THE GROUP CONFIGURATION IN SIMPLE THEORIES AND ITS APPLICATIONS

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ABSTRACT. In recent work, the authors have established the group configuration theorem for simple theories, as well as some of its main applications from geometric stability theory, such as the binding group theorem, or, in the ω-categorical case, the characterization of the forking geometry of a finitely based non-trivial locally modular regular type as projective geometry over a finite field and the equivalence of pseudolinearity and local modularity.

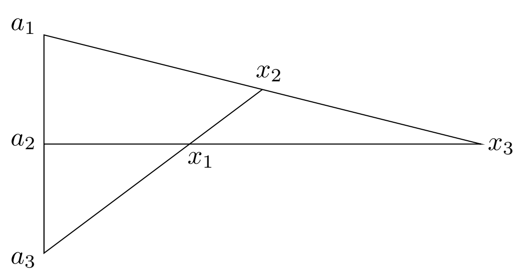
The proof necessitated an extension of the model-theoretic framework to include *almost hyper-imaginaries*, and the study of *polygroups*.

1. The stable case

The group configuration theorem, proved in full generality by Hrushovski [Hru02] following ideas of Zilber [Zil84] is one of the cornerstones of geometric stability theory (see [Pil96] for a comprehensive exposition); it can be seen as an abstract version of the classical problem of coordinatizing a projective geometry [vN60] (see also [Tom01]). It states that if some dependence/independence situation (the group configuration) exists then there is a non-trivial group behind it. More precisely, if a stable structure contains a 6-tuple $(a_1, a_2, a_3, x_1, x_2, x_3)$ such that

(1) any one of a_1, a_2, a_3 is in the closure of the other two,
(2) $x_i \in \mathrm{cl}(a_j, x_k)$ for $\{i, j, k\} = \{1, 2, 3\}$,
(3) all other triples and all pairs are independent,

then there is a type-definable group acting definably, transitively and faithfully on a type-definable set equivalent to $\mathrm{tp}(x_1)$. Here closure can mean algebraic closure, or p-closure for some regular type p (or in fact more general P-closure, as in [Wag97, Wag01]). The conditions are best visualized in a diagram, where lines indicate dependence:



Note that if we have a group G acting transitively on a set X in a simple theory, we can take two independent group generic g, h and a set generic x independent of gh and put $a_1 = g$, $a_2 = h$, $a_3 = hg$, $x_2 = x$, $x_3 = gx$, $x_1 = hgx$, in order to obtain a group configuration.

The proof of the group configuration theorem in the stable case proceeds by replacing the original elements by interalgebraic ones (over some independent parameters), such that in addition x_i and x_j are inter*definable* over a_k for $\{i, j, k\} = \{1, 2, 3\}$. We then have definable invertible (partial) functions $f_{jk} : \mathrm{lstp}(x_j) \to \mathrm{lstp}(x_k)$ defined over a_i, such that if the argument x is independent of f_i so is the image $f_i(x)$ (*generic* functions). Using definability of types and stationarity, the relation "$f(x) = f'(x)$ for generic x" is a definable equivalence relation, the *germ* of f, which is compatible with composition. One can now apply the Weil-Hrushovski group chunk theorem [Hru02] to the set

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of germs of functions $f^{-1} \circ f'$, where f and f' are two independent realisations of $\mathrm{lstp}(f_{12})$, in order to obtain, a type-definable group acting on $\mathrm{lstp}(x_1)$.

Perhaps the most important application of the group configuration theorem is the characterisation of the forking geometry of a non-trivial locally modular regular type $p \in S(\emptyset)$ in a stable theory. Recall that p is *k-linear* if $\mathrm{SU}_p(\mathrm{Cb}(q)) \leq k$ for any completion $q(x,y) \supseteq p(x) \cup p(y)$ of SU_p-rank 1; it is *pseudolinear* if it is k-linear for some k. Note that k-linearity for $k \geq 2$ implies non-triviality, and 1-linearity is equivalent to *local modularity*; a 1-linear type of SU-rank 1 is one-based. A k-linear non-trivial regular type p gives rise to a group configuration with $\mathrm{SU}_p(x_i) = 1$ and $\mathrm{SU}_p(a_i) \leq k$, and thus to a group G of SU_p-rank $\leq k$ acting transitively and faithfully on a set X of SU_p-rank 1. If p is not $(k-1)$-linear, then $\mathrm{SU}_p(G) = k$. However, such transitive actions have been completely classified by Hrushovski [Hru02] in the stable case: either G is abelian and $k = 1$, or there is a definable field and the action is either affine ($k = 2$) or projective ($k = 3$). Since a field is not pseudolinear (the canonical base of a polynomial curve of degree k with independent generic coefficients has rank k), the latter two cases are impossible, and pseudolinearity implies local modularity. Moreover, the geometry of a locally modular non-trivial regular type must be that of a locally modular regular group, which on the connected component is that of a vector space over a division ring, again by a theorem of Hrushovski [Hru87].

The second application, for simple theories, of the group configuration occurs at a point where it is not necessary in the stable context. Recall that if p is a type over A and π a partial type over A, then p is *π-internal* if for any $a \models p$ there is $B \mathop{\smile\!\!\!\!\!\!|}_A a$ and a tuple $\bar{c}$ of realizations of π such that $a \in \mathrm{dcl}(AB\bar{c})$. On the other hand, p is *almost foreign* to π if any realisation of p is independent *over* A of any realisation of π. The binding group theorem, again first proved by Zilber [Zil80] in the ω-categorical context and generalized by Hrushovski [Hru02], states that if a strong type p is π-internal and almost orthogonal to π^ω, then the group $\mathrm{aut}(p/A \cup \pi)$ of permutations of the realizations of p induced by automorphisms fixing A and all realizations of π is type-definable and acts transitively on p. Note that for regular p we can then use the classification of all possible actions; in particular we see that at most 3 parameters from p are needed to witness internality; moreover, if $\mathrm{aut}(p/A \cup \pi)$ is not abelian, there is an interpretable field. The original proof by Hrushovski again used germs of generic functions; Poizat [Poi87] gave a direct interpretation of the group, which however relies heavily on stationarity of types and does not generalize to simple theories. Worse still: adding a generic bipartite graph between p and π will destroy $\mathrm{aut}(p/A \cup \pi)$ while preserving independence. So in order to understand internality and almost foreignness, we shall have to look for a different group.

2. The group configuration theorem for simple structures

We shall follow the terminology of [Wag00]. In particular, the class of a modulo an equivalence relation (or even just a reflexive symmetric relation) E will be denoted by a_E. Throughout, we shall assume that the ambient theory is simple.

2.1. **Germs.** In a stable theory, the germ of a generic function f was defined as its class modulo the relation "f and g agree on some $x \mathop{\smile\!\!\!\!\!\!|} fg$". If $\mathrm{tp}(x)$ is stationary, this is indeed an equivalence relation, which is definable by definability of types; note that the germ of f is equal to $\mathrm{Cb}(x, f(x)/f)$. In a simple theory, it is *a priory* only type-definable (whence the immediate need to consider hyperimaginaries); for non-stationary $\mathrm{tp}(x)$ it need not be transitive. Moreover, contrary to the stable case, there is no reason why $f(x)$ should be in $\mathrm{dcl}(x, \mathrm{Cb}(x, f(x)/f))$: the theory only yields $f(x) \in \mathrm{bdd}(x, \mathrm{Cb}(x, f(x)/f))$. We are thus lead to consider *multifunctions*.

Definition 1. A partial type $\pi(x)$ over A has *definable independence* if for any partial type $\pi'(y)$ over A the set $\pi(x) \wedge \pi'(y) \wedge x \mathop{\smile\!\!\!\!\!\!|}_A y$ is type-definable.

Remark 1. A complete type has definable independence. More generally, if $\pi(x)$ and $\pi'(y)$ have definable independence, so does $\pi(x) \wedge \pi'(y) \wedge x \mathop{\smile\!\!\!\!\!\!|} y$; if $\pi(x)$ has definable independence and $\pi'(x) \vdash \pi(x)$, then π' has definable independence.

Definition 2. A partial type $\pi(x, y, z)$ is an *invertible generic action* if

(1) $\mathrm{Func}(\pi) = \pi \restriction_x$ and $\mathrm{Arg}(\pi) = \pi \restriction_y$ have definable independence,

(2) π implies that x, y, z are pairwise independent,
(3) if $f \models \mathrm{Func}(\pi)$, for any y there are at most boundedly many z, and for any z there are at most boundedly many y, such that $\models \pi(f, y, z)$.

We also say that a partial generic action is:

- *complete* if $\pi(f, x, y)$ is a (consistent) Lascar strong type over f for every $f \models \mathrm{Func}(\pi)$,
- *reduced* if it is complete and for any $f \models \mathrm{Func}(\pi)$ we have $f = \mathrm{Cb}(\pi(f, x, y))$.

Remark 2. (1) If π is an invertible complete reduced generic action, so is $\pi^{-1}(x, y, z) = \pi(x, z, y)$.
(2) The second part of condition (3) means that π (resp. any $f \models \mathrm{Func}(\pi)$) is *invertible*.
(3) If π is not complete, we can construct its *completion* $\underline{\pi}$ as follows: Let $E(xyz, x'y'z')$ be the type-definable equivalence relation stating that $x = x'$ and there are x-indiscernible sequences $(y_i z_i : i < \omega)$ and $(y'_i z'_i : i < \omega)$ with $yz = y_0 z_0$, $y'z' = y'_0 z'_0$ and $y_1 z_1 = y'_1 z'_1$. This uniformly type-defines the equivalence relation "$x = x'$ and $\mathrm{lstp}(yz/x) = \mathrm{lstp}(y'z'/x')$"; putting $\underline{x} = (xyz)_E$, we obtain that $\underline{x} \in \mathrm{bdd}(x)$. Define $\underline{\pi}(\underline{x}, y', z')$ as $\exists x' \, [\pi(x', y', z') \wedge (x'y'z')_E = \underline{x}]$; if $fab \models \pi$ and $p = \mathrm{lstp}(ab/f)$, we write f_p for $(fab)_E$. Then $\models \underline{\pi}(f_p, a', b')$ iff $\models \pi(f, a', b')$ and $\mathrm{lstp}(a'b'/f) = p$ (so $\underline{\pi}$ is complete). Finally, if $f \models \mathrm{Func}(\pi)$, put
$$\underline{f} = \{(fab)_E : \models \pi(f, a, b)\} = \{f_p : p = \mathrm{lstp}(ab/f) \text{ for some } ab \models \pi(f, x, y)\},$$
the bounded set of completions of f.
(4) If π is complete but not reduced, we can construct its *reduction* $\bar{\pi}$ as follows: If $f, g \models \mathrm{Func}(\pi)$, define $f \sim_0 g$ if $\pi(f, x, y)$ and $\pi(g, x, y)$ have a common non-forking extension. Since π is complete, this type-definable relation is generically transitive, and its two-step iterate is a type-definable equivalence relation E by [Wag00, Lemma 3.3.1]. Putting $\bar{f} = f_E$ and $\bar{\pi}(\bar{x}, y, z) = \exists x' \, [\pi(x, y, z) \wedge E(x, x')]$, we obtain $\models \bar{\pi}(\bar{f}, a, b)$ iff $\models \pi(f, a, b)$ and $\bar{f} = \mathrm{Cb}(ab/f)$ (so $\bar{\pi}$ is reduced). We put $\hat{f} = \{\bar{f'} : f' \in \underline{f}\}$. Reduced functions are called *germs*, and we denote $\mathrm{Germ}(\pi) = \mathrm{Func}(\bar{\pi})$.
(5) We have only required that $\pi(f, y, z)$ be a Lascar strong type for any $f \models \mathrm{Func}(\pi)$. If $\pi(x, a, z)$ or $\pi(x, y, b)$ is a Lascar strong type for any $a \models \mathrm{Arg}(\pi)$ and $b \models \mathrm{Val}(\pi) = \pi \restriction_z$, we say that π is *strong on the left* (resp. *right*). Clearly, we may apply the completion process also to the argument and value variables in π, and *strengthen* any invertible complete reduced generic action.

Definition 3. A hyperimaginary a is *quasi-finite* if it is in the definable closure of a finite real tuple.

Remark 3. If π is an invertible complete reduced generic action on quasi-finite sorts, the completion and strengthening processes preserve quasi-finiteness: If $\models \pi(f, a, b)$, then $\underline{f} = \mathrm{bdd}(f) \cap \mathrm{dcl}(fab)$, $\underline{a} = \mathrm{bdd}(a) \cap \mathrm{dcl}(fab)$, and $\underline{b} = \mathrm{bdd}(b) \cap \mathrm{dcl}(fab)$. Moreover, as $\mathrm{Cb}(p) \in \mathrm{dcl}(A)$ for any type p over A which is Lascar strong, reduction preserves quasi-finiteness as well.

Definition 4. If π and π' are two invertible complete reduced generic actions with $\mathrm{Val}(\pi) = \mathrm{Arg}(\pi')$, their *composition* $\pi' \circ \pi$ is defined as the reduction of the completion of $x \perp\!\!\!\perp x' \wedge \exists u \, [\, u \perp\!\!\!\perp xx' \wedge \pi(x, y, u) \wedge \pi'(x', u, z) \,]$.

Lemma 1. [Ben02, Proposition 2.8] *The composition of two invertible complete reduced generic actions is again an invertible complete reduced action. For independent $f \models \mathrm{Germ}(\pi)$ and $g \models \mathrm{Germ}(\pi')$ any $h \in \widehat{g \circ f}$ is in $\mathrm{Germ}(\pi' \circ \pi)$.*

Definition 5. A composition $\pi' \circ \pi$ of two invertible complete reduced generic actions is *generic* if for independent $f \in \mathrm{Germ}(\pi)$ and $g \in \mathrm{Germ}(\pi')$ any $h \in \widehat{g \circ f}$ is independent of f and of g.

Theorem 2. [Ben02, Corollary 4.8] *Let $C = (a_1 a_2 a_3 x_1 x_2 x_3)$ be a group configuration with respect to* $\mathrm{bdd}(-)$. *Put $x'_i = \mathrm{bdd}(x_i) \cap \mathrm{dcl}(C)$, and $a'_i = \mathrm{Cb}(x'_j x'_k / a_i)$ for $\{i, j, k\} = \{1, 2, 3\}$. Then $\pi = \mathrm{lstp}(a'_i x'_j x'_k)$ is an invertible complete reduced generic action which is left and right strong, and the composition $\pi^{-1} \circ \pi$ is generic.*

The proof is analogous to the stable case, except that we don't have to worry about replacing bounded by definable closure. In fact, if the original configuration is merely with respect to p-closure for some regular type p, we find $A \mathop{\smile\!\!\!\!\!\!\,|} C$ such that C is a group configuration with respect to bdd$(-)$ *over* A; if C is quasi-finite, we can choose A quasi-finite.

Definition 6. An invertible complete reduced generic action π is a *generic multi-chunk* if $\text{Arg}(\pi)$ is a Lascar strong type, and $\pi = \pi^{-1} = \pi \circ \pi$.

Note that in this case $\text{Arg}(\pi) = \text{Val}(\pi)$.

Theorem 3. [Ben02, Corollary 3.8] *Let π be an invertible complete reduced generic action which is strong on the left, and such that $Germ(\pi)$ and $Arg(\pi)$ are Lascar string types. If the composition $\hat{\pi} = \pi^{-1} \circ \pi$ is generic, then $\hat{\pi}$ is a generic multi-chunk.*

Theorem 4. [Ben02, Theorem 3.9] *Let π be a generic multi-chunk, and $P = Germ(\pi)$. Then composition $\pi \circ \pi = \pi$ induces a hyperdefinable multifunction $* : P \times P \to P$, which is defined up to a bounded non-zero number of possible values, satisfying*

(1) Generic independence: *If $f \mathop{\smile\!\!\!\!\!\!\,|} g$ and $h \in f * g$, then $h \mathop{\smile\!\!\!\!\!\!\,|} f$ and $h \mathop{\smile\!\!\!\!\!\!\,|} g$.*
(2) Generic associativity: *If $f, g, h \in P$ are independent, then $(f * g) * h = f * (g * h)$ (as sets).*
(3) Generic inverse: *For any $f \mathop{\smile\!\!\!\!\!\!\,|} g$ we have $h \in f * g$ iff $f \in h * g^{-1}$ iff $g \in f^{-1} * h$.*

We shall call such a structure $\langle P, * \rangle$ a *generic polygroup chunk*. It almost satisfies the hypothesis of the Weil-Hrushovski group chunk theorem [Wag00, Theorem 4.7.1], only that multiplication is many-valued.

2.2. **Almost hyperimaginaries.**
In order to deal with many-valued multiplication, one has to extend the logic to include a special sort of ultra-imaginaries.

Definition 7. Let $(I, \leq)$ be a directed partial order, and X a sort (real, imaginary, or hyperimaginary).

(1) An equivalence relation on X is *invariant* if it is automorphism-invariant. A class modulo an invariant equivalence relation is called an *ultraimaginary*.
(2) A *graded equivalence relation (g.e.r.)* $R = \bigvee_I R_i = R_I$ on X is the union of reflexive symmetric type-definable relations $(R_i : i \in I)$ on X satisfying:
   (a) If $i \leq j$ then R_j is coarser than R_i.
   (b) For every i, j there is $k \geq i, j$ with $xR_iyR_jz \Longrightarrow xR_kz$.
(3) An invariant equivalence relation R is *almost type-definable* if there is a type-definable symmetric and reflexive relation R' finer than R such that any R-class can be covered by boundedly many R'-classes. If in addition R is graded and R' is finer than some R_i, then we say that it is *gradedly almost type-definable (above i)*.
(4) A class modulo a (graded) invariant equivalence relation is called a *(graded) ultraimaginary*. A class modulo a (gradedly) almost type-definable equivalence relation is called a (graded) almost hyperimaginary.
(5) Let $R = R_I$ and $R' = R'_J$ be g.e.r.'s on sorts X and Y respectively, $f(x, y)$ a type-definable relation on $X \times Y$, and put $f(x) = \{y \in Y : \models f(x, y)\}$.
   (a) f defines a *gradedly type-definable partial multi-map* $\bar{f} : X/R \to Y/R'$, if
      (i) there is some R'_0 such that for every $x \in X$ there is a bounded set of elements $y_\alpha \in f(x)$ with $f(x) \subseteq \bigcup_\alpha y_{\alpha R'_0}$, and
      (ii) for every $i \in I$ there is $j \in J$ such that $f(x_{R_i}) \subseteq f(x)_{R'_j}$ for every $x \in X$.
   (b) If in the above we need at most a single y_α, then f defines a *gradedly type-definable partial map*.
   (c) If in addition $f(x) \neq \emptyset$ for every $x \in X$, then f defines a *gradedly type-definable total multi-map* or *map*, as the case may be.
(6) Two gradedly type-definable multi-maps $\bar{f}$ and $\bar{f}'$ are gradedly equal if there is i such that $f(x) \subseteq f'(x)_{R_i}$ and $f'(x) \subseteq f(x)_{R_i}$ for every $x \in X$.

Remark 4. (1) It is easy to see that every ultraimaginary can be graded, and every almost type definable graded equivalence relation is equivalent to a gradedly almost type-definable equivalence relation.

(2) If a graded equivalence relation R is on a hyperimaginary sort modulo some type-definable equivalence relation E, one can incorporate E into R and assume that R lives on real tuples.

Definition 8. Two ultraimaginaries a_R and b_R have the same (Lascar strong) *type* over a hyperimaginary c if there are representatives which do.
They are *independent* over c, denoted $a_R \mathop{\smile\!\!\!\!\!\!|}_c b_R$, if they have representatives which are.

Clearly, this coincides with the definitions for hyperimaginaries. Moreover, two ultraimaginaries have the same type (Lascar strong type) over c if and only if they are conjugate under a (strong) c-automorphism. Independence is particularly well-behaved for almost hyperimaginaries.

Lemma 5. [BTW04, Lemma 1.8] *The following are equivalent:*

(1) *R is an (I-gradedly) almost type-definable equivalent relation.*
(2) *There is a type-definable reflexive symmetric relation R' finer than R (finer than some R_i), such that whenever $a_R \mathop{\smile\!\!\!\!\!\!|}_c b$ for some hyperimaginaries b, c, then there is $a'R'a$ with $a' \mathop{\smile\!\!\!\!\!\!|}_c b$.*
(3) *There are a cardinal κ and a type-definable reflexive symmetric relation R' finer than R (finer than some R_i), such that within an R-class there are no κ disjoint R'-classes.*
(4) *There are a cardinal κ and a type-definable reflexive symmetric relation R'' finer than R (finer than some R_i), such that among any κ R-equivalent elements there are necessarily two which satisfy R''.*

We thus get a "first-order" characterisation of independence for almost hyperimaginaries:

Lemma 6. [BTW04, Lemma 1.9] *Assume that R' witnesses that R is almost type-definable. Write $p(x, y) = \mathrm{tp}(ab/c)$, $p'(x, y) = p(x_{R'}, y_{(R'^2)})$. Then $a_R \mathop{\smile\!\!\!\!\!\!|}_c b_R$ if and only if $p'(x, b)$ does not divide over c.*

Independence for ultraimaginaries satisfies symmetry, a suitable form (since we do not consider types *over* ultraimaginaries) of transitivity, local character, extension, and the Independence Theorem. For almost hyperimaginaries, we also have local character.

Definition 9. An equational polystructure is given by a theory whose language is purely functional apart from a binary relation $\in$, without equality, and whose axioms are universal quantifications over formulas of the form $\bigwedge x_i \in \tau_i \to \bigvee y_j \in \sigma_j$ where τ_i and σ_j are terms. The interpretation, however, is that every function is multi-valued, and everything on the right-hand side of a $\in$ symbol is considered as a set (variables being singletons).
An ultradefinable equational structure S in a given theory is given by a definable set S_0, some I-g.e.r. R on S_0 such that $S = S_0/R$, and for each n-ary function symbol f a gradedly definable map $f^S : (S_0/R)^n \to S_0/R$, such that:

- For every axiom $(\bigwedge x_n \in \tau_n \to \bigvee y_m \in \sigma_m)$ and every $i \in I$ there is $j \in I$, such that for every substitution of elements from S for the variables in the axiom, if the conditions hold up to R_i (that is, for every n there is $x'_n \in_i x_n$ such that $x'_n \in \tau_n$), then one of the conclusions holds up to R_j.

Remark 5. We have formulated the definitions for one-sorted structures, the adaptations needed for many-sorted ones, such as polyspaces, being obvious.

Example 1. A polygroup [Cor93] is a structure with a binary multifunction $\cdot$ and a unary multifunction $^{-1}$, satisfying:

(1) $u \in (x \cdot y) \cdot z \leftrightarrow u \in x \cdot (y \cdot z)$,
(2) $z \in x \cdot y^{-1} \leftrightarrow x \in z \cdot y$,
(3) $z \in x^{-1} \cdot y \leftrightarrow y \in x \cdot z$.

A polygroup with identity carries in addition a 0-ary multifunction e satisfying:

4. $z \in x \cdot e \leftrightarrow z \in x$,
5. $z \in e \cdot x \leftrightarrow z \in x$.

Example 2. Let G be a group, and H a (not necessarily normal) subgroup. The double coset space $G/\!\!/H$ is a polygroup with the multioperation $HaH * HbH := \{HahbH : h \in H\}$.

Example 3. A projective geometry is an incidence system (P, L, I) consisting of a set of points P, a set of lines L and an incidence relation $I \subseteq P \times L$ satisfying the following axioms:

(1) any line contains at least three points;
(2) two distinct points a, b are contained in a unique line denoted by $L(a, b)$;
(3) if a, b, c, d are distinct points and $L(a, b)$ intersects $L(c, d)$, then $L(a, c)$ must intersect $L(b, d)$ (Pasch axiom).

Let $P' := P \cup \{e\}$, where e is not in P, and define:

- for $a \neq b \in P$, $a \circ b := L(a, b) \setminus \{a, b\}$;
- for $a \in P$, if any line contains exactly three points, put $a \circ a := \{e\}$, otherwise $a \circ a := \{a, e\}$;
- for $a \in P'$, $e \circ a = a \circ e := \{a\}$.

Then it is easily verified that $(P', \circ)$ is a polygroup.

2.3. **Generic types.** Let $P = P_0 = P_0/R_I$ be a gradedly almost hyperdefinable polygroup. Generic types and elements of P are defined in analogy to the hyperdefinable case [Pil96, Pil98, Wag01].

Definition 10. A *generic* element of P is an element g_R such that whenever $g_R \downarrow h$ for $h \in P_0$ and $k \in h \cdot g$, then $k_R \downarrow h$.

The basic theory of generics holds with this definition.

Lemma 7. [BTW04, Lemma 2.13]

(1) *Let g_R be a generic element of P, and assume that $h_R \downarrow g_R$ and $k_R \in h_R \cdot g_R$. Then k_R is generic.*
(2) *If $g_R \in P$ is generic, then so is g_R^{-1}.*
(3) *If $g_R \in P$ is generic, then whenever $h \downarrow g_R$ for $h \in P_0$ and $k \in g \cdot h$, we have $k_R \downarrow h$.*

Existence of generic types is proved for polygroups by means of a suitable family of stratified ranks. For convenience, we give names to certain elements of P_0 and I:

(1) We fix $0 \in I$ and an infinite cardinal ν such that every R-class can be covered by ν R_0-classes, and every operation ($\bar{\cdot}$ or $\overline{^{-1}}$) has at most ν values.
(2) We choose $1 \in I$ such that $(g_{R_0} \cdot h)_{R_0} \subseteq (g \cdot h)_{R_1}$, and $h \in [(g^{-1} \cdot g) \cdot h]_{R_1}$ for all $g, h \in P_0$.

Definition 11. Let $k < \omega$, and $\varphi(x, y)$, $\psi(y_0, \ldots, y_k)$ be formulas. We say that ψ is a *k-inconsistency witness* for φ if $\psi(\bar{y}) \wedge \bigwedge_{j<k} \varphi(x, y_j)$ is contradictory.

Definition 12. Let ψ be a k-inconsistency witness for $\varphi(x_{R_0}, y)$. We define a local rank $D_P(-, \varphi, \psi)$ with values in $\omega + 1$ on (consistent) partial types (with parameters) extending $x \in P_0$:

- $D_P(\pi, \varphi, \psi) \geq n + 1$ if there are a sequence $(c_\ell : \ell < \omega)$, any k-subsequence of which satisfies ψ, and $g \in P_0$, such that $D_P(\pi(x) \wedge \varphi((g \cdot x)_{R_1}, c_\ell), \varphi, \psi) \geq n$ for all $\ell < \omega$.

Lemma 8. [BTW04, Section 2.2] *$D_P(-, \varphi, \psi)$ takes finite values and is left-translation-invariant. Moreover, the condition $D_P(\pi(x, a), \varphi, \psi) \geq n$ is type-definable on the parameter a. The family of local ranks $D_P(-, \varphi, \psi)$ witnesses dividing. Hence generic types exist; they are precisely those who have the same stratified ranks as P (over $\emptyset$). Thus the set $g(P)$ of all the representatives of generic elements is a gradedly almost hyperdefinable generic (poly-)group chunk.*

This also means that the independence of generic elements is type-definable, using stratified ranks. For generic chunks, we have:

Lemma 9. [BTW04, Lemma 2.22] *Let $S = S_0/R_I$ be a generic chunk, and let w be some sort. Then there is a partial type $\Phi(x, w)$ such that $\Phi(a, d)$ if and only if $a \in S_0$ and $a_R \downarrow d$.*
Moreover, if R' is an almost type-definable equivalence relation on w, then there is $\Phi'(x, y)$ such that $\Phi(a, d)$ if and only if $a \in S_0$ and $a_R \downarrow d_{R'}$.

2.4. **The core equivalence and blow-up.** We shall now deal with the fact that multiplication is many-valued and that, as in example 3, we can have $c, c' \in a*b$ with $c \mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$|$}} c'$. To that end, we shall first divide out by an almost type-definable equivalence relation with bounded classes, the *core equivalence*, and then replace the generic polygroup chunk by a bounded cover. Both processes clearly preserve rank (if defined). However, as the core equivalence is not type-definable, it is at this stage that we have to introduce almost hyperimaginaries.

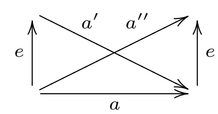
Definition 13. Let $S = S_0/R$ be an I-gradedly almost hyperdefinable generic polygroup chunk. For $a, b \in S_0$ and $i \in I$, we say that $a \sim_{i1} b$ if there is $g \perp ab$ such that $a, b \in_i g \cdot h$ for some h. The n-closure of $\sim_{i1}$ is $\sim_{in}$, and $\sim$ is $\bigvee_{in} \sim_{in}$.
S is *coreless* if the core equivalence is the same as R, that is for every $(i, n) \in I \times \omega$ there is $j \in I$ such that R_j is coarser than $\sim_{in}$.

Lemma 10. [BTW04, Lemma 2.28] *Let $S = S_0/R$ be an I-gradedly almost hyperdefinable polygroup chunk. Then $\sim$ is an $(I \times \omega)$-gradedly almost type-definable equivalence relation on S coarser than R, the* core equivalence. *Every $\sim$-class contains boundedly many R-classes.*
$S_0/\!\sim$ is coreless. Any almost hyperdefinable group chunk is coreless.

We now fix an I-gradedly almost hyperdefinable coreless generic polygroup chunk $S = \langle S_0/R, \cdot, ^{-1} \rangle$, as well as some R_0 which witnesses that R is almost type-definable. We shall no longer distinguish between the multiplication and inverse on S_0, and the maps induced on S_0/R.

Definition 14. We fix some $e \in S_0$, and put $S_0' = \{a \in S_0 : a_R \perp e_R\}$.

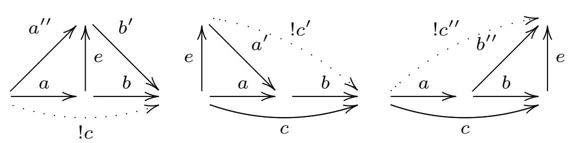
(1) Define $\tilde{S}_0 = \{(a, a', a'') : a \in S_0', a' \in e^{-1} \cdot a \text{ and } a'' \in a \cdot e\}$ and $\tilde{S} = \tilde{S}_0/R$.



(We follow a tacit understanding that R may also stand for $R \times R \times R$, where this is clear from the context.)

(2) A triplet $\tilde{a} = (a, a', a'') \in \tilde{S}_0$ is called a *blow-up* of a. Conversely, we define the *blow-up* map $\pi : \tilde{S}_0 \to S_0'$ by $\pi(a, a', a'') = a$.

(3) Given $\tilde{a}_R \perp_e \tilde{b}_R$, we wish to define $\tilde{a} \cdot \tilde{b}$. First, we choose $c \in a \cdot b \cap (a'' \cdot b')_{R_1}$, and then $c' \in e^{-1} \cdot c \cap (a' \cdot b)_{R_1}$ and $c'' \in c \cdot e \cap (a \cdot b'')_{R_1}$, for suitable $1 \in I$. Set $\tilde{a} \cdot \tilde{b}$ to be the set of all $\tilde{c} = (c, c', c'')$ obtained in this manner.



(4) Recall that the inverse is a gradedly definable map, so it is only defined up to some R_i. Thus, for $\tilde{a} = (a, a', a'') \in \tilde{S}_0$, we can define its inverse as:

$$\tilde{a}^{-1} = \{(b, b', b'') \in \tilde{S}_0 : b \in a^{-1}, b' \in a''^{-1}_{R_j}, b'' \in a'^{-1}_{R_j}\}$$

for $j \in I$ big enough to make sure that $\tilde{a}^{-1}$ cannot be empty; re-choosing R_0 we may assume that $j \leq 0$.

Remark 6. It should be noted that this blow-up is similar to the procedure by which one obtains a dcl-configuration from an acl-configuration in a stable theory: one adds an independent base point e, over which any point a is interbounded with its blow-up $\tilde{a}$; however, $\tilde{a}$ contains two independent points (over $\emptyset$) of the original set. A (coordinate of a) point in the product of $\tilde{a} \cdot \tilde{b}$ is obtained as the intersection of two lines/products in the original configuration/chunk.

Theorem 11. [BTW04, Proposition 3.5 and Theorem 3.6] $\langle \tilde{S}, \cdot, ^{-1} \rangle$ *is an almost hyperdefinable generic group chunk (with inversion) over* e. *Moreover, the blow-up map induces a gradedly type-definable surjective bounded-to-one homomorphism* $\bar{\pi} : \tilde{S} \to S'_0/R$, *such that if* $a_R \mathop{\smile\!\!\!\!\!\mid}_e b_R$, $c \in a \cdot b$, *and a blow-up* $\tilde{c}$ *is given, then there are blow-ups* $\tilde{a}_R \mathop{\smile\!\!\!\!\!\mid}_e \tilde{b}_R$ *such that* $\tilde{c}_R = \tilde{a}_R \cdot \tilde{b}_R$.

2.5. **The group chunk theorem.** In order to finish the construction, it is now sufficient to prove an almost hyperdefinable version of the group chunk theorem.

Definition 15. Let $\langle \tilde{S}_0/R, \cdot, ^{-1} \rangle$ be an I-gradedly almost hyperdefinable group chunk. We define a relation $R' = \bigvee_I R'_i$ on $\tilde{S}_0^2$ as follows: We say that $(a,b)R'_i(a',b')$ if there are x, y such that:

(1) $x_R \mathop{\smile\!\!\!\!\!\mid} aba'b'$ and $y_R \mathop{\smile\!\!\!\!\!\mid} aba'b'$.

(2) $a \cdot x R_i a' \cdot y$ and $b \cdot x R_i b' \cdot y$ (where AR_iB means that aR_ib for some $a \in A$ and $b \in B$).

We write $[a,b]_i = (a,b)/R'_i$, and $[a,b] = (a,b)/R'$.

Lemma 12. [BTW04, Lemma 4.3] R' *is an* I*-gradedly almost type-definable equivalence relation.*

Theorem 13. [BTW04, Theorem 4.1 and 4.4] *Let* $\tilde{S} = \langle \tilde{S}_0/R, \cdot, ^{-1} \rangle$ *be an* I*-gradedly almost hyperdefinable group chunk, and* R' *as above. Then* $G = \tilde{S}_0^2/R'$ *is a gradedly almost hyperdefinable group, with product* $[a,b] \cdot [b,c] = [a,c]$ *and inverse* $[a,b]^{-1} = [b,a]$. *There is a gradedly type-definable map* $\sigma : \tilde{S} \to G$ *whose image generates* G, *and the couple* (G, σ) *is gradedly unique as such.*
If $P = P_0/R''$ *is a coreless polygroup, and* $\tau : \tilde{S}_0/R \to P_0/R''$ *is a gradedly type-definable generic homomorphism (i.e.* $\tau(a^{-1}) = \tau(a)^{-1}$, *and* $\tau(a \cdot b) \in \tau(a) \cdot \tau(b)$, *gradedly, for any independent* $a, b \in \tilde{S}_0$*) such that every element in the image of* τ *is generic, then there is a unique gradedly type-definable homomorphism* $\hat{\tau} : G \to P$ *with* $\tau = \hat{\tau} \circ \sigma$. *Moreover, for every* $g \in G$ *(we omit the subscript* R'*), if we write it as* $a \cdot b$ *where these are generics each of which is independent of* g, *then* $\hat{\tau}(g) = \tau(a) \cdot \tau(b) \cap \mathrm{dcl}(g) = \tau(a) \cdot \tau(b) \cap \mathrm{bdd}(g)$.

This can be used to see that every coreless almost hyperdefinable generic polygroup chunk comes from a double coset space:

Theorem 14. [Ben03, Theorem 1.12] *Let* S *be a coreless generic polygroup chunk,* G *the corresponding almost hyperdefinable group as given by Theorems 11 and 13, and* $H \subseteq G$ *the set* $\{\sigma(a, a'_0, a'') \cdot \sigma(a, a'_1, a'')^{-1} : (a, a'_0, a'')_R, (a, a'_1, a'')_R \in \tilde{S}\}$. *Then* H *is a bounded subgroup of* G, *and the set* $gen(G/\!\!/H)$ *of generic elements is isomorphic to* S.

Finally, given a generic multi-chunk π, it is also possible to recover the action:

Theorem 15. [TW03, Theorem 6] *Let* π *be a generic multi-chunk. Then there is an almost hyperdefinable group* G *acting transitively and faithfully on an almost hyperdefinable set* X. *If* π *is quasi-finite, so are* G *and* X. *Moreover, over some independent parameters a generic element of* G *is interbounded with a realization of* $Germ(\pi)$, *and a generic element of* X *is interbounded with a realization of* $Arg(\pi)$.

In particular, the rank of G (if defined) is equal to the rank of $\mathrm{Germ}(\pi)$, and the rank of X is equal to the rank of $\mathrm{Arg}(\pi)$.

3. Applications

3.1. **One-basedness and pseudolinearity.** In this section we shall use SU_p-rank, Lascar rank relativized to some type p, and p-closure $\mathrm{cl}_p(-)$, as developed in [Wag97, Sections 3.5 and 3.6] and [Wag01, Section 5].

Definition 16. Let p be a regular type over A, and $k < \omega$. We say that p is *k-linear* if $\mathrm{SU}_p(\mathrm{Cb}(q)/A) \leq k$ for any completion $q(x,y) \supseteq p(x) \cup p(y)$ of SU_p-rank 1, and k is minimal possible; it is *pseudolinear* if it is k-linear for some k.
We say that p is *locally modular* if any tuples $\bar{a}$ and $\bar{b}$ of realizations of p are independent over $\mathrm{cl}_p(A\bar{a}) \cap \mathrm{cl}_p(A\bar{b})$. It is *one-based* if $\bar{a}$ and $\bar{b}$ are independent over $\mathrm{acl}^{heq}(A\bar{a}) \cap \mathrm{acl}^{heq}(A\bar{b})$.
Finally, p is *trivial* if any pairwise independent set of realisations of p is independent.

Remark 7. (1) Since $q(x,y) \supseteq p(x) \cup p(y)$ is p-internal, $\mathrm{SU}_p(q) = w_p(q)$.
(2) A type $p \in S(A)$ is 1-linear if and only if it is locally modular.
(3) If $\mathrm{SU}(p) = 1$, then local modularity equals one-basedness.
(4) Local modularity refers to the (pre-)geometry of p^{heq}, not of p.

Theorem 16. [TW03, Theorem 9] *Let $p \in S(\emptyset)$ be a k-linear regular type, with $k > 1$ or $k = 1$ and p non-trivial. Then there is an almost hyperdefinable p-connected group G of* SU_p*-rank k acting transitively and faithfully on an almost hyperdefinable set X of* SU_p*-rank* 1*. If p is quasi-finite, so are G and X; only finitely many parameters are needed.*

Remark 8. In fact X is generically *p-pure*, i.e. a forking extension of a type in X has SU_p-rank 0. Similarly, a group G is *p-connected* if a forking extension of any generic type has smaller SU_p-rank.

In the ω-categorical case, a quasi-finite hyperimaginary is in fact imaginary, so we can use known results about such group actions, and describe the possibilities.

Theorem 17. [TW03, Theorem 14] *Let p be a finitely based locally modular regular Lascar strong type in an ω-categorical theory. Then the geometry associated to (some non-forking extension of) p is either trivial, or projective geometry over a finite field.*

Remark 9. (1) When $\mathrm{SU}(p) = 1$, Theorem 17 has also been shown by dePiro and Kim [dPK03] by a direct reconstruction of the geometry. Moreover, they note that the geometry of a one-based SU-rank 1 type over $\emptyset$ in an ω-categorical theory has a stable reduct preserving independence, namely the reduct to the structure induced by the geometry.
(2) In [Vas01], Vassiliev studies the theory of a generic pair of SU-rank 1 structures. He shows that the pair has SU-rank at most ω; rank 1 in the trivial and rank 2 in the linear non-trivial case; moreover, in the ω-categorical non-trivial locally modular case he interprets a projective geometry in the pair structure.
(3) In the non-ω-categorical case, one can likewise hope to generalize the theory of groups with locally modular generic types to the almost hyperdefinable context, in order to characterize the geometry of a locally modular non-trivial regular Lascar strong type as projective geometry over a division ring (work in progress).

As for the equivalence of pseudolinearity and local modularity, we have:

Theorem 18. [TW03, Theorem 11] *Let $p \in S(\emptyset)$ be a finitely based pseudolinear regular type in an ω-categorical theory. Then p is locally modular.*

The proof proceeds as in the stable case by analysing the action of the group G on the set X given by Theorem 16 in a counterexample with $k \geq 2$. However, in order to derive a contradiction, one needs G to be nilpotent; the only known sufficient condition in a simple theory is ω-categoricity [Mac88, Theorem 1.2], where the group will even be virtually central-by-abelian by a suitable generalisation of [EW00].

3.2. **The binding group.**

Definition 17. Let p be a type over a set A, and Σ an A-invariant family of partial types. We say that p is

- (*almost*) *Σ-internal* if for every realization $a \models p$ there is $B \mathop{\smile\!\!\!\!\!|}_A a$ and realizations $\bar{c}$ of types in Σ over B, such that $a \in \mathrm{dcl}(B\bar{c})$ (resp. $a \in \mathrm{bdd}(B\bar{c})$).
- (*almost*) *generated over* Σ if there is $B \supseteq A$ such that for any realization $a \models p$ there are realizations $\bar{c}$ of types in Σ over B with $a \in \mathrm{dcl}(B\bar{c})$ (resp. $a \in \mathrm{bdd}(B\bar{c})$).
- *almost orthogonal to* Σ^ω if for any $a \models p$ and any tuple $\bar{c}$ consisting of realizations of partial types in Σ we have $a \mathop{\smile\!\!\!\!\!|}_A \bar{c}$.

From now on, Σ will be a family of partial types over $\emptyset$, and S the set of realizations of Σ (in the monster model).

Lemma 19. [BW04, Lemma 2.2] *Let a be any hyperimaginary, and $c = \mathrm{Cb}(a/S)$. Then there exists $\bar{c} \in \mathrm{dcl}(a) \cap \mathrm{dcl}(S)$ such that for every automorphism σ*

$$\sigma(\bar{c}) = \bar{c} \Longleftrightarrow \mathrm{tp}(\sigma(c)/S) = \mathrm{tp}(c/S).$$

One then has $\bar{c} \in \mathrm{dcl}(c)$, $c \in \mathrm{bdd}(\bar{c})$, $\mathrm{bdd}(c) = \mathrm{bdd}(\bar{c})$, and $a \mathop{\smile\!\!\!\!\!\!\mid}_{\bar{c}} S$.

We denote such a $\bar{c}$ by $\mathrm{Cb}_\Sigma(a)$. Clearly, it is unique. In a stable structure, for every two tuples a and a' the following are equivalent:

(1) a and a' are conjugate under an automorphism fixing S pointwise.
(2) $\mathrm{tp}(a/S) = \mathrm{tp}(a'/S)$.
(3) $\mathrm{Cb}_\Sigma(a) = \mathrm{Cb}_\Sigma(a')$.

However, in the simple case, only the implications from top to bottom need hold. The classical definition of the the binding group as $\mathrm{aut}(p/A \cup S)$ corresponds to the first condition. The group $\mathrm{Pél}(p, \Sigma)$ of elementary permutations of p over $A \cup S$ proposed in [SW02] corresponds to second condition, but, as remarked in that paper, it still seems to be too small, as it can also be destroyed by adding a random bipartite graph between p and Σ. We shall now construct a polygroup corresponding to the third and weakest condition.

Suppose $p \in S(\emptyset)$ is Lascar strong and almost orthogonal to Σ^ω. Put

$$R = \{\mathrm{tp}(ab) : a \models p, a \mathop{\smile\!\!\!\!\!\!\mid} b, a \in \mathrm{bdd}(bS)\},$$

and for $r(y, x) \in R$ define

$$\pi_r(xx', y, z) = r(yx) \wedge r(zx') \wedge \mathrm{Cb}_\Sigma(yx) = \mathrm{Cb}_\Sigma(zx') \wedge x \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{Cb}_\Sigma(yx)} x'.$$

Theorem 20. [BW04, Corollary 2.8, Theorem 2.9] *$\pi_r \approx \pi_{r'}$ for any $r, r' \in R$. If $\pi = \overline{\pi_r}$ for some (any) $r \in R$, the composition π^2 is generic, and $\pi \approx \pi^2$. Hence π is a generic multi-chunk. If P is the set of its germs with product given by composition, then P is a generic polygroup chunk with* $\mathrm{SU}(P) \geq \mathrm{SU}(p)$.
Moreover, if p is in a real sort, then P is in a finitary sort.

We can now apply Theorem 15 and obtain an almost hyperdefinable group G acting transitively and faithfully on an almost hyperdefinable set X, such that a generic element of G is interbounded (over independent parameters) with a realization of $\mathrm{Germ}(\pi)$, and a generic element of X is interbounded (over independent parameters) with a realisation of $\mathrm{Arg}(\pi) = p$.

Remark 10. If the theory was stable to start with, and p is Σ-internal, then $\langle G, X\rangle$ will be isomorphic to $\langle \mathrm{aut}(p/S), p\rangle$.

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