifiers meaningless. An empty ultra-product can also be obtained with unbounded structures, for example $M_n = E \smallsetminus B(n)$ where E is a Banach space and $B(n)$ is its open ball of radius n. (These and other pathological examples were pointed out to the originally over-optimistic author by C. Ward Henson.)

Definition 3.3. Say that a family of conditions $\Sigma = \{\varphi_i \leq r_i : i \in \lambda\}$ is *approximately finitely satisfiable* if for every finite $w \subseteq \lambda$ and $\varepsilon > 0$, the family $\Sigma_0 = \{\varphi_i \leq r_i + \varepsilon : i \in w\}$ is satisfiable.

Corollary 3.4. *If a set of sentential conditions (i.e., conditions without free variables) is approximately finitely satisfied in a family of structures, then it is satisfied in some ultra-product of these structures.*

Proof. Standard. $\blacksquare_{3.4}$

Corollary 3.5 (Bounded compactness for unbounded continuous logic)**.** *Let $\mathcal{L}$ be an unbounded signature, $r \in \mathbb{R}^+$, and let Σ be a family of conditions in the free variables $x_{<n}$. Then $\Sigma \cup \{\nu(x_i) \leq r : i < n\}$ is satisfiable of and only if it is approximately finitely satisfiable.*

As usual, a theory is a set of sentential conditions. The complete theory of a structure M, elementary equivalence and elementary embeddings are defined as usual.

Corollary 3.6. *Two structures M and N are elementarily equivalent if and only if M embeds elementarily into an ultra-power of N.*

Proof. One direction is clear. For the other we observe that if M and N are elementarily equivalent, then the elementary diagram of M is approximately finitely satisfiable in N. Indeed, let $\bar{a} \in M$ and say that $\varphi(\bar{a}) = 0$. Let also $\varepsilon > 0$ and $r = \nu(\bar{a})$. Then $N \vDash \inf_{\bar{x}}^{r,r+\varepsilon} \varphi(\bar{x}) = 0$, so there are $\bar{b} \in N$ such that $\nu(\bar{b}) < r + \varepsilon$ and $\varphi(\bar{b}) < \varepsilon$. $\blacksquare_{3.6}$

We could prove an analogue of the Shelah-Keisler theorem that if N and M are elementarily equivalent then they have isomorphic ultra-powers. We give a more elementary proof of a lesser result, which will suffice just as well later on.

Lemma 3.7.
(i) *Two models M and N are elementarily equivalent if and only if there are sequences $M = M_0 \preceq M_1 \preceq \ldots$ and $N = N_0 \preceq N_1 \preceq \ldots$ where each M_{n+1} (N_{n+1}) is an ultra-power of M_n (N_n) and $\bigcup_{n\in\mathbb{N}} M_n \simeq \bigcup_{n\in\mathbb{N}} N_n$ (so their completions are isomorphic as well).*
(ii) *A class of structures $\mathcal{K}$ is elementary if and only if it is closed under elementary equivalence and ultra-products.*

Proof. For the first item, right to left by the elementary chain lemma, which is proved as usual. For left to right, assume that $M \equiv N$. Then there is an ultra-power $N_1 = N^{\mathscr{U}}$ and an elementary embedding $f_0 : M \to N_1$. Then $(M, M) \equiv (N_1, f_0(M))$ (in a language with all elements of M named) so there exists an ultra-power $M_1 = M^{\mathscr{U}'}$ and an elementary embedding $g_0 : N_1 \to M_1$ such that $g_0 \circ f_0 = \mathrm{id}_M$. Proceed in this manner to obtain the sequences.

The second item is standard. $\blacksquare_{3.7}$

It is easily verified that any theory is logically equivalent to one which only consists of conditions of the form $\varphi = 0$. A *universal* theory is one which only consists of conditions of the form $\sup_{\bar x} \varphi(\bar x) = 0$ where φ is quantifier-free (and syntactically bounded and eventually constant in each x_i). Observe that:

- For any formula φ we can express $\forall \bar x\, \varphi(\bar x) = 0$ by the universal axiom scheme $\sup_{\bar x}^{n,n+1} \varphi(\bar x) = 0$.
- If t and s are terms we can express $\forall \bar x\, t = s$ by $\forall \bar x\, d(t,s) = 0$.
- If φ and ψ are formulae we can express $\forall \bar x\, \varphi \geq \psi$ by $\forall \bar x\, \psi \dot{-} \varphi = 0$.

Example 3.8. We can continue Example 2.5 and give the (universal) theory of the class of Banach spaces:

$$\begin{aligned}
&\langle \textit{Universal equational axioms of a vector space.} \rangle \\
&\forall x\, s\|x\| \leq \|m_r(x)\| \leq s'\|x\| && s, s' \text{ dyadic}, s \leq |r| \leq s' \\
&\forall xy\, \|x+y\| \leq \|x\| + \|y\| \\
&\forall xy\, d(x,y) = \|x + m_{-1}(y)\|.
\end{aligned}$$

More generally, it will be convenient to write

$$\big(\sup_x{}^r \inf_y{}^s \ldots \varphi\big) = 0, \qquad \text{or even} \qquad \forall^{<r} x\, \exists^{\leq s} y \ldots \big(\varphi = 0\big)$$

for the axiom scheme

$$\sup_x{}^{r-\varepsilon,r} \inf_y{}^{s,s+\varepsilon} \ldots \varphi = 0, \qquad \varepsilon > 0.$$

Notice that in the $\forall\exists$ notation, the universal quantifier holds literally, while the existential quantifiers holds in an approximate sense, with respect to the quantification radius as well as with respect to the value of φ (which may both be slightly bigger than s or 0, respectively.)

Example 3.9 (Measure algebras). Let $\mathcal{L} = \{0, \vee, \wedge, \smallsetminus\}$, where 0 is a constant symbol, $\vee, \wedge, \smallsetminus$ are binary function symbols. We use $\mu(x)$ as shorthand for $d(x,0)$, and take it to be the gauge. All the continuity moduli are standard.

The universal theory of measure algebras (which are the topic of [Fre04]) consists of:

$$\begin{aligned}
&\langle \textit{Universal equational axioms of relatively complemented distributive lattices} \rangle, \\
&\forall xy\, \mu(x) + \mu(y) = \mu(x \wedge y) + \mu(x \vee y), \\
&\mu(0) = 0, \\
&\forall xy\, d(x,y) = \mu(x \smallsetminus y) + \mu(y \smallsetminus x).
\end{aligned}$$

We can further say that a measure algebra is *atomless* by the axiom scheme:

$$\forall^{<n} x\, \exists^{\leq n} y\, |\mu(x \wedge y) - \mu(x)/2| = 0, \qquad n \in \mathbb{N}.$$

Example 3.10 (Replacing a function with its graph). Let $\mathcal{L}$ be an unbounded signature, $f \in \mathcal{L}$ an n-ary function symbol. We define its *graph* to be the $(n+1)$-ary predicate $G_f(\bar x, y) = d(f(\bar x), y)$. Since it is defined by a formula it respects a continuity modulus under ν uniformly in all $\mathcal{L}$-structures, and we may add it to the language. The axiom scheme $\forall \bar x y\, G_f(\bar x, y) = d(f(\bar x), y)$ is universal.

We may further drop f from the language. Indeed, we observe that a predicate G_f is the graph of a function f with continuity modulus δ_f if and only if the following theory holds. The second axiom ensures that in the third axiom there actually exists a unique $y = f(\bar x)$ such that $G_f(\bar x, y) = 0$. Then the first two axioms imply that G_f is the graph of f, and the two last axioms together ensure that f

respects δ_f under ν.

$$\begin{array}{lll}
\forall \bar{x}, y, z & G_f(\bar{x}, y) \leq G_f(\bar{x}, z) + d(y, z) & \\
\forall \bar{x}, y, z & d(y, z) \leq G_f(\bar{x}, y) + G_f(\bar{x}, z) & \\
\forall^{<\varepsilon^{-1}} \bar{x}\, \exists^{\leq \delta_f(\varepsilon)^{-1}} y & G_f(\bar{x}, y) = 0 & \varepsilon > 0 \\
\forall^{<\varepsilon^{-1}} \bar{x}\bar{y}\, \forall^{<\delta_f(\varepsilon)^{-1}+1} z & \big(\delta_f(\varepsilon) \dot{-} d(\bar{x}, \bar{y})\big) \wedge \big(G_f(\bar{x}, z) \dot{-} G_f(\bar{y}, z) \dot{-} \varepsilon\big) = 0 & \varepsilon > 0
\end{array}$$

Types and type spaces are defined more or less as usual:

Definition 3.11. Fix an unbounded signature $\mathcal{L}$.

(i) Given an n-tuple $\bar{a}$, we define its type $p(\bar{x}) = \mathrm{tp}(\bar{a})$ as usual as the set of all $\mathcal{L}$-conditions in the variables $x_{<n}$ satisfied by $\bar{a}$. The type $p(\bar{x})$ determines the value of $\varphi(\bar{a})$ for every formula φ, and we may write $\varphi^p = \varphi(\bar{x})^{p(\bar{x})} = \varphi(\bar{a})$.

(ii) A complete n-type (in $\mathcal{L}$) is the type of some n-tuple. By Corollary 3.5, this is the same as a maximal finitely consistent set of conditions $p(x_{<n})$ such that for some $r \geq 0$ we have $\nu(x_i) \leq r \in p$ for all $i < n$.

(iii) The set of all n-types is denoted S_n. The set of all n-types containing a theory T (equivalently: realised in models of T) is denoted $\mathrm{S}_n(T)$.

(iv) For every condition s in the free variables $x_{<n}$, $[s]^{\mathrm{S}_n(T)}$ (or just $[s]$, if the ambient type space is clear from the context) denotes the set of types $\{p \in \mathrm{S}_n(T) \colon s \in p\}$.

(v) The family of all sets of the form $[s]^{\mathrm{S}_n(T)}$ forms a base of closed sets for the *logic topology* on $\mathrm{S}_n(T)$. It is easily verified to be Hausdorff.

For each $n \in \mathbb{N}$, we can define $\nu \colon \mathrm{S}_n(T) \to \mathbb{R}$ by $\nu(p) = \bigvee_{i<n} \nu(x_i)^p$. With this definition, $(\mathrm{S}_n(T), d, \nu)$ is a gauged space. Applying previous definitions we have:

$$\mathrm{S}_n^{\nu \leq r}(T) = \bigcap_{i<n} \big[\nu(x_i) \leq r\big] = \left[\Big(\bigvee_{i<n} \nu(x_i)\Big) \leq r\right].$$

By Corollary 3.5, $\mathrm{S}_n^{\nu \leq r}(T)$ is compact. If $\mathrm{S}_n(T) = \mathrm{S}_n^{\nu \leq r}(T)$ for some r, then $\mathrm{S}_n(T)$ is compact. Conversely, if $\mathrm{S}_n(T)$ is compact for $n \geq 1$, then ν is necessarily bounded on models of T, so there is some r such that $T \vdash \sup_x \nu(x) \wedge (r+1) \leq r$ and $\mathrm{S}_m(T) = \mathrm{S}_m^{\nu \leq r}(T)$ for all $m \in \mathbb{N}$. In this case all the other symbols are also bounded in models of T, so up to re-scaling everything into $[0, 1]$ we are in the case of standard continuous first order logic.

In the non compact case we still have $\mathrm{S}_n(T) = \bigcup_r \mathrm{S}_n^{\nu \leq r}(T)$. Thus each $p \in \mathrm{S}_n(T)$ there is r such that $p \in \mathrm{S}_n^{\nu \leq r}(T)$, and $\mathrm{S}_n^{\nu \leq r+1}(T)$ is a compact neighbourhood of p (since it contains the open set $[(\bigvee \nu(x_i)) < r+1]$). Therefore $\mathrm{S}_n(T)$ is locally compact.

4. On the relation with Henson's positive bounded logic

We sketch out here how unbounded continuous logic generalises, in an appropriate sense, Henson's logic of approximate satisfaction of positive bounded formulae in Banach space structures. For this purpose we assume familiarity with the syntax and semantics of Henson's logic (see for example [HI02]).

The classical presentation of Henson's logic involves a purely functional signature $\mathcal{L}_H$ with a distinguished sort for $\mathbb{R}$. There is no harm in assuming that the distinguished sort only appears as the target sort of some function symbols (otherwise we can add a second copy and a single function symbol for the identity mapping into the copy, and treat the copy as the distinguished sort). Also, there is no harm in replacing $\mathbb{R}$ with $\mathbb{R}^+$.

We can therefore define an unbounded continuous signature $\mathcal{L}$ by dropping the distinguished sort and replacing all function symbols into it with $\mathbb{R}^+$-valued predicate symbols. As every sort is assumed to be normed, we identify ν with $\|\cdot\|$. While a signature in Henson's logic does not specify continuity moduli, in every class under consideration each symbol satisfies some continuity modulus uniformly under $\|\cdot\|$ which we may use (or else the logic would fail to describe the class). It is a known fact that there exists a (universal) $\mathcal{L}_H$-theory, call it T_0, whose models are precisely the structures respecting these continuity moduli under $\|\cdot\|$.

From now on by "structure" we mean a model of T_0, or equivalently a $\mathcal{L}$-structure (as these can be identified). The ambiguity concerning whether a structure is a Henson or unbounded continuous structure is further justified by the fact that the definitions of isomorphism and ultra-products in either logic coincide. As we can moreover prove Lemma 3.7 for Henson's logic just as well, we conclude:

Theorem 4.1. *A class of structures $\mathcal{K}$ is elementary in Henson's logic if and only if it is elementary in unbounded continuous logic.*

Recall:

Fact 4.2. *Let $X = \bigcup_{n\in\mathbb{N}} X_n$ be a topological space where each X_n is closed and X_{n+1} is a neighbourhood of X_n. Then a subset $F \subseteq X$ is closed if and only if $F \cap X_n$ is for all n.*

An n-type is the same thing as a complete theory with n new constant symbols (more precisely, a type p with $\nu(p) \leq r$ corresponds to a complete theory with new constants symbols with continuity moduli $\delta_a \leq \frac{1}{r}$).

Corollary 4.3. *Two n-tuples in a structure have the same type in one logic if and only if they have the same type in the other, and this identification induces a homeomorphism $\mathrm{S}_n^{\mathcal{L}_H}(T_0) \simeq \mathrm{S}_n^{\mathcal{L}}$.*

Proof. The first statement is by Theorem 4.1. Also, a set $X \subseteq \mathrm{S}_n^{\|\cdot\|\leq r}$ is closed if and only if the class $\{(M, \bar{a})\colon \mathrm{tp}(\bar{a}) \in X\}$ is elementary: the bounds on the norm are needed since we need to impose bounds on the norms of constant symbols. It follows from Theorem 4.1 that the bijection $\mathrm{S}_n^{\mathcal{L}_H}(T_0) \simeq \mathrm{S}_n^{\mathcal{L}}$ is a homeomorphism when restricted to $\mathrm{S}_n^{\|\cdot\|\leq n}$. Now use Fact 4.2 and the fact that $\mathrm{S}_n^{\|\cdot\|\leq r}$ is compact and $\mathrm{S}_n^{\|\cdot\|<r}$ is open in both topologies to conclude that this is a global homeomorphism. $\blacksquare_{4.3}$

This can be restated as:

Corollary 4.4. *For every set $\Sigma(\bar{x})$ of $\mathcal{L}_H$-formulae there exists a set $\Gamma(\bar{x})$ of $\mathcal{L}$-conditions, and for every set $\Gamma(\bar{x})$ of $\mathcal{L}$-conditions there exists a set $\Sigma(\bar{x})$ of $\mathcal{L}_H$-formulae, such that for every structure M and $\bar{a} \in M$:*

$$M \vDash_A \Sigma(\bar{a}) \quad \Longleftrightarrow \quad M \vDash \Gamma(\bar{a}).$$

Remark 4.5. In Henson's logic, the bounded quantifier $\forall^{\leq r}x$ ($\exists^{\leq r}x$) mean "for all (there exists) x such that $\|x\| \leq r$". Thus Henson's logic coincides with unbounded continuous logic of normed structures where $\nu = \|\cdot\|$. One may generalise Henson's logic to allow an arbitrary ν and obtain full equivalence of the two logics.

For the benefit of the reader who finds this proof a little too obscure, let us give one direction explicitly. We know that every formula in Henson's logic is equivalent to one in prenex form

$$\forall^{\leq r_0}x_0\exists^{\leq r_1}x_1 \ldots \varphi(\bar{x}, \bar{y}),$$

where φ is a positive Boolean combination of atomic formulae of the form $t_i(\bar{x}, \bar{y}) \geq r_i$ or $t_i \leq r_i$. Every term t_i can be identified with an atomic $\mathcal{L}$-formula, and replacing t_i with $t_i \dot{-} r_i$ or with $r_i \dot{-} t_i$, we may assume all these atomic formulae are of the form $t_i \leq 0$. Since $(t_i \leq 0) \wedge (t_j \leq 0) \Longleftrightarrow (t_i \vee t_j) \leq 0$ and $(t_i \leq 0) \vee (t_j \leq 0) \Longleftrightarrow (t_i \wedge t_j) \leq 0$, we can find a single t such that $\varphi(\bar{x}, \bar{y})$ is equivalent to $t \leq 0$. We thus reduced to:

$$\forall^{\leq r_0}x_0\,\exists^{\leq r_1}x_1 \ldots \big(t(\bar{x}, \bar{y}) \leq 0\big).$$

We can view t as a quantifier-free $\mathcal{L}$-formula, in which case the above holds approximately if and only if the following holds (with the notation preceding Example 3.9):

$$\forall^{<r_0} x_0 \, \exists^{\leq r_1} x_1 \ldots \, t(\bar{x}, \bar{y}) = 0.$$

Thus the approximate satisfaction of a $\mathcal{L}_H$-formula, and therefore of a partial type, are equivalent to the satisfaction of a partial type in $\mathcal{L}$.

5. Emboundment

As we mentioned earlier, the multi-sorted approach to unbounded structures allows us to reduce many issues concerning unbounded structures to their well-established analogues in bounded continuous logic, but this does not work well for perturbations when we wish to perturb ν itself. In addition, if the bounded balls are not definable in the unbounded structure then their introduction as sorts adds unexpected structure – this may happen, for example, when considering a field equipped with a valuation in $\mathbb{R}$ as an unbounded metric structure.

We could of course generalise everything we did to the unbounded case, but that would be extremely tedious to author and reader alike. Instead, we seek a universal reduction of unbounded logic to the more familiar (and easier to manipulate) bounded one. This reduction goes through a construction which we call *emboundment*. Thus, for example, a bounded set $X \subseteq M^n$ in an unbounded structure is said to be *definable* (a term we knowingly used above without a definition) if it is definable in the embounded structure M^∞. An easy verification yields that this is equivalent to the predicate $d(\bar{x}, X)$ being definable in M, i.e., a uniform limit of formulae on every bounded set. (See [Bena] for definable sets in bounded structures.)

One naïve approach would be to choose a continuous function mapping $\mathbb{R}^+$ into $[0, 1]$, say $\theta(x) = 1 - e^{-x}$, and apply it to all the predicate symbols: for every $\mathcal{L}$-structure M we define M^θ as having the same underlying set, and for every predicate symbol P we define $P^{M^\theta}(\bar{a}) = \theta(P^M(\bar{a}))$. It can be verified that $\theta(x + y) \leq \theta(x) + \theta(y)$ for all $x, y \geq 0$ (this is true when $x = 0$, and the partial derivative with respect to x of the left hand side is smaller). It follows that d^{M^θ} is a metric:

$$\begin{aligned} d^{M^\theta}(a, b) &= \theta(d^M(a, b)) \leq \theta(d^M(a, c) + d^M(c, b)) \\ &\leq \theta(d^M(a, c)) + \theta(d^M(c, b)) = d^{M^\theta}(a, b) + d^{M^\theta}(a, b). \end{aligned}$$

Of course d^{M^θ} needs not be a complete metric, so we obtain new elements when passing to the completion. Similarly, if $T^\theta = \mathrm{Th}\{M^\theta \colon M \vDash T\}$, then we have a natural embdding of $\mathrm{S}_n(T)$ in $\mathrm{S}_n(T^\theta)$, and it can be verified that the latter is the Stone-Čech compactification of the former. This is essentially the same thing as allowing ∞ as a legitimate truth value (since θ extends to a homeomorphism $[0, \infty] \to [0, 1]$). As usual with the Stone-Čech compactification, this adds too many new types to be manageable. In short, this naïve construction does yield bounded structures but it is not at all clear that the structures (or theories) thus obtained are meaningful. For example, even the following is not clear (to the author), and one would expect it to be false:

Question 5.1. Is every model of T^θ of the form M^θ, where $M \vDash T$?

For a better approach, we take a second look on the construction of unbounded logic and its semantics, as well as on the construction of unbounded ultra-products. Throughout these constructions appeared a formal infinity element ∞, which, while not a member of the structures, was treated for many intents and purposes as if it were. Indeed, the quantifier semantics included ∞ in the set over which quantification takes place, and the ultra-product construction could be restated informally as "add ∞, take a usual ultra-product, then take ∞ out". In particular, unbounded structures may be formally empty since, from a practical point of view, they still always contain the ideal point at infinity.

With this motivation in mind, we seek to equip each unbounded structure M with a new metric, denoted d^{M^∞} such that every sequence (a_n) in M which goes to infinity in the sense that $\nu(a_n) \to \infty$, is Cauchy in d^{M^∞}, converging to a new element representing the formal infinity. Such a metric is naturally bounded. Moreover, every predicate on M which is uniformly continuous under ν can be modified to yield a bounded predicate which is in uniformly continuous in the usual sense with respect to d^{M^∞}. On the other hand, this does not work well for function symbols (for example, we cannot give a sense to $\infty + \infty$ in the embowndment of a Banach space). We shall therefore replace every function symbol in the language with its graph $G_f(\bar{x}, y) = d(f(\bar{x}), y)$ as in Example 3.10, and assume that the signature $\mathcal{L}$ is purely relational. We then define

$$\mathcal{L}^\infty = \mathcal{L} \cup \{\infty\}$$

where ∞ is a new constant symbol. We may consider $\mathcal{L}$ to consist, as a set, of its non distinguished symbols alone, in which case ν gets dropped (or more precisely, both d and ν are dropped, and then $\mathcal{L}^\infty$ is equipped with its own distinguished distance symbol d). Whether or not ν is kept will be of no essential difference to the construction. We do not specify at this point the uniform continuity moduli, but we shall show below that such moduli can be chosen that do fit our purpose.

For every $\mathcal{L}$-structure M we define an $\mathcal{L}^\infty$-structure M^∞. Its domain is the set $M \cup \{\infty\}$. For elements coming from M we interpret the symbols as follows (we recall that $d(a, \infty) = \nu(\infty) = \infty$, $\theta(\infty) = 1$, and $\nu(\bar{x}) = \bigvee \nu(x_i)$):

$$d^{M^\infty}(a, b) = \frac{\theta \circ d^M(a, b)}{e^{\nu^M(a) \wedge \nu^M(b)}}, \qquad P^{M^\infty}(\bar{a}) = \frac{\theta \circ P^M(\bar{a})}{e^{\nu^M(\bar{a})}}, \qquad (P \neq d).$$

So in particular:

$$d^{M^\infty}(a, \infty) = e^{-\nu^M(a)}, \qquad P^{M^\infty}(\ldots, \infty, \ldots) = 0, \qquad (P \neq d).$$

Notice that if we interpreted d^{M^∞} as with other symbols we would have $d^{M^\infty}(a, \infty) = 0$ for all a, and thus not obtain a metric. Conversely, we can reconstruct M from M^∞, first recovering ν^M from $d^{M^\infty}(x, \infty)$ and then recovering d^M and P^M from d^{M^∞} and P^{M^∞}, respectively, using the fact that $\theta^{-1}(y) = -\ln(1 - y)$.

Let us show that d^{M^∞} is a metric. The only non trivial property to verify is the triangle inequality, namely

$$\frac{\theta \circ d^M(a, c)}{e^{\nu^M(a) \wedge \nu^M(c)}} \leq \frac{\theta \circ d^M(a, b)}{e^{\nu^M(a) \wedge \nu^M(b)}} + \frac{\theta \circ d^M(b, c)}{e^{\nu^M(b) \wedge \nu^M(c)}}.$$

If b has the smallest gauge among the three then this follows from the fact that $\theta \circ d^M$ is a metric, which we verified earlier. Otherwise we may assume without loss of generality that a has the smallest gauge, say r. Let $t = d^M(a, b)$, $s = d^M(b, c)$. Then $\nu^M(b) \leq r + t$ and $d^M(a, c) \leq t + s$, and it is enough to verify that

$$\frac{\theta(t + s)}{e^r} = e^{-r} - e^{-r-t-s} = e^{-r} + e^{-r-t} + e^{-r-t} - e^{-r-t-s} = \frac{\theta(t)}{e^r} + \frac{\theta(s)}{e^{r+t}}.$$

Once we know that d^∞ is a metric it is clear that $a_n \to \infty$ in d^{M^∞} if and only if $\nu^M(a_n) \to \infty$.

Example 5.2. Let M be a bounded structure, and turn it into an unbounded structure M' as in Example 2.4. Then $M \cong (M')^\infty \smallsetminus \{\infty\}$, so all we did was add a single isolated point with distance 1 to the original structure.

Lemma 5.3. *The gauged space (M, d^M, ν^M) and the bounded metric space (M, d^{M^∞}) are related as follows:*

(i) *We have $d^M \geq d^{M^\infty}$ on all of M, and the two metrics are uniformly equivalent on every bounded subset of M (bounded in the sense of M).*
(ii) *For every $r' > r$ the ν-ball $M^{\nu<r'}$ contains a uniform d^{M^∞}-neighbourhood of $M^{\nu\leq r}$ (of radius $\frac{\theta(r'-r)}{e^r} = e^{-r} - e^{-r'}$).*

Proof. The inequality $d^M \geq d^{M^\infty}$ is immediate. Let us fix $r \geq 0$ and let $a \in M^{\nu\leq r}$, $b \in M$. Then by definition $d^{M^\infty}(a,b) \geq \frac{\theta\circ d^M(a,b)}{e^r}$. Thus, for all $\varepsilon > 0$

$$d^{M^\infty}(a,b) < \frac{\theta(\varepsilon)}{e^r} \quad \Longrightarrow \quad d^M(a,b) < \varepsilon,$$

concluding the proof of the first item. This also proves the third item, since

$$B_{d^{M^\infty}}\left(M^{\nu\leq r}, \frac{\theta(r'-r)}{e^r}\right) \subseteq B_d(M^{\nu\leq r}, r'-r) \subseteq M^{\nu<r'}. \qquad \blacksquare_{5.3}$$

Proposition 5.4. *For every $\mathcal{L}$-structure M, M^∞ as defined above is an $\mathcal{L}^\infty$-structure, called the* emboundment *of M. That is to say that M^∞ is complete, and that we can complete the definition of $\mathcal{L}^\infty$ choosing uniform continuity moduli for its symbols which are satisfied in every M^∞.*

Proof. For completeness, let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in d^{M^∞}. If $\nu(a_n) \to \infty$ (where again, $\nu(\infty) = \infty$) then $a_n \to \infty$ in d^∞. Otherwise, there is r such that $a_n \in M^{\nu<r}$ infinitely often. Passing to a sub-sequence, we may assume that the entire sequence fits inside $M^{\nu<r}$. By Lemma 5.3(i) the sequence is Cauchy in d^M and therefore admits a limit in M, which is necessarily also its limit in d^{M^∞}.

For uniform continuity, let $P \in \mathcal{L}$ be an $(n+1)$-ary predicate symbol. Let $\varepsilon > 0$ be given, and we wish to find $\delta > 0$ such that for all $a, b \in M^\infty$

$$d^{M^\infty}(a,b) \leq \delta \quad \Longrightarrow \quad \sup_{\bar{x}} |P(a,\bar{x}) - P(b,\bar{x})|^{M^\infty} \leq \varepsilon.$$

First, if $\nu(a), \nu(b) \geq -\ln\varepsilon$ (where $\nu(\infty) = \infty$) then the above is satisfied regardless of $d^{M^\infty}(a,b)$. Otherwise, without loss of generality we have $\nu(a) < -\ln\varepsilon$. If in addition $d^{M^\infty}(a,b) < \varepsilon - \varepsilon^2$ then $\nu(b) < -2\ln\varepsilon$ by Lemma 5.3(ii). Since P^M and ν^M are uniformly continuous with respect to d^M on $M^{\nu<-2\ln\varepsilon}$, so is P^{M^∞}. By Lemma 5.3(i), P is uniformly continuous with respect to d^{M^∞} on $M^{\nu<-2\ln\varepsilon}$, whence the existence of δ as desired. $\blacksquare_{5.4}$

It is straightforward to verify that the emboundment construction commutes with the ultra-product construction, since everything is continuous:

$$\left(\prod M_i/\mathscr{U}\right)^\infty = \prod M_i^\infty/\mathscr{U}.$$

In particular, all the tuples (a_i) such that $\lim_{\mathscr{U}} \nu^{M_i}(a_i) = \infty$, which were dropped during the construction of $\prod M_i/\mathscr{U}$, satisfy $[a_i] = [\infty^{M_i}] = \infty$ in $\prod M_i^\infty/\mathscr{U}$.

Similarly, emboundment commutes with unions of increasing chains, and by Lemma 3.7 we have $M \equiv N \Longleftrightarrow M^\infty \equiv N^\infty$ for any two $\mathcal{L}$-structures M and N. If $N \subseteq M$ then working with $\mathcal{L}(N)$ we get $N \preceq M \Longleftrightarrow N^\infty \preceq M^\infty$. Similarly, if $N' \preceq M^\infty$ where N' is an $\mathcal{L}^\infty$-structure then we can recover an $\mathcal{L}$-structure on $N = N' \smallsetminus \{\infty\} \subseteq M$, so $N' = N^\infty$ and $N \preceq M$.

Proposition 5.5. *Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures, and let*

$$\mathcal{K}^\infty = \{M^\infty : M \in \mathcal{K}\}.$$

Then $\mathcal{K}$ is elementary if and only if $\mathcal{K}^\infty$ is.

Proof. Assume $\mathcal{K}$ is elementary. Then, by the arguments above, $\mathcal{K}^\infty$ is closed under ultra-products, isomorphism and elementary substructures. It is therefore elementary. Similarly for the converse. $\blacksquare_{5.5}$

By Proposition 5.5 we may replace every $\mathcal{L}$-theory T (in unbounded logic) with its emboundment $T^\infty = \mathrm{Th}_{\mathcal{L}^\infty}(\mathrm{Mod}(T)^\infty)$, which is a theory in standard bounded logic. By naming constants we further see that $\mathcal{L}$-types of tuples in M are in bijection with $\mathcal{L}^\infty$-types of tuples in $M^\infty \smallsetminus \{\infty\}$ (i.e., in M again, but this time viewed as a subset of a $\mathcal{L}^\infty$-structure).

Given a tuple $\bar a \in M^\infty$, let $w = w(\bar a) = \{i < n\colon a_i \neq \infty\}$. Then we may identify $\mathrm{tp}^{M^\infty}(\bar a)$ with the pair $(w, \mathrm{tp}^M(a_{\in w}))$. We can therefore express the set of types $\mathrm{S}_n(T^\infty)$ as $\bigcup_{w\subseteq n}\{w\} \times \mathrm{S}_{|w|}(T)$. For $w \subseteq n$, $r \in \mathbb{R}^+$ and $\varphi(x_{\in w}) \in \mathcal{L}$, define:

$$V_{n,w,r,\varphi} = \left\{ (v, q(x_{\in v})) \in \mathrm{S}_n(T^\infty)\colon \begin{array}{l} w \subseteq v \subseteq n,\ \varphi^q < 1, \\ \bigwedge_{i\in v\smallsetminus w} \nu(x_i)^q > r \end{array} \right\}.$$

Given a type $(w, p) \in \mathrm{S}_n(T^\infty)$, one can verify that the family of all sets of the form $V_{n,w,r,\varphi}$ where $\varphi^p = 0$ forms a base of neighbourhoods for (w, p). In particular, the natural inclusion $\mathrm{S}_n(T) \hookrightarrow \mathrm{S}_n(T^\infty)$, consisting of sending $p \mapsto (n, p)$, is an open topological embedding. In case T is complete (so $|\mathrm{S}_0(T)| = 1$), this embedding for $n = 1$ is a single point compactification of $\mathrm{S}_1(T)$ obtained by adding the type at infinity. We may therefore also refer to T^∞ as the *compactification* of T.

Once we understand types we know what saturation means. Among other things we have:

Lemma 5.6. *An $\mathcal{L}$-structure M is approximately $\aleph_0$-saturated if and only if M^∞ is.*

Proof. Follows from the facts that there is a unique point at infinity, which belongs to M^∞, and that in the neighbourhood of every other point d^M and d^{M^∞} are equivalent. $\blacksquare_{5.6}$

Finally, we point out that the theory T is bounded to begin with if and only if the point at infinity in models of T^∞ is isolated, in analogy with what happens when one attempts to add a point at infinity to a space which is already compact.

6. Perturbations of unbounded structures

We now adapt the framework of perturbation of bounded metric structures to unbounded structures, essentially by reducing the unbounded case to the bounded one through emboundment. For this purpose we assume close familiarity with the original development in [Benc]. We fix an unbounded theory T and its emboundment T^∞.

Definition 6.1. A *perturbation pre-radius* for T is defined as for a bounded theory, i.e., as a family $\rho = \{\rho_n \subseteq \mathrm{S}_n(T)\colon n \in \mathbb{N}\}$ containing the diagonals. We define X^ρ, $\mathrm{Pert}_\rho(M, N)$, $\mathrm{BiPert}_\rho(M, N)$, $\langle\rho\rangle$, $[\![\rho]\!]$ as in [Benc].

Let ρ be a perturbation pre-radius for T. We can always extend it to a perturbation radius ρ^* for T^∞ by:

$$\rho_n^\infty = \overline{\{((w, p), (w, q)) \in \mathrm{S}_n(T^\infty)\colon w \subseteq n, (p, q) \in \rho_{|w|}\}}.$$

Clearly, this is a perturbation pre-radius for T^∞. Conversely, if ρ' is a perturbation pre-radius for T^∞ then its restriction to $\mathrm{S}(T)$, denoted $\rho'{\restriction}_{\mathrm{S}(T)}$, is a perturbation pre-radius for T, and as the inclusion $\mathrm{S}_n(T) \subseteq \mathrm{S}_n(T^\infty)$ is open we have the identity:

$$\rho^\infty{\restriction}_{\mathrm{S}(T)} = \rho.$$

Also, as every $f \in \mathrm{Pert}_\rho(M, N)$ extends to $f \cup (\infty \mapsto \infty) \in \mathrm{Pert}_{\rho^\infty}(M^\infty, N^\infty)$, we also have $\langle\rho^\infty\rangle{\restriction}_{\mathrm{S}(T)} \geq \langle\rho\rangle$.

We define perturbation radii for T directly by reduction to T^∞:

Definition 6.2. (i) Let ρ' be a perturbation pre-radius for T^∞. We say that ρ' *separates infinity* if for all $f \in \mathrm{Pert}_{\rho'}(M^\infty, N^\infty)$ and $a \in M^\infty$:

$$a = \infty \Longleftrightarrow f(a) = \infty.$$

(ii) A perturbation pre-radius ρ for T is a *perturbation radius* if ρ^∞ is a perturbation radius for T^∞ which separates infinity.

Definition 6.3. A *perturbation pre-system* for T is a decreasing family $\mathfrak{p}$ of perturbation pre-radii satisfying downward continuity, symmetry, triangle inequality and strictness as in [Benc, Definition 1.23]. It is a *perturbation system* if $\mathfrak{p}(\varepsilon)$ is a perturbation radius for all ε, i.e., if $\mathfrak{p}^\infty$ is a perturbation system separating infinity for T^∞.

We turn to characterise perturbation radii as in [Benc], and establish more precisely the relation between perturbations of T and of T^∞.

Definition 6.4. Let ρ a perturbation pre-radius for T.

(i) We say that ρ *respects infinity* if for all $r \in \mathbb{R}^+$ there exists $r' \in \mathbb{R}^+$ such that

$$[\nu(x) \geq r']^\rho \subseteq [\nu(x) \geq r] \qquad \text{and} \qquad [\nu(x) \leq r]^\rho \subseteq [\nu(x) \leq r'].$$

(ii) We define when ρ *respects equality*, *respects* $\exists$, or is *permutation-invariant* as in the bounded case.

Proposition 6.5. *Let ρ be a perturbation pre-radius for T. The the following are equivalent:*

(i) *ρ is a perturbation radius.*

(ii) *ρ respects infinity, and for every $n, m \in \mathbb{N}$ and mapping $\sigma \colon n \to m$, the induced mapping $\sigma^* \colon \mathrm{S}_m(T) \to \mathrm{S}_n(T)$ satisfies that for all $p \in \mathrm{S}_m(T)$:*

$$\sigma^*(p^\rho) = \sigma^*(p)^\rho.$$

(I.e., $\sigma^* \circ \rho_m = \rho_n \circ \sigma^*$ as multi-valued functions).

(iii) *ρ respects ∞, $=$, $\exists$, and is permutation-invariant.*

(iv) *ρ^∞ separates ∞, respects $=$ and $\exists$ and is permutation-invariant.*

Proof. (i) $\Longrightarrow$ (ii). Assume ρ is a perturbation radius, so ρ^∞ is a perturbation radius respecting infinity. If ρ does not respect infinity, then by definition of ρ^∞ we have in ρ_1^∞ a pair (p,q) where p is the type of a finite elements and $q = \mathrm{tp}(\infty)$ or vice versa, contradicting the assumption on ρ^∞.

Since ρ^∞ is a perturbation radius, for all $\sigma \colon n \to m$ we have in $\mathrm{S}(T^\infty)$: $\sigma^* \circ \rho_m^\infty = \rho_n^\infty \circ \sigma^*$. As ρ^∞ also separates infinity we can restrict this to $\mathrm{S}(T)$ and obtain $\sigma^* \circ \rho_m = \rho_n \circ \sigma^*$.

(ii) $\Longrightarrow$ (iii). By restricting to the case where σ is the mapping $2 \to 1$, $n \hookrightarrow n+1$, or a permutation of $n \in \mathbb{N}$.

(iii) $\Longrightarrow$ (iv). By a mirror-image to the argument above, if ρ respects ∞ then ρ^∞ must separate ∞.

We claim that since ρ respects ∞ and $\exists$ and is permutation-invariant, we have for all $n \in \mathbb{N}$:

$$\rho_n^\infty = \big\{((w,p),(w,q)) \in \mathrm{S}_n(T^\infty) \colon w \subseteq n, (p,q) \in \rho_{|w|}\big\}$$

(i.e., the right hand side is a closed set). Indeed, assume we have pairs $((w_i,p_i),(w_i,q_i))$ for $i \in I$ and $\mathscr{U}$ is an ultra-filter on I, and let $((v,p),(u,q)) = \lim_{\mathscr{U}}((w_i,p_i),(w_i,q_i))$. We need to show that $v = u$ and $(p,q) \in \rho_{|v|}$. First, as there are finitely many possibilities for $w_i \subseteq n$ we may assume that $w_i = w \subseteq n$ for all i. Then we might as well assume $w = n$ throughout.

For $s \subseteq n$, let p_i^s and q_i^s be the restrictions of p_i and q_i, respectively, to $x_{\in s}$. As ρ respect $\exists$ and is permutation-invariant, $(p_i^s, q_i^s) \in \rho_{|s|}$. As ρ respects infinity we have:

$$k \notin v \Longleftrightarrow p_i^{\{k\}} \to_{\mathscr{U}} \mathrm{tp}(\infty) \Longleftrightarrow q_i^{\{k\}} \to_{\mathscr{U}} \mathrm{tp}(\infty) \Longleftrightarrow k \notin u.$$

Therefore $v = u$, and as $\rho_{|v|}$ is closed $(p, q) = \lim_{\mathscr{U}}(p_i^v, q_i^v) \in \rho_{|v|}$. This proves our claim.

It is now immediate that as ρ respects $=$ and $\exists$ and is permutation-invariant, the same holds of ρ^∞.

(iv) $\Longrightarrow$ (i). Since then ρ^∞ is a perturbation radius. $\blacksquare_{6.5}$

Corollary 6.6. *Perturbation systems $\mathfrak{p}$ for T are in a natural one-to-one correspondence with families $\{d_{\mathfrak{p},n} \colon n \in \mathbb{N}\}$, in which each $d_{\mathfrak{p},n}$ is a $[0,\infty]$-valued metric on $\mathrm{S}_n(T)$, and such that:*

(i) *For every n, the set $\big\{(p, q, \varepsilon) \in \mathrm{S}_n(T)^2 \times \mathbb{R}^+ \colon d_{\mathfrak{p},n}(p, q) \leq \varepsilon\big\}$ is closed.*

(ii) *For every $n, m \in \mathbb{N}$ and mapping $\sigma \colon n \to m$, the induced mapping $\sigma^* \colon \mathrm{S}_m(T) \to \mathrm{S}_n(T)$ satisfies for all $p \in \mathrm{S}_m(T)$ and $q \in \mathrm{S}_n(T)$:*

$$d_{\mathfrak{p},m}(p, (f^*)^{-1}(q)) = d_{\mathfrak{p},n}(f^*(p), q).$$

(Here we follow the convention that $d_{\mathfrak{p},m}(p, \varnothing) = \inf \varnothing = \infty$.)

(iii) *For every $r \in \mathbb{R}^+$ there is $r' \in \mathbb{R}^+$ such that if $p, q \in \mathrm{S}_1(T)$ and $d_{\mathfrak{p},1}(p, q) \leq r$, then*

$$\nu(x)^p \geq r' \Longrightarrow \nu(x)^q \geq r$$

Similarly, perturbation pre-systems are in one-to-one correspondence with families of metrics satisfying the first condition alone.

Proof. Same as [Benc, Lemma 1.24], where condition (iii) corresponds to the requirement that every $\mathfrak{p}(\varepsilon)$ respect infinity. $\blacksquare_{6.6}$

Let us fix a perturbation system $\mathfrak{p}$ for T, and let $\mathfrak{p}^\infty$ be the corresponding perturbation system for T^∞. As for plain approximate $\aleph_0$-saturation, we have

Lemma 6.7. *A model $M \vDash T$ is $\mathfrak{p}$-approximately $\aleph_0$-saturated if and only if M^∞ is $\mathfrak{p}^\infty$-approximately $\aleph_0$-saturated.*

Proof. As for Lemma 5.6. $\blacksquare_{6.7}$

In particular, and two separable $\mathfrak{p}$-approximately $\aleph_0$-saturated models of T must be $\mathfrak{p}$-isomorphic.

Similarly:

Lemma 6.8. *Two models $M, N \vDash T$ are $\mathfrak{p}$-isomorphic if and only if M^∞ and N^∞ are $\mathfrak{p}^\infty$-isomorphic. The theory T is $\mathfrak{p}$-$\aleph_0$-categorical if and only if T^∞ is $\mathfrak{p}^\infty$-$\aleph_0$-categorical.*

We conclude that [Benc, Theorem 3.5] holds as stated for unbounded structures:

Theorem 6.9. *Let T be a complete countable unbounded theory, $\mathfrak{p}$ a perturbation system for T. Then the following are equivalent:*

(i) *The theory T is $\mathfrak{p}$-$\aleph_0$-categorical.*

(ii) *For every $n \in \mathbb{N}$, finite $\bar{a}$, $p \in \mathrm{S}_n(\bar{a})$ and $\varepsilon > 0$, the set $[p^{\mathfrak{p}(\varepsilon)}(\bar{x}^\varepsilon, \bar{a}^\varepsilon)]$ has non empty interior in $\mathrm{S}_n(\bar{a})$.*

(iii) *Same restricted to $n = 1$.*

Proof. The idea is to reduce to [Benc, Theorem 3.5]. Most of the reduction is in the preceding results: T is complete if and only if T^∞ is, T is $\mathfrak{p}$-$\aleph_0$-categorical if and only if T^∞ is $\mathfrak{p}^\infty$-$\aleph_0$-categorical, etc. The last thing to check is that the property

$$(*) \qquad p(\bar{x}, \bar{a}) \in \mathrm{S}_n(\bar{a}), \varepsilon > 0 \qquad \Longrightarrow \qquad [p^{\mathfrak{p}(\varepsilon)}(\bar{x}^\varepsilon, \bar{a}^\varepsilon)]^\circ \neq \varnothing$$

holds for $T, \mathfrak{p}$ if and only it holds for $T^\infty, \mathfrak{p}^\infty$.

Indeed, assume first $(*)$ holds for $T^\infty, \mathfrak{p}^\infty$. Let $\bar{a} \in M \vDash T$, $p(\bar{x}, \bar{a}) \in \mathrm{S}_n(\bar{a})$. Then $\bar{a}$ can be viewed also as a tuple in $M^\infty \vDash T^\infty$, and we can identify $p(\bar{x}, \bar{a})$ with a type $p^\infty(\bar{x}, \bar{a}) \in \mathrm{S}_n^{\mathcal{L}^\infty}(\bar{a})$. Then $p^{\mathfrak{p}(\varepsilon)}(\bar{x}, \bar{y})$ and $p^{\infty\mathfrak{p}^\infty(\varepsilon)}(\bar{x}, \bar{y})$ coincide more or less by definition, and fit in $\mathrm{S}^{\nu \leq r}(T)$ for some $r \in \mathbb{R}^+$. It is not true that $p^{\mathfrak{p}(\varepsilon)}(\bar{x}^\varepsilon, \bar{y}^\varepsilon)$ and $p^{\infty\mathfrak{p}^\infty(\varepsilon)}(\bar{x}^\varepsilon, \bar{y}^\varepsilon)$ coincide since in the metrics on models of T and T^∞ differ. But as everything fits inside some ν-ball, and the two metrics are uniformly equivalent on every ν-ball, we can still find $\varepsilon' > 0$ such that

$$[\mathfrak{p}(\varepsilon)(\bar{x}^\varepsilon, \bar{y}^\varepsilon)]^\circ \supseteq [p^{\infty\mathfrak{p}^\infty(\varepsilon')}(\bar{x}^{\varepsilon'}, \bar{y}^{\varepsilon'})]^\circ \neq \varnothing.$$

For the converse, consider a finite tuple $\bar{a} \in M^\infty \vDash T^\infty$, and a type $p(\bar{x}, \bar{a}) \in \mathrm{S}_n^{\mathcal{L}^\infty}(\bar{a})$. As ∞ is definable in T^∞ (it is the unique element satisfying $P_1(x) = 0$, for example) we never need it as a parameter, so we may assume that $\bar{a} \in M$. Assume first that $p(\bar{x}, \bar{a})$ says that all x_i are finite as well. Then in fact $p(\bar{x}, \bar{a}) \in \mathrm{S}_n^{\mathcal{L}}(\bar{a})$, and we conclude as above by the uniform equivalence of the metric. In the general case we may need to write $p(\bar{x}, \bar{y})$ as (w, q) where $w \subseteq |\bar{x}, \bar{y}|$, and $q \in \mathrm{S}_{|w|}(T)$. Then q is a type of finite elements and is taken care of by the previous case, while the infinite coordinates are taken care of by the fact that ∞ is definable, so $[d^{\mathcal{L}^\infty}(x, \infty) < \varepsilon]$ defines an open set in $\mathrm{S}^{\mathcal{L}^\infty}(\bar{a})$. $\blacksquare_{6.9}$

The discussion at the end of [Benc, Section 3], and in particular the characterisation of $\mathfrak{p}$-$\aleph_0$-categoricity for an open perturbation system $\mathfrak{p}$ by coincidence of topologies ([Benc, Theorem 3.15]), can be transferred to an unbounded theory T via reduction to T^∞ in precisely the same way.

7. An example: Henson's categoricity theorem

Let T_0 be the (unbounded) theory of pure Banach spaces as given in Example 3.8.

Definition 7.1. Let E and F be Banach spaces (i.e., models of T_0). Say that a mapping $f : E \to F$ is an *ε-isomorphism* if it is an isomorphism of the underlying vector spaces, and satisfies in addition:

$$\forall v \in E \qquad e^{-\varepsilon}\|v\| \leq \|f(v)\| \leq e^\varepsilon \|v\|.$$

Definition 7.2. Let $\bar{a} \in E_0 \vDash T_0$. Define the *Banach-Mazur distance* between two types $p, q \in \mathrm{S}_n(\bar{a})$, denoted $d_{BM,n}(p, q)$, as the minimal $\varepsilon > 0$ such that there exist models $(E, \bar{a}), (F, \bar{a}) \vDash \mathrm{Th}(E_0, \bar{a})$, and tuples $\bar{b} \in E$, $\bar{c} \in F$ realising p and q, respectively, and an ε-isomorphism $f : E \to F$ fixing $\bar{a}$ and sending $\bar{b}$ to $\bar{c}$. If no such $\varepsilon > 0$ exists then $d_{BM,n}(p, q) = \infty$.

The following result is very similar to an unpublished result communicated to the author orally by C. Ward Henson. It is one of the original motivations for the present paper as well as for [Benc].

Corollary 7.3. *Let T be a complete theory of Banach spaces with no additional structure (i.e., a completion of T_0). Then the following are equivalent:*

(i) *If E and F are two separable models of T, then for every $\varepsilon > 0$ there exists an ε-isomorphism (i.e., a bijective ε-embedding) from E to F.*
(ii) *For $n \in \mathbb{N}$ and finite tuple $\bar{a} \in E \vDash T$, let $\mathrm{S}_n^*(\bar{a})$ be the space of types of n-tuples which are linearly independent over $\bar{a}$. Then every Banach-Mazur ball in $\mathrm{S}_n^*(\bar{a})$ has non empty interior in the logic topology on $\mathrm{S}_n^*(\bar{a})$.*

Proof. First we observe that the Banach-Mazur distance defines a perturbation system BM by Corollary 6.6. Therefore, by Theorem 6.9, the first condition is equivalent to the one saying that for all $\varepsilon > 0$ and $p(\bar{x}, \bar{a}) \in \mathrm{S}_n(\bar{a})$: $[p^{BM(\varepsilon)}(\bar{x}^\varepsilon, \bar{a}^\varepsilon)]^\circ \neq \varnothing$ in $\mathrm{S}_n(\bar{a})$. We need to show that this is equivalent to the second condition. Since the Banach-Mazur perturbation preserves linear dependencies we may drop superfluous parameters and always assume that the tuple $\bar{a}$ is linearly independent. Thus, if $p(\bar{x}, \bar{a}) \in \mathrm{S}^*(\bar{a})$ then $p(\bar{x}, \bar{y}) \in \mathrm{S}^*(T)$.

Observe also that $\mathrm{S}_n^*(\bar{a})$ is a dense open subset of $\mathrm{S}_n(\bar{a})$ (indeed, it is metrically dense there in the usual metric on types). It follows that a subset $X \subseteq \mathrm{S}_n^*(\bar{a})$ has the same interior in $\mathrm{S}_n(\bar{a})$ and in $\mathrm{S}_n^*(\bar{a})$, so we may simply speak of its interior. Moreover, a subset $X \subseteq \mathrm{S}_n(\bar{a})$ has non empty interior if and only if $X \cap \mathrm{S}_n^*(\bar{a})$ has.

For left to right, let us show that if $p \in \mathrm{S}^*_n(T)$ and $\varepsilon > 0$ then there exists $\delta > 0$ such that $[p(\bar{x}^\delta)] \subseteq [p^{BM(\varepsilon)}]$. So let $\Lambda = \{\lambda \in \mathbb{F}^n \colon \sum |\lambda_i| = 1\}$, i.e., the (compact) space of all formal linear combinations of n variables of $\|\cdot\|_1$-norm 1, and let $s = \min\{\|\lambda(\bar{x})\|^{p(\bar{x})} \colon \lambda \in \Lambda\} > 0$. We claim that $\delta = \frac{s\varepsilon}{2n} > 0$ will do.

Indeed, let $q \in [p(\bar{x}^\delta)]$. Let E be a model, $\bar{b}, \bar{c} \in E$ such that $\bar{b} \vDash p$, $\bar{c} \vDash q$ and $\|b_i - c_i\| \leq \delta$ for all $i < n$. For $i < n$ define a linear functional $\eta_i \colon \mathrm{Span}(\bar{b}) \to \mathbb{F}$ by $\eta_i\left(\sum \lambda_j b_j\right) = \lambda_i$. Then $\|\eta_i\| \leq s^{-1}$, and by the Hahn-Banach Theorem we may extend them to $\tilde{\eta}_i \colon E \to \mathbb{F}$ such that $\|\tilde{\eta}_i\| \leq s^{-1}$. Define a linear operator $S \colon E \to E$ by $S(x) = \sum_i \tilde{\eta}_i(x)(b_i - c_i)$. Then a simple calculation shows that $S(b_i) = b_i - c_i$ and $\|S\| \leq \varepsilon/2$. Assuming ε was small enough to begin with (which we may), $I - S$ is invertible, its inverse being $I + S + S^2 + \ldots$. Finally, for all $v \in E$:

$$e^{-\varepsilon}\|v\| \leq (1 - \varepsilon/2)\|v\| \leq \|v - S(v)\| \leq (1 + \varepsilon/2)\|v\| \leq e^{\varepsilon}\|v\|.$$

We conclude that $I - S$ is an ε-automorphism sending $\bar{b}$ to $\bar{c}$, so $q \in p^{BM(\varepsilon)}$.

Re-choosing our numbers we find $\varepsilon/2 > \delta > 0$ such that that $[p(\bar{x}^\delta)] \subseteq [p^{BM(\varepsilon/2)}(\bar{x})]$, so $[p(\bar{x}^\delta)]^{BM(\delta)} \subseteq [p^{BM(\varepsilon)}(\bar{x})]$. As the former has non empty interior so does the latter (in $\mathrm{S}_n(T)$ as well as when restricted to $\mathrm{S}^*_n(T)$). When considering parameters we have $p(\bar{x}, \bar{a}) \in \mathrm{S}^*_n(\bar{a})$ such that $p(\bar{x}, \bar{y}) \in \mathrm{S}^*_{n+m}(T)$, so we find $\delta > 0$ such that $[p(\bar{x}^\delta, \bar{y}^\delta)]^{BM(\delta)} \subseteq [p^{BM(\varepsilon)}(\bar{x}, \bar{y})]$, and thus $[p(\bar{x}^\delta, \bar{a}^\delta)]^{BM(\delta)} \subseteq [p^{BM(\varepsilon)}(\bar{x}, \bar{a})]$, concluding as above.

For the other direction, let us show that for all $p \in \mathrm{S}_n(T)$ and $\varepsilon > 0$, $[p^{BM(\varepsilon)}(\bar{x}^\varepsilon)]^\circ \neq \varnothing$. Assume first that $p \in \mathrm{S}^*_n(T)$. Then $[p^{BM(\varepsilon)}]^\circ \neq \varnothing$ in $\mathrm{S}^*_n(T)$, and therefore in $\mathrm{S}_n(T)$, as $\mathrm{S}^*_n(T)$ is open in $\mathrm{S}_n(T)$. In case $p \notin \mathrm{S}^*_n(T)$ we need to be more delicate. Up to a permutation of the variables we may assume that p is of the form $p(x_{<m}, y_{<k})$, where $m + k = n$, $q(\bar{x}) = p{\restriction}_{\bar{x}} \in \mathrm{S}^*_m(T)$, and $p \vdash \bigwedge_{i<k}(y_i = \lambda_i(\bar{x}))$ for some linear combinations λ_i.

Then we know there is a formula $\varphi(\bar{x})$ such that $\varnothing \neq [\varphi < 1/2] \subseteq q^{BM(\varepsilon)}$. Then in $\mathrm{S}_n(T)$ we have:

$$\begin{aligned} \varnothing \neq [\varphi(\bar{x}) < 1/2] \cap \bigcap_{i<k} [d(y_i, \lambda_i(\bar{x})) < \varepsilon] \\ \subseteq [p^{BM(\varepsilon)}(\bar{x}, \bar{y}^\varepsilon)] \\ \subseteq [p^{BM(\varepsilon)}(\bar{x}^\varepsilon, \bar{y}^\varepsilon)]. \end{aligned}$$

Indeed, if $p' \in [\varphi(\bar{x}) < 1/2] \cap \bigcap_{i<k}[d(y_i, \lambda_i(\bar{x})) < \varepsilon]$, then there is $p'' \in [p'(\bar{x}, \bar{y}^\varepsilon)]$ such that $p'{\restriction}_{\bar{x}} = p''{\restriction}_{\bar{x}}$, and $p'' \vdash \bigwedge_{i<k}(y_i = \lambda_i(\bar{x}))$. As $\varphi(\bar{x})^{p''} < 1/2$, we have $p''{\restriction}_{\bar{x}} \in q^{BM(\varepsilon)}$. We by variable-invariance we may find $p''' \in (p'')^{BM(\varepsilon)}$ such that $p'''{\restriction}_{\bar{x}} = q$. As the linear structure is left untouched by the Banach-Mazur perturbation we must have $p'''(\bar{x}, \bar{y}) \vdash \bigwedge_{i<k}(y_i = \lambda_i(\bar{x}))$, so in fact $p''' = p$, as required.

The case with parameters is proved identically (with each y_i being equal to a linear combination of $\bar{x}$ and $\bar{a}$). $\blacksquare_{7.3}$

References

[BBHU08] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with Applications to Algebra and Analysis, volume 2 (Zoé Chatzidakis, Dugald Macpherson, Anand Pillay, and Alex Wilkie, eds.), London Math Society Lecture Note Series, vol. 350, Cambridge University Press, 2008, pp. 315–427.

[Bena] Itaï Ben Yaacov, *Definability of groups in $\aleph_0$-stable metric structures*, Journal of Symbolic Logic, to appear, arXiv:0802.4286.

[Benb] ———, *Model theoretic properties of metric valued fields*, in preparation, arXiv:0907.4560.

[Benc] ———, *On perturbations of continuous structures*, submitted, arXiv:0802.4388.

[BU] Itaï Ben Yaacov and Alexander Usvyatsov, *Continuous first order logic and local stability*, Transactions of the American Mathematical Society, to appear, arXiv:0801.4303.

[Fre04] D. H. Fremlin, *Measure theory volume 3: Measure algebras*, Torres Fremlin, 25 Ireton Road, Colchester CO3 3AT, England, 2004.

[HI02] C. Ward Henson and José Iovino, *Ultraproducts in analysis*, Analysis and Logic (Catherine Finet and Christian Michaux, eds.), London Mathematical Society Lecture Notes Series, no. 262, Cambridge University Press, 2002.

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