MODEL THEORETIC FORCING IN ANALYSIS

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Abstract. We present a framework for model theoretic forcing in a non first order context, and present some applications of this framework to Banach space theory.

Introduction

In this paper we introduce a framework of model theoretic forcing for metric structures, i.e., structures based on metric spaces. We use the language of infinitary continuous logic, which we define below. This is a variant of finitary continuous logic which is exposed in [BU] or [BBHU08].

The model theoretic forcing framework introduced here is analogous to that developed by Keisler [Kei73] for structures of the form considered in first order model theory.

The paper concludes with an application to separable quotients of Banach spaces. The long standing Separable Quotient Problem is whether for every nonseparable Banach space X there exists an operator $T\colon X \to Y$ such that $T(X)$ is a separable, infinite dimensional Banach space. We prove the following result (Theorem 5.4): If X is an infinite dimensional Banach space and $T\colon X \to Y$ is a surjective operator with infinite dimensional kernel, then there exist Banach spaces $\hat{X}, \hat{Y}$ and a surjective operator $\hat{T}\colon \hat{X} \to \hat{Y}$ such that

(i) $\hat{X}$ has density character ω_1,
(ii) The range of $\hat{T}$ is separable,
(iii) (X, Y, T) and $(\hat{X}, \hat{Y}, \hat{T})$ are elementarily equivalent as metric structures.

The paper is organized as follows. In Section 1 we introduce the syntax that will be used in the paper. In Section 2, we introduce model theoretic forcing for metric structures. In Section 3 we focus our attention on two particular forcing properties. These properties are used in Section 4 to prove the general Omitting Types Theorem. The last section, Section 5, is devoted to the aforementioned application to separable quotients.

For the exposition of the material we focus on one-sorted languages. However, as the reader will notice, the results presented here hold true, mutatis mutandis, for multi-sorted contexts. In fact, the structures used in the last section are multi-sorted.

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1. Preliminaries

Recall that if $f\colon (X,d) \to (X',d')$ is a mapping between two metric spaces, then f is uniformly continuous if and only if there exists a mapping $\delta\colon (0,\infty) \to (0,\infty]$ such that for all $x, y \in X$ and $\epsilon > 0$,

$$d(x,y) < \delta(\epsilon) \Longrightarrow d'(f(x),f(y)) \leq \epsilon. \tag{1}$$

If (1) holds, we say that δ is a *uniform continuity modulus* and that f *respects* δ. The choice of strict and weak inequalities here is so that the property of respecting δ is preserved under certain important constructions (e.g., completions and ultraproducts).

Let $\delta'\colon (0,\infty) \to (0,\infty]$ be any mapping, and define:

$$\delta(\epsilon) = \sup\{\delta'(\epsilon') \mid 0 < \epsilon' < \epsilon\}. \tag{2}$$

Then δ and δ' are equivalent as uniform continuity moduli, in the sense that a function f respects δ if and only if it respects δ'. In addition we have

$$\delta(\epsilon) = \sup\{\delta(\epsilon') \mid 0 < \epsilon' < \epsilon\}, \tag{3}$$

i.e., δ is increasing and continuous on the left. As a consequence, (1) is equivalent to the apparently stronger version:

$$d(x,y) < \delta(\epsilon) \Longrightarrow d'(f(x),f(y)) < \epsilon. \tag{4}$$

From this point on, when referring to a uniform continuity modulus δ, we mean one that satisfies (3).

In this section we introduce infinitary continuous formulas. For a general text regarding continuous structures and finitary continuous first order formulas we refer the reader to Sections 2 and 3 of [BU] or Sections 2–6 of [BBHU08].

Recall that a continuous signature $\mathcal{L}$ consists of the following data:

- For each n, a set of n-ary function and predicate symbols.
- A distinguished binary predicate symbol d.
- For each n-ary symbol s and $i < n$, a uniform continuity modulus for the ith argument denoted $\delta_{s,i}$.

A continuous $\mathcal{L}$-structure is a set M equipped with interpretations of the symbols of the language:

- Each n-ary function symbol is interpreted by an n-ary function:
$$f^M\colon M^n \to M.$$
- Each n-ary predicate symbol is interpreted by a continuous n-ary predicate:
$$P^M\colon M^n \to [0,1].$$
- The interpretation d^M of the distinguished symbol d is a complete metric.
- For each n-ary symbol s and $i < n$, the interpretation s^M, viewed as a function of its ith argument, respects the uniform continuity modulus $\delta_{s,i}$.

It is proved in [BU] that the following system of connectives is *full*:

$$x \mapsto \neg x, \qquad x \mapsto \frac{x}{2}, \qquad (x,y) \mapsto x \dot{-} y := \max(x-y, 0)$$

This means that for every $n \geq 1$, the family of functions from $[0,1]^n \to [0,1]$ which can be written using these three operations is dense in the class of all continuous functions $[0,1]^n \to [0,1]$. For the purposes of this paper (namely, to simplify the treatment of forcing, in Section 2), it is convenient to use the connective $\dot{+}$ instead of $\dot{-}$. Note that this causes no loss in expressive power, since $x \dot{-} y = \neg(\neg x \dot{+} y)$.

In this paper we extend the class of first order continuous formulas by considering formulas that may contain the infinitary connectives $\bigwedge$ and $\bigvee$, where for a set of formulas Φ, $\bigwedge_{\varphi\in\Phi}\varphi$ and $\bigvee_{\varphi\in\Phi}\varphi$ stand for $\sup\{\varphi \mid \varphi \in \Phi\}\varphi$ and $\inf\{\varphi \mid \varphi \in \Phi\}$, respectively. Because of the infinitary nature of this language, in order to form formulas with these connectives, one needs to be particularly careful about the uniform continuity moduli of the terms and formulas with respect to each variable, denoted $\delta_{\tau,x}$ and $\delta_{\varphi,x}$, respectively; thus, we have the following definition.

Definition 1.1. Let $\mathcal{L}$ be a continuous signature. We define the formulas of $\mathcal{L}_{\omega_1,\omega}$. Simultaneously, for each variable x, each term τ and each formula φ of $\mathcal{L}_{\omega_1,\omega}$ we define uniform continuity moduli $\delta_{\tau,x}$ and $\delta_{\varphi,x}$. Both definitions are inductive.

- A variable is a term, with $\delta_{x,x} = \mathrm{id}$ and $\delta_{x,y} = \infty$ for $y \neq x$.
- If f is an n-ary function symbol and $\tau_0, \ldots, \tau_{n-1}$ are terms, then $f\tau_0 \ldots \tau_{n-1}$ is a term. If τ is a term of this form,
$$\delta_{\tau,x}(\epsilon) = \sup_{\epsilon_0+\ldots+\epsilon_{n-1}<\epsilon} \min\{\delta_{\tau_i,x} \circ \delta_{f,i}(\epsilon_i) \mid i < n\}.$$
Here we follow the convention that $\delta(\infty) = \infty$.
- If P is an n-ary predicate symbol and $\tau_0, \ldots \tau_{n-1}$ are terms, then $P\tau_0 \ldots \tau_{n-1}$ is a formula (called an *atomic formula*). The definition of $\delta_{P\tau_0\ldots\tau_{n-1},x}$ is formally identical that of $\delta_{f\tau_0\ldots\tau_{n-1},x}$.
- If φ and ψ are formulas then so are $\neg\varphi$, $\frac{1}{2}\varphi$ and $\varphi \dot{+} \psi$. We have:
$$\begin{aligned}
\delta_{\neg\varphi,x}(\epsilon) &= \delta_{\varphi,x}(\epsilon) \\
\delta_{\frac{1}{2}\varphi,x}(\epsilon) &= \delta_{\varphi,x}(2\epsilon) \\
\delta_{\varphi\dot{+}\psi,x}(\epsilon) &= \sup_{\epsilon_0+\epsilon_1<\epsilon} \min\{\delta_{\varphi,x}(\epsilon_0), \delta_{\psi,x}(\epsilon_1)\}.
\end{aligned}$$
- Let Φ be a countable set of formulas in a finite tuple of free variables $\bar{x}$. For each variable x, let $\delta'_{\bigwedge\Phi,x} = \inf_{\varphi\in\Phi}\delta_{\varphi,x} \colon (0,\infty) \to [0,\infty]$. If $\delta'_{\bigwedge\Phi,x}(\epsilon) > 0$ for all $\epsilon > 0$ and $x \in \bar{x}$, then $\bigwedge\Phi$ is a formula, also denoted $\bigwedge_{\varphi\in\Phi}\varphi$. Its uniform continuity moduli are given by
$$\delta_{\bigwedge\Phi,x}(\epsilon) = \sup\{\delta'_{\bigwedge\Phi,x}(\epsilon') \colon 0 < \epsilon' < \epsilon\},$$
so that (3) is satisfied.

- If φ is a formula and x a variable, then $\inf_x \varphi$ is a formula. For $y \neq x$ we have $\delta_{\inf_x \varphi, y} = \delta_{\varphi,y}$, while $\delta_{\inf_x \varphi, x} = \infty$.

Notation 1.2. Rather than putting $\bigvee$ and sup in our language we define them as abbreviations:

$$\bigvee \Phi := \neg \bigwedge_{\varphi \in \Phi} \neg\varphi$$
$$\sup_x \varphi := \neg \inf_x \neg\varphi.$$

If M is an $\mathcal{L}$-structure and $\varphi(x_0, \ldots, x_{n-1}) \in \mathcal{L}_{\omega_1,\omega}$, one constructs the interpretation $\varphi^M \colon M^n \to [0,1]$ in the obvious manner. By induction on the structure of φ one also shows that for each variable x, φ^M is uniformly continuous in x respecting $\delta_{\varphi,x}$.

Finitary continuous first order formulas, as defined in [BU] and [BBHU08], are constructed in the same manner, with the exclusion of the infinitary connectives $\bigwedge$ and $\bigvee$ (i.e., only using the connectives $\neg, \frac{1}{2}, \dotplus$, or equivalently $\neg, \frac{1}{2}, \dot{-}$). We observe that $\varphi \wedge \psi$ is equivalent to $\varphi \dot{-} (\varphi \dot{-} \psi)$, so finitary instances of $\bigwedge$ and $\bigvee$ are allowed there as well. The set of all such formulas is denoted $\mathcal{L}_{\omega,\omega}$.

Definition 1.3. Let $\mathcal{L}$ be a continuous signature and let φ be an $\mathcal{L}_{\omega_1,\omega}$-formula. The set of *subformulas* of φ denoted $\mathrm{sub}(\varphi)$, is defined inductively as follows.

- If P is a predicate symbol and $\tau_0, \ldots \tau_{n-1}$ are terms, then $\mathrm{sub}(P\tau_0 \ldots \tau_{n-1}) = \{P\tau_0 \ldots \tau_{n-1}\}$.
- $\mathrm{sub}(\neg\varphi) = \{\neg\varphi\} \cup \mathrm{sub}(\varphi)$ and $\mathrm{sub}(\frac{1}{2}\varphi) = \{\frac{1}{2}\varphi\} \cup \mathrm{sub}(\varphi)$.
- $\mathrm{sub}(\varphi \dotplus \psi) = \{\, \varphi \dotplus \psi \,\} \cup \mathrm{sub}(\varphi) \cup \mathrm{sub}(\psi)$.
- $\mathrm{sub}(\bigwedge_{\varphi\in\Phi} \varphi) = \{\, \bigwedge_{\varphi\in\Phi} \varphi \,\} \cup \bigcup_{\varphi\in\Phi} \mathrm{sub}(\varphi)$.
- $\mathrm{sub}(\inf_x \varphi) = \{\, \inf_x \varphi \,\} \cup \mathrm{sub}(\varphi)$.

$\mathcal{L}_{\omega_1,\omega}$ need not be countable if $\mathcal{L}$ is countable. Nevertheless, it is often sufficient to work with countable fragments of $\mathcal{L}_{\omega_1,\omega}$:

Definition 1.4. A *fragment* of $\mathcal{L}_{\omega_1,\omega}$ is subset of $\mathcal{L}_{\omega_1,\omega}$ which contains all atomic formulas and is closed under subformulas and substitution of terms for free variables.

Remark 1.5. Every countable subset of $\mathcal{L}_{\omega_1,\omega}$ is contained in a countable fragment of $\mathcal{L}_{\omega_1,\omega}$.

For the next three sections (that is, the rest of the paper minus the last section), $\mathcal{L}$ will denote a fixed countable continuous signature, and $\mathcal{L}_A$ will denote a fixed countable fragment of $\mathcal{L}_{\omega_1,\omega}$. We will let $C = \{c_i \mid i < \omega\}$ be a set of new constant symbols, and $\mathcal{L}(C) = \mathcal{L} \cup C$. An $\mathcal{L}(C)$-structure M will be called *canonical* if the set $\{c_i^M \mid i < \omega\}$ is dense in M.

By $\mathcal{L}_A(C)$ we will denote the smallest countable fragment of $\mathcal{L}_{\omega_1,\omega}(C)$ that contains $\mathcal{L}_A$; notice that $\mathcal{L}_A(C)$ is obtained allowing closing $\mathcal{L}_A$ under substitution of constant symbols from C for free variables.

We will also use the following notation:

- The set of all sentences in $\mathcal{L}_A(C)$ will be denoted $\mathcal{L}_A^s(C)$.
- The set of all atomic sentences in $\mathcal{L}_A(C)$ will be denoted $\mathcal{L}_A^{as}(C)$.
- The set of variable-free terms in $\mathcal{L}(C)$ will be denoted $\mathcal{T}(C)$.

2. Forcing

Definition 2.1. A *forcing property* for $\mathcal{L}_A$ is a triplet $(\mathbb{P}, \leq, f)$ where $(\mathbb{P}, \leq)$ is a partially ordered set. The elements of $\mathbb{P}$ are called *conditions*. For each condition p, f assigns a mapping $f_p\colon \mathcal{L}_A^{as}(C) \to [0,1]$ satisfying the following conditions.

(i) $p \leq q$ implies $f_p \leq f_q$ i.e., $f_p(\varphi) \leq f_q(\varphi)$ for all $\varphi \in \mathcal{L}_A^{as}(C)$.

(ii) Given $p \in \mathbb{P}$, $\epsilon > 0$, $\tau, \sigma \in \mathcal{T}(C)$, and an atomic $\mathcal{L}(C)$-formula $\varphi(x)$ there are $q \leq p$ and $c \in C$ such that:

$$f_q(d(\tau, c)) < \epsilon,$$
$$f_q(d(\tau, \sigma)) < f_p(d(\sigma, \tau)) + \epsilon,$$

and if $f_p(d(\tau, \sigma)) < \delta_{\varphi,x}(\epsilon)$,

$$f_q(\varphi(\sigma)) < f_p(\varphi(\tau)) + \epsilon.$$

For the rest of this section, $(\mathbb{P}, \leq, f)$ will denote a fixed forcing property.

Definition 2.2. Let $p \in \mathbb{P}$ be a condition and $\varphi \in \mathcal{L}_A^s(C)$ a sentence. We define $F_p(\varphi) \in [0,1]$ by induction on φ. For φ atomic,

$$F_p(\varphi) = f_p(\varphi).$$

Otherwise,

$$\begin{array}{lcl} F_p(\neg\varphi) & = & \neg \inf_{q \leq p} F_q(\varphi) \\ F_p(\frac{1}{2}\varphi) & = & \frac{1}{2} F_p(\varphi) \\ F_p(\varphi \dot{+} \psi) & = & F_p(\varphi) \dot{+} F_p(\psi) \\ F_p(\bigwedge \Phi) & = & \inf_{\varphi \in \Phi} F_p(\varphi) \\ F_p(\inf_x \varphi(x)) & = & \inf_{c \in C} F_p(\varphi(c)). \end{array}$$

If $r \in \mathbb{R}$ and $F_p(\varphi) < r$ we say that p *forces* that $\varphi < r$, in symbols $p \Vdash \varphi < r$.

Remark 2.3. Let $p \in \mathbb{P}$ be a condition, $\varphi \in \mathcal{L}_A^s(C)$ a sentence, and $r \in \mathbb{R}$. Then,

$$\begin{array}{lcl} p \Vdash \varphi < r & \iff & f_p(\varphi) < r, \text{ if } \varphi \text{ is atomic} \\ p \Vdash \frac{1}{2}\varphi < r & \iff & p \Vdash \varphi < 2r \\ p \Vdash \neg\varphi < r & \iff & (\exists s > 1 - r)(\forall q \leq p)(q \nVdash \varphi < s) \\ p \Vdash (\varphi \dot{+} \psi) < r & \iff & (\exists s)(p \Vdash \varphi < s \text{ and } p \Vdash \psi < r - s) \\ p \Vdash \bigwedge \Phi < r & \iff & (\exists \varphi \in \Phi)(p \Vdash \varphi < r) \\ p \Vdash \inf_x \varphi(x) < r & \iff & (\exists c \in C)(p \Vdash \varphi(c) < r). \end{array}$$

Remark 2.4. The forcing relation $\Vdash$ can be defined inductively, without reference to the function $F_p(\varphi)$, by the list of equivalences in the preceding remark. One can then define $F_p(\varphi)$ as $\inf\{\, r \in \mathbb{R} \mid p \Vdash \varphi < r \,\}$.

The following basic properties will be used many times.

Lemma 2.5. *For all p, φ,*

(i) $F_p(\varphi) \in [0,1]$.
(ii) $q \leq p \Longrightarrow F_q(\varphi) \leq F_p(\varphi)$.
(iii) $F_p(\varphi) + F_p(\neg\varphi) \geq 1$.

Proof. The first two items are by induction on the structure of φ. The last one follows directly from the definition. ■$_{2.5}$

Definition 2.6. We also define F^w_p by:

$$F^w_p(\varphi) = \sup_{q \leq p} \inf_{q' \leq q} F_{q'}(\varphi).$$

If $r \in \mathbb{R}$ and $F^w_p(\varphi) < r$ we say that p *weakly forces* that $\varphi < r$, in symbols $p \Vdash^w \varphi < r$.

By Lemma 2.5, $F^w_p(\varphi) \leq F_p(\varphi)$.

Remark 2.7. The weak forcing relation $\Vdash^w$ can be defined without reference to the function F^w_p as follows: $p \Vdash^w \varphi < r$ if and only if $(\exists s < r)(\forall q \leq p)(\exists q' \leq q)(q' \Vdash \varphi < s)$. We can then define $F^w_p(\varphi)$ as $\inf\{\, r \mid p \Vdash^w \varphi < r \,\}$.

Lemma 2.8. *Let $p \in \mathbb{P}$, $\varphi \in \mathcal{L}^s_A(C)$ and $r \in \mathbb{R}$. Then*

$$F^w_p(\varphi) = \sup_{q \leq p} F^w_q(\varphi) = \sup_{q \leq p} \inf_{q' \leq q} F^w_{q'}(\varphi).$$

Proof. That $F^w_p(\varphi) = \sup_{q \leq p} F^w_q(\varphi)$ follows easily from the definitions, and $\sup_{q \leq p} F^w_q(\varphi) \geq \sup_{q \leq p} \inf_{q' \leq q} F^w_{q'}(\varphi)$ is immediate. Finally:

$$\begin{aligned}\sup_{q \leq p} \inf_{q' \leq q} F^w_{q'}(\varphi) &= \sup_{q \leq p} \inf_{q' \leq q} \sup_{q'' \leq q'} \inf_{q''' \leq q''} F_{q'''}(\varphi) \geq \sup_{q \leq p} \inf_{q' \leq q} \inf_{q''' \leq q'} F_{q'''}(\varphi) \\ &= \sup_{q \leq p} \inf_{q' \leq q} F_{q'}(\varphi) = F^w_p(\varphi).\end{aligned}$$ ■$_{2.8}$

Proposition 2.9. *The weak forcing function F^w obeys the following inductive rules:*

$$\begin{array}{lcl} F^w_p(\neg\varphi) & = & \neg \inf_{q \leq p} F^w_q(\varphi) \\ F^w_p(\frac{1}{2}\varphi) & = & \frac{1}{2} F^w_p(\varphi) \\ F^w_p(\varphi \dot{+} \psi) & = & \sup_{q \leq p} \inf_{q' \leq q} F^w_{q'}(\varphi) \dot{+} F^w_{q'}(\psi) \\ F^w_p(\bigwedge \Phi) & = & \sup_{q \leq p} \inf_{q' \leq q} \inf_{\varphi \in \Phi} F^w_{q'}(\varphi) \\ F^w_p(\inf_x \varphi(x)) & = & \sup_{q \leq p} \inf_{q' \leq q} \inf_{c \in C} F^w_{q'}(\varphi(c)). \end{array}$$

Proof. For $\neg\varphi$ and $\frac{1}{2}\varphi$ this follows from a straightforward calculation. For example:

$$F_p^w(\neg\varphi) = \sup_{q\leq p}\inf_{q'\leq q} F_{q'}(\neg\varphi) = \sup_{q\leq p}\inf_{q'\leq q} \neg \inf_{q''\leq q'} F_{q'}(\varphi)$$
$$= \neg \inf_{q\leq p}\sup_{q'\leq q}\inf_{q''\leq q'} F_{q'}(\varphi) = \neg \inf_{q\leq p} F_q^w(\varphi),$$

For the other three, the inequality $\geq$ is obtained substituting the definition of F_p^w on the left hand side and using the fact that $F_p \geq F_p^w$. For $\leq$, we first use Lemma 2.8 to replace each occurrence of F_p^w on the left hand side with $\sup_{q<p}\inf_{q'\leq q} F_{q'}^w$. Thus, it will suffice to show that:

$$\begin{array}{lcl} F_p^w(\varphi \dot{+} \psi) & \leq & F_p^w(\varphi) \dot{+} F_p^w(\psi) \\ F_p^w(\bigwedge \Phi) & \leq & \inf_{\varphi\in\Phi} F_p^w(\varphi) \\ F_p^w(\inf_x \varphi(x)) & \leq & \inf_{c\in C} F_p^w(\varphi(c)). \end{array}$$

For $\dot{+}$ assume $F_p^w(\varphi) = r$ and $F_p^w(\psi) = s$. Then for all $\epsilon > 0$ and for all $q \leq p$ there is $q_0' \leq q$ such that $F_{q_0'}(\varphi) < r + \epsilon$, and as $q_0' \leq p$ there is $q' \leq q_0'$ such that $F_{q'}(\psi) < s + \epsilon$. Then $F_{q'}(\varphi \dot{+} \psi) < r + s + 2\epsilon$, yielding $F_{q'}^w(\varphi \dot{+} \psi) \leq r + s$.

For $\bigwedge \Phi$ and $\inf_x \varphi(x)$ it is a straightforward quantifier exchange argument, e.g.:

$$F_p^w(\bigwedge \Phi) = \sup_{q\leq p}\inf_{q'\leq q}\inf_{\varphi\in\Phi} F_p(\varphi) \leq \inf_{\varphi\in\Phi}\sup_{q\leq p}\inf_{q'\leq q} F_p(\varphi) = \inf_{\varphi\in\Phi} F^w(\varphi). \qquad \blacksquare_{2.9}$$

Lemma 2.10. *For all $p \in \mathbb{P}$ and τ: $F_p^w(\inf_x d(\tau, x)) = 0$.*

Proof. If not then $F_p^w(\inf_x d(\tau, x)) = \sup_{q\leq p}\inf_{q'\leq q}\inf_{c\in C} f_p(d(\tau, x)) > 0$. But this contradicts the definition of forcing property. $\blacksquare_{2.10}$

Definition 2.11. A nonempty $G \subseteq \mathbb{P}$ is *generic* if:

(i) It is directed downwards, i.e., for all $p, q \in G$ there is $p' \in G$ such that $p' \leq p, q$.
(ii) It is closed upwards, i.e., if $p \in G$ and $q \geq p$ then $q \in G$.
(iii) For every $\varphi \in \mathfrak{L}_A^s(C)$ and $r > 1$ there is $p \in G$ such that $F_p(\varphi) + F_p(\neg\varphi) < r$.

If G is a generic set and $\varphi \in \mathfrak{L}_A^s(C)$ we define

$$\varphi^G = \inf_{p\in G} F_p(\varphi).$$

Proposition 2.12. *Every condition belongs to a generic set.*

Proof. Fix $p \in \mathbb{P}$. Let $(\,(r_n, \varphi_n) \colon n < \omega\,)$ enumerate all pairs (r, φ), where $r \in \mathbb{Q}$, $r > 1$, and $\varphi \in \mathfrak{L}_A^s(C)$. Construct a sequence $p_0 \geq p_1 \geq \ldots \geq p_n \geq \ldots$ in $\mathbb{P}$ as follows. We start with $p_0 = p$. Assume p_n has already been chosen. By definition $F_{p_n}(\neg\varphi_n) + \inf_{q\leq p_n} F_q(\varphi_n) = 1 < r_n$, so we can choose $p_{n+1} \leq p_n$ such that $F_{p_n}(\neg\varphi_n) + F_{p_{n+1}}(\varphi_n) < r_n$, whereby $F_{p_{n+1}}(\neg\varphi_n) + F_{p_{n+1}}(\varphi_n) < r_n$. Define

$$G = \{\, q \in \mathbb{P} \mid q \geq p_n \text{ for some } n \,\}.$$

Then G is generic, and $p \in G$. $\blacksquare_{2.12}$

Lemma 2.13. *Let G be generic and $\varphi \in \mathfrak{L}_A^s(C)$. Then $\varphi^G = \inf_{p\in G} F_p^w(\varphi)$.*

Proof. The inequality $\geq$ is immediate since $F^w_p(\varphi) \leq F_p(\varphi)$. For the other, assume $\varphi^G > \inf_{p\in G} F^w_p(\varphi)$, so there are $\epsilon > 0$ and $p \in G$ such that $\varphi^G - \epsilon > F^w_p(\varphi)$. As G is generic there is $q \in G$ such that $F_q(\varphi) + F_q(\neg\varphi) < 1 + \epsilon$, and as $p \in G$ we may assume $q \leq p$. We obtain

$$F^w_p(\varphi) \geq \inf_{q'\leq q} F_{q'}(\varphi) = 1 - F_q(\neg\varphi) > F_q(\varphi) - \epsilon \geq \varphi^G - \epsilon > F^w_p(\varphi),$$

a contradiction. ■2.13

Lemma 2.14. *If G is generic and $\varphi \in \mathcal{L}^s_A(C)$, then $(\neg\varphi)^G = 1 - \varphi^G$.*

Proof. From Lemma 2.5 we have $\varphi^G + (\neg\varphi)^G \geq 1$, while $\varphi^G + (\neg\varphi)^G \leq 1$ follows from Definition 2.11. ■2.14

Lemma 2.15. *Let G be a generic set and $\tau, \sigma \in \mathfrak{T}(C)$. Then:*

(i) *For every $\epsilon > 0$ there is $c_{\tau,\epsilon,G} \in C$ such that $d(\tau, c_{\tau,\epsilon,G})^G < \epsilon$.*
(ii) *$d(\tau,\sigma)^G = d(\sigma,\tau)^G$.*
(iii) *For every atomic $\mathcal{L}(C)$-formula $\varphi(x)$, if $d(\tau,\sigma)^G < \delta_{\varphi,x}(\epsilon)$ then $|\varphi(\tau)^G - \varphi(\sigma)^G| < \epsilon$.*

Proof. For (i), observe that $(\inf_x d(\tau, x))^G = 0$ by Lemma 2.10 and Lemma 2.13, so there is $p \in G$ such that $F_p(\inf_x d(\tau, x)) < \epsilon$, and thus there exists $c \in C$ such that $d(\tau, c)^G \leq F_p(d(\tau, c)) < \epsilon$. The other two statements follow directly from Lemma 2.13 and the definition of forcing property. ■2.15

Lemma 2.16. *Let M^G_0 be the term algebra $\mathfrak{T}(C)$ equipped with the natural interpretation of the function symbols, and interpreting the predicate symbols by: $P^{M^G_0}(\bar\tau) = P(\bar\tau)^G$. Then M^G_0 is a pre-$\mathcal{L}(C)$-structure, and its completion M^G is a canonical structure.*

Proof. First we use Lemma 2.15 to show that $d^{M^G_0}$ is a pseudometric. Symmetry is Lemma 2.15(ii). The triangle inequality follows from Lemma 2.15(iii), keeping in mind that $\delta_{d(x,\sigma),x} = \mathrm{id}$. That $d^{M^G_0}(\tau,\tau) = 0$ follows from the triangle inequality and Lemma 2.15(i). Finally, by Lemma 2.15(iii), every symbol respects its uniform continuity modulus. Thus M^G_0 is a pre-structure, and we can define M^G to be its completion.

That C^{M^G} is dense in M^G now follows from Lemma 2.15(i). ■2.16

Theorem 2.17. *For all $\varphi \in \mathcal{L}^s_A(C)$ we have $\varphi^{M^G} = \varphi^G$.*

Proof. By induction on φ:

(i) For φ atomic, this is immediate from the construction of M^G.
(ii) For $\frac{1}{2}\varphi$, $\varphi \dot{+} \psi$ and $\bigwedge \Phi$, this is immediate from the definition of forcing and the induction hypothesis.
(iii) For $\neg\varphi$, this is immediate from Lemma 2.14 and the induction hypothesis.
(iv) For $\inf_x \varphi(x)$, it follows from the definition of forcing and the induction hypothesis that $(\inf_x \varphi)^G = \inf\{\varphi(c)^{M^G} \mid c \in C\}$. Since C^{M^G} is dense in M^G and $\varphi(x)^{M^G}$ is uniformly continuous in x, the latter is equal to $(\inf_x \varphi)^{M^G}$. ■2.17

3. The forcing Properties $\mathcal{P}(\mathcal{M})$ and $\mathcal{P}(\mathcal{M},\Sigma)$

If $\mathcal{M}$ is class of $\mathcal{L}$-structures, we denote by $\mathcal{M}(C)$ the class of all structures of the form $(M, a_c)_{c\in C_0}$, where M is in $\mathcal{M}$ and C_0 is a finite subset of C; such a structure is regarded naturally as an $\mathcal{L}(C_0)$-structure by letting a_c be the interpretation of c in M, for each $c \in C_0$.

Let Σ be a class of formulas of $\mathcal{L}_A$ that contains all the atomic formulas and is closed under subformulas, and let $\Sigma(C)$ denote the subset of $\mathcal{L}(C)$ obtained from formulas φ in Σ by replacing finitely many free variables of φ with constant symbols from C.

The forcing property $\mathcal{P}(\mathcal{M},\Sigma)$ is defined as follows. The conditions of $\mathcal{P}(\mathcal{M},\Sigma)$ are the finite sets of the form

$$\{\, \varphi_1 < r_1, \ldots, \varphi_n < r_n \,\},$$

where $\varphi_1, \ldots, \varphi_n \in \Sigma(C)$ and there exist $M \in \mathcal{M}(C)$ such that $\varphi_i^M < r_i$, for $i = 1, \ldots, n$. The partial order $\leq$ on conditions is reverse inclusion, i.e., if p, q are conditions of $\mathcal{P}(\mathcal{M},\Sigma)$, then $p \leq q$ if and only $p \supseteq q$. If p is a condition of $\mathcal{P}(\mathcal{M}_\Delta,\Sigma)$ and φ is an atomic sentence of $\mathcal{L}(C)$, we define

$$f_p(\varphi) = \begin{cases} \min\{r \leq 1 \mid \varphi < r \in p\}, & \text{if } \{r \leq 1 \mid \varphi < r \in p\} \neq \varnothing, \\ 1, & \text{otherwise.} \end{cases}$$

When Σ is the set of all atomic $\mathcal{L}$-formulas, the forcing property $\mathcal{P}(\mathcal{M},\Sigma)$ is denoted simply $\mathcal{P}(\mathcal{M})$.

The main result of this section is Proposition 3.4, below, which characterizes weak forcing for the forcing property $\mathcal{P}(\mathcal{M},\Sigma)$; for the proof, we need two lemmas.

Definition 3.1. We extend the definition of f_p above to all sentences of $\Sigma(C)$:

$$H_p(\varphi) = \begin{cases} \min\{r \leq 1 \mid \varphi < r \in p\}, & \text{if } \{r \leq 1 \mid \varphi < r \in p\} \neq \varnothing, \\ 1, & \text{otherwise.} \end{cases}$$

We define H_p^w accordingly: $H_p^w(\varphi) = \sup_{q\leq p} \inf_{p\leq q} H_p(\varphi)$.

Clearly if $q \leq p$ then $H_q(\varphi) \leq H_p(\varphi)$ and $H_q^w(\varphi) \leq H_p^w(\varphi)$, whereby for all p: $H_p^w(\varphi) \leq H_p(\varphi)$.

Lemma 3.2. *For all $p \in \mathcal{P}(\mathcal{M},\Sigma)$ and $\varphi \in \Sigma(C)$:*

$$\begin{aligned} H_p^w(\varphi) &= \inf\{r \in [0,1] \mid (\forall q \leq p)(q \cup \{\varphi < r\} \in \mathcal{P}(\mathcal{M},\Sigma))\} \\ &= \sup\{r \in [0,1] \mid p \cup \{\neg\varphi < 1 - r\} \in \mathcal{P}(\mathcal{M},\Sigma)\} \end{aligned}$$

(Here $\inf \varnothing = 1$, $\sup \varnothing = 0$.)

Proof. The first equality is a mere rephrasing: $H_p^w(\varphi) \leq r$ if and only if $\inf_{q'\leq q} H_p(\varphi) \leq r$ for all $q \leq p$, i.e., if and only if $q \cup \{\varphi < r\} \in \mathcal{P}(\mathcal{M},\Sigma)$ for all $q \leq p$.

For the second equality: Assume first that $q = p \cup \{\neg\varphi < 1 - r\} \in \mathcal{P}(\mathcal{M},\Sigma)$. Then $q \leq p$ but $q \cup \{\varphi < r\} \notin \mathcal{P}(\mathcal{M},\Sigma)$. This gives $\geq$. Now assume $p \cup \{\neg\varphi < 1 - r\} \notin \mathcal{P}(\mathcal{M},\Sigma)$.

Then $p \cup \{\neg\varphi < 1 - r\}$ cannot be realized in the given class. Thus, for every $q \leq p$, as q can be realized, it is realized in a model where $\varphi \leq r$. Thus $q \cup \{\varphi < s\} \in \mathcal{P}(\mathcal{M}, \Sigma)$ for all $q \leq p$ and $s > r$. This gives $\leq$. ■3.2

Proposition 3.3. *The functions H^w_p satisfy the properties stated for F^w_p in Lemma 2.8 and Proposition 2.9, i.e.:*

$$\begin{array}{lcl}
H^w_p(\varphi) & = & \sup_{q\leq p} H^w_q(\varphi) = \sup_{q\leq p} \inf_{q'\leq q} H^w_{q'}(\varphi) \\
H^w_p(\neg\varphi) & = & \neg \inf_{q\leq p} H^w_q(\varphi) \\
H^w_p(\frac{1}{2}\varphi) & = & \frac{1}{2} H^w_p(\varphi) \\
H^w_p(\varphi \dot{+} \psi) & = & \sup_{q\leq p} \inf_{q'\leq q} H^w_{q'}(\varphi) \dot{+} H^w_{q'}(\psi) \\
H^w_p(\bigwedge \Phi) & = & \sup_{q\leq p} \inf_{q'\leq q} \inf_{\varphi\in\Phi} H^w_{q'}(\varphi) \\
H^w_p(\inf_x \varphi(x)) & = & \sup_{q\leq p} \inf_{q'\leq q} \inf_{c\in C} H^w_{q'}(\varphi(c)).
\end{array}$$

Proof. The first property is proved precisely as in Lemma 2.8.

For $\neg$: it follows from Lemma 3.2 that $H^w_p(\neg\varphi) = \neg \inf_{q\leq p} H_q(\varphi)$, and we conclude as in the proof of Proposition 2.9.

For $\frac{1}{2}$: observe that $q \cup \{\varphi < r\} \in \mathcal{P}(\mathcal{M}, \Sigma)$ if and only if $q \cup \{\frac{1}{2}\varphi < \frac{1}{2}r\} \in \mathcal{P}(\mathcal{M}, \Sigma)$ and apply Lemma 3.2.

For the last three we reduce as in the proof of Proposition 2.9 to showing that:

$$\begin{array}{lcl}
H^w_p(\varphi \dot{+} \psi) & \leq & H^w_p(\varphi) \dot{+} H^w_p(\psi) \\
H^w_p(\bigwedge \Phi) & \leq & \inf_{\varphi\in\Phi} H^w_p(\varphi) \\
H^w_p(\inf_x \varphi(x)) & \leq & \inf_{c\in C} H^w_p(\varphi(c)).
\end{array}$$

For $\dot{+}$ this follows from Lemma 3.2. For $\bigwedge$ and inf the quantifier exchange argument from proof of the corresponding items in Proposition 2.9 works here too. ■3.3

Proposition 3.4. *Suppose that p is a condition in the forcing property $\mathcal{P}(\mathcal{M}, \Sigma)$ and σ is a sentence of $\Sigma(C)$. Then $F^w_p(\sigma) = H^w_p(\sigma)$.*

Proof. For atomic σ the equality is immediate. We then proceed by induction on σ, noting that Proposition 2.9 on the one hand and Proposition 3.3 on the other tell us that F^w_p and H^w_p obey the same inductive definitions. ■3.4

4. Generic Models and sup ⋀ inf-Formulas

Recall from Section 1 that the expressions $\bigvee \Phi$ and $\sup_x \varphi$ are regarded abbreviations of $\neg \bigwedge_{\varphi\in\Phi} \neg\varphi$ and $\neg \inf_x \neg\varphi$ respectively.

Proposition 4.1. *Let $(\mathbb{P}, \leq, f)$ be a forcing property for $\mathcal{L}_A(C)$ and let $p \in \mathbb{P}$. Then*

(i) $F_p(\bigvee \Phi) = \sup_{\varphi\in\Phi} F^w_p(\varphi)$.
(ii) $F_p(\sup_x \varphi(x)) = \sup_{c\in C} F^w_p(\varphi(c))$.

Proof. The proofs are straightforward applications of the definitions: for (i),

$$\begin{aligned} F_p(\bigvee \Phi) = F_p(\neg \bigwedge_{\varphi\in\Phi} \neg\varphi) &= \neg \inf_{q\le p} F_q(\bigwedge_{\varphi\in\Phi} \neg\varphi) \\ &= \neg \inf_{q\le p} \inf_{\varphi\in\Phi} F_q(\neg\varphi) \\ &= \neg \inf_{q\le p} \inf_{\varphi\in\Phi} \neg \inf_{q'\le q} F_{q'}(\varphi) \\ &= \sup_{q\le p} \sup_{\varphi\in\Phi} \inf_{q'\le q} F_{q'}(\varphi) \\ &= \sup_{\varphi\in\Phi} \sup_{q\le p} \inf_{q'\le q} F_{q'}(\varphi) \\ &= \sup_{\varphi\in\Phi} F_p^w(\varphi), \end{aligned}$$

and for (ii),

$$\begin{aligned} F_p(\sup_x \varphi(x)) = F_p(\neg \inf_x \neg\varphi(x)) &= \neg \inf_{q\le p} F_q(\inf_x \neg\varphi(x)) \\ &= \neg \inf_{q\le p} \inf_{c\in C} F_q(\neg\varphi(c)) \\ &= \neg \inf_{q\le p} \inf_{c\in C} \neg \inf_{q'\le q} F_{q'}(\varphi(c)) \\ &= \sup_{q\le p} \sup_{c\in C} \inf_{q'\le q} F_{q'}(\varphi(c)) \\ &= \sup_{c\in C} \sup_{q\le p} \inf_{q'\le q} F_{q'}(\varphi(c)) \\ &= \sup_{c\in C} F_{q'}^w(\varphi(c)). \end{aligned}$$

■4.1

Notation 4.2. If Φ is a finite set of formulas, say $\Phi = \{\varphi_1, \dots, \varphi_n\}$, we write

$$\varphi_1 \wedge \cdots \wedge \varphi_n \quad \text{and} \quad \varphi_1 \vee \cdots \vee \varphi_n$$

as abbreviations of $\bigwedge \Phi$ and $\bigvee \Phi$, respectively.

Proposition 4.3. *If $(\mathbb{P}, \le, f)$ is a forcing property for $\mathfrak{L}_A(C)$ and $p \in \mathbb{P}$, then*

$$F_p(\varphi_1 \vee \cdots \vee \varphi_n) = \max_i F_p^w(\varphi_i).$$

Proof. By Proposition 4.1. ■4.3

Definition 4.4. Let Σ be a class of formulas of $\mathfrak{L}_A$ which contains all atomic formulas and is closed under subformulas. A $\sup\bigwedge\inf$*-formula over* Σ is an $\mathfrak{L}_A$-formula of the form

$$\sup_{x_1} \dots \sup_{x_m} \bigwedge_{n<\omega} \inf_{y_1} \dots \inf_{y_{i(n)}} (\sigma_{n,1}(\bar{x}, \bar{y}_n) \vee \cdots \vee \sigma_{n,j(n)}(\bar{x}, \bar{y}_n)),$$

where $\sigma_{n,\nu}$ belongs to Σ for $n < \omega$ and $\nu = 1, \dots, j(n)$, $\bar{x} = x_1, \dots, x_m$, and $\bar{y}_n = y_1, \dots, y_{i(n)}$.

Proposition 4.5. *Let Σ be a class of formulas of $\mathcal{L}_A$ which contains all atomic formulas and is closed under subformulas. Suppose that φ is a $\sup\bigwedge\inf$-formula over Σ, of the form*

$$\sup_{x_1}\dots\sup_{x_m}\bigwedge_{n<\omega}\inf_{y_1}\dots\inf_{y_{i(n)}}(\sigma_{n,1}(\bar{x},\bar{y}_n)\vee\dots\vee\sigma_{n,j(n)}(\bar{x},\bar{y}_n)),$$

where $\sigma_{n,\nu}$ belongs to Σ for $n<\omega$ and $\nu=1,\dots,j(n)$, $\bar{x}=x_1,\dots,x_m$, and $\bar{y}_n=y_1,\dots,y_{i(n)}$. Then, if $(\mathbb{P},\leq,f)$ is a forcing property for $\mathcal{L}_A(C)$ and $p\in\mathbb{P}$,

$$F_p(\varphi)=\sup_{\substack{\bar{c}\in C^m\\ q\leq p}}\inf_{\substack{q'\leq q\\ \bar{d}\in C^{i(n)}\\ n<\omega}}\max_{1\leq\nu\leq j(n)}F^w_{q'}(\,\sigma_{n,\nu}(\bar{c},\bar{d})\,).$$

Proof. We use Propositions 4.1 and 4.3 to compute $F_p(\varphi)$:

$$\begin{aligned}
&F_p(\,\sup_{x_1}\dots\sup_{x_m}\bigwedge_{n<\omega}\inf_{y_1}\dots\inf_{y_{i(n)}}(\sigma_{n,1}(\bar{x},\bar{y}_n)\vee\dots\vee\sigma_{n,j(n)}(\bar{x},\bar{y}_n)\,))\\
&=\sup_{\bar{c}\in C^m}F^w_p(\,\bigwedge_{n<\omega}\inf_{y_1}\dots\inf_{y_{i(n)}}(\sigma_{n,1}(\bar{c},\bar{y}_n)\vee\dots\vee\sigma_{n,j(n)}(\bar{c},\bar{y}_n))\,) && \text{(by 4.1)}\\
&=\sup_{\bar{c}\in C^m}\sup_{q\leq p}\inf_{q'\leq q}F_{q'}(\,\bigwedge_{n<\omega}\inf_{y_1}\dots\inf_{y_{i(n)}}(\sigma_{n,1}(\bar{c},\bar{y}_n)\vee\dots\vee\sigma_{n,j(n)}(\bar{c},\bar{y}_n))\,)\\
&=\sup_{\bar{c}\in C^m}\sup_{q\leq p}\inf_{q'\leq q}\inf_{n<\omega}\inf_{\bar{d}\in C^{i(n)}}F_{q'}(\,\sigma_{n,1}(\bar{c},\bar{d})\vee\dots\vee\sigma_{n,j(n)}(\bar{c},\bar{d})\,)\\
&=\sup_{\bar{c}\in C^m}\sup_{q\leq p}\inf_{q'\leq q}\inf_{n<\omega}\inf_{\bar{d}\in C^{i(n)}}\max_{1\leq\nu\leq j(n)}F^w_{q'}(\,\sigma_{n,\nu}(\bar{c},\bar{d})\,) && \text{(by 4.3).}
\end{aligned}$$

■$_{4.5}$

Remark 4.6. If p is a condition in the forcing property $\mathcal{P}(\mathcal{M},\Sigma)$, then $F^w_p(\varphi)=H^w_p(\varphi)$, by Proposition 3.4. Hence, if φ is as in Proposition 4.5,

$$F_p(\varphi)=\sup_{\substack{\bar{c}\in C^m\\ q\leq p}}\inf_{\substack{q'\leq q\\ \bar{d}\in C^{i(n)}\\ n<\omega}}\max_{1\leq\nu\leq j(n)}H^w_{q'}(\,\sigma_{n,\nu}(\bar{c},\bar{d})\,).$$

Recall (Section 1) that C denotes countable set of constants not in $\mathcal{L}$ and that $\mathcal{L}(C)=\mathcal{L}\cup C$. As in Section 3, if $\mathcal{M}$ is a class of $\mathcal{L}$-structures, $\mathcal{M}(C)$ denotes the class of structures of the form $(M,a_c)_{c\in C_0}$, where M is in $\mathcal{M}$ and C_0 is a finite subset of C. If Γ is a set of inequalities of the form $\varphi<r$, where φ is an $\mathcal{L}_A(C)$-formula and r is a real number, we will say that Γ is *satisfiable* in $\mathcal{M}$ if there exists a structure M in $\mathcal{M}(C)$ such that $\varphi^M<r$ for every inequality $\varphi<r$ in Γ.

Let Σ be class of formulas of $\mathcal{L}_A$ that contains all the atomic formulas and is closed under subformulas. A *finite Σ-piece of* $\mathcal{M}$ is a finite set p of inequalities of the form $\varphi<r$, where $\varphi\in\Sigma$, such that p that is satisfiable in $\mathcal{M}$.

Corollary 4.7 (Omitting Types Theorem)**.** *Let* $(\varphi_n \mid n < \omega)$ *be a sequence of* $\mathcal{L}$*-formulas such that for each* $n < \omega$ φ_n *is a* $\sup\bigwedge\inf$*-formula over* Σ*, of the form*

$$\sup_{x_1} \dots \sup_{x_{m(n)}} \psi_n(x_1, \dots, x_{m(n)}),$$

where for each $n < \omega$ ψ_n *is of the form*

$$\bigwedge_{k<\omega} \inf_{y_1} \dots \inf_{y_{i(n,k)}} (\sigma_{n,k,1}(\bar{x}_n, \bar{y}_{n,k}) \vee \dots \vee \sigma_{n,k,j(n,k)}(\bar{x}_n, \bar{y}_{n,k})),$$

with $\bar{x}_n = x_1, \dots, x_{m(n)}$*, and* $\bar{y}_{n,k} = y_1, \dots, y_{i(n,k)}$*, and let* $(r_n \mid n < \omega)$ *be a sequence of real numbers such that for every finite* Σ*-piece* p *of* $\mathcal{M}$ *and every* $\bar{c}_n \in C^{m(n)}$*, the set* $p \cup \{\psi_n(\bar{c}_n) < r_n\}$ *is satisfiable in* $\mathcal{M}$*. Then there exists a canonical* $\mathcal{L}(C)$*-structure* M *such that* $\varphi_n^M \leq r_n$ *for every* $n < \omega$*.*

Proof. Let p be a condition in the forcing property $\mathcal{P}(\mathcal{M}, \Sigma)$. Fix a condition $q \leq p$, $n < \omega$ and $\bar{c}_n \in C^{m(n)}$. Since $q \cup \{\psi_n(\bar{c}_n) < r_n\}$ is satisfiable in $\mathcal{M}$, there exist $k < \omega$ and $\bar{d}_{n,k} \in C^{i(n,k)}$ such that

$$q \cup \{\, \sigma_{n,k,1}(\bar{c}_n, \bar{d}_{n,k}) \vee \dots \vee \sigma_{n,k,j(n,k)}(\bar{c}_n, \bar{d}_{n,k}) < r_n \,\}$$

is satisfiable in $\mathcal{M}$. Let

$$q' = q \cup \{\, \sigma_{n,k,1}(\bar{c}_n, \bar{d}_{n,k}) < r_n, \dots, \sigma_{n,k,j(n,k)}(\bar{c}_n, \bar{d}_{n,k}) < r_n \,\}.$$

Then, q is a condition in $\mathcal{P}(\mathcal{M}, \Sigma)$, and by Lemma 3.2,

$$\max_{1\leq\nu\leq j(n,k)} H^w_{q'}(\, \sigma_{n,k,\nu}(\bar{c}_n, \bar{d}_{n,k})\,) < r_n,$$

so

$$\inf_{\substack{q'\leq q \\ \bar{d}\in C^{i(n,k)} \\ k<\omega}} \max_{1\leq\nu\leq j(n,k)} H^w_{q'}(\, \sigma_{n,k,\nu}(\bar{c}_n, \bar{d}_{n,k})\,) < r_n.$$

Thus, by Remark 4.6, $F_p(\varphi_n) \leq r_n$. Let G be a generic set for $\mathcal{P}(\mathcal{M}, \Sigma)$ (the existence of G is guaranteed by Proposition 2.12). For every $n < \omega$, $\varphi_n^G = \inf_{p\in G} F_p(\varphi_n) \leq r_n$ (see Definition 2.11). Let now M^G be as in Lemma 2.16. Then, by Theorem 2.17, $\varphi_n^{M^G} = \varphi_n^G \leq r_n$. ■$_{4.7}$

Remark 4.8. The reader may worry about the fact that the assumptions of Corollary 4.7 involve strict inequalities while the conclusion only yields a weak inequality. In fact it would be enough to assume a weak inequality, i.e., that $p \cup \{\psi_n(\bar{c}_n) \leq r_n\}$ is satisfiable in $\mathcal{M}$ for every p and ψ_n as in the statement of Corollary 4.7, or more precisely, that $p \cup \{\psi_n(\bar{c}_n) < r_n + \epsilon\}$ is satisfiable for every $\epsilon > 0$. Indeed, in this case we would be able to find M in which $\varphi_n^M \leq r_n + 2^{-m}$ for all n, m, i.e., such that $\varphi_n^M \leq r_n$ for all n.

5. Application: Separable Quotients

If φ is a formula of $\mathcal{L}_{\omega_1,\omega}$, we will say that φ is *finitary* if all the occurrences of $\bigwedge$ in φ are finitary, i.e., if whenever $\bigwedge_{\psi\in\Phi}\psi$ is a subformula of φ, the set Φ is finite. We recall from Section 1 that the set of all finitary formulas is denoted $\mathcal{L}_{\omega,\omega}$.

If M and N are $\mathcal{L}$-structures, M and N are said to be *elementary equivalent*, written $M \equiv N$, if $\varphi^M = \varphi^N$ for every finitary $\mathcal{L}$-sentence φ. Thus $M \equiv N$ if and only if $\varphi^M < r$ implies $\varphi^N < r$ for every finitary $\mathcal{L}$-sentence φ and every rational number r. If M is a substructure of N, M is said to be an *elementary substructure* of N if $(M, a \mid a \in M) \equiv (N, a \mid a \in M)$.

A Banach space $(X, \|\cdot\|)$ can be regarded as a metric structure in a number of ways. A natural approach is to introduce for each nonnegative rational r a distinct sort for the closed ball $B_X(r)$ of radius r around 0; the metric on $B_X(r)$ is given by the norm $\|\cdot\|$; in the structure we also include:

- the inclusion maps $I_{r,s}\colon B_X(r) \to B_X(s)$ for $r < s$,
- the vector addition, which maps $B_X(r) \times B_X(s)$ onto $B_X(r+s)$,
- for each $\lambda \in \mathbb{Q}$, the scalar multiplication by λ, which maps $B_X(r)$ onto $B_X(|\lambda|r)$,
- the normalized norm predicate $\|\cdot\|/r$, which maps $B_X(r)$ onto the interval $[0,1]$,
- the normalized distance predicate on $B_X(r)$ defined by $d(x,y) = \|x-y\|/(2r)$ (as $x - y \in B_X(2r)$).

Notice that with the normalized norm and distance, all symbols are 1-Lipschitz, meaning that the identity function $\delta(\epsilon) = \epsilon$ is a modulus of uniform continuity for each and every one of them.

Other ways of regarding Banach space as metric structures are discussed in Section 3 of [BU] and in Section 4 [Ben].

If X and Y are Banach spaces and $T\colon X \to Y$ is a Banach space operator, we denote by (X, Y, T) the structure that includes, in addition to the Banach space structure of X and the Banach space structure on Y, in separate sorts, the operator T as a family of functions between the appropriate sorts, i.e., from $B_X(r)$ to $B_Y(s)$ if $s \geq \|T\|r$. If T is non zero, the function $\delta(\epsilon) = \epsilon/\|T\|$ is a modulus of uniform continuity for T.

Proposition 5.1. *Let $X, Y, \hat{X}, \hat{Y}$ be Banach spaces and let $T\colon X \to Y$ and $\hat{T}\colon \hat{X} \to \hat{Y}$ be bounded linear operators such that $(X, Y, T) \equiv (\hat{X}, \hat{Y}, \hat{T})$. Then T is surjective if and only if $\hat{T}$ is surjective.*

Proof. For a real number $r \geq 0$, let $B_X(r)$ and $B_Y(r)$ denote the closed balls of radius r around 0 in X and Y, respectively. The proof of the Open Mapping Theorem shows that $T\colon X \to Y$ is surjective if and only if the following holds: for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\forall y \in B_Y(\delta(\epsilon))\ \exists x \in B_X(1)\ (\,\|T(x) - y\| \leq \epsilon\,).$$

We can assume that $\delta(\epsilon) < 1$ for $\epsilon < 1$. Thus, T is surjective if and only if for every ϵ with $0 < \epsilon < 1$ we have $\varphi_\epsilon^{(X,Y,T)} = 0$, where φ_ϵ is the following finitary sentence (the

variables x and y are of sort $B_X(1)$ and $B_Y(1)$, respectively):

$$\sup_y \inf_x \min\big(\ \delta(\epsilon) \dot{-} \|y\|, \|T(x) - y\| \dot{-} \epsilon\ \big).$$

(Or, if one wishes to be pedantic, replace $\|T(x) - y\| \dot{-} \epsilon$ with $\frac{1}{2}\|T(x) - y\| \dot{-} \frac{\epsilon}{2}$.) Hence, if $(X, Y, T) \equiv (\hat{X}, \hat{Y}, \hat{T})$ and T is surjective, $\hat{T}$ is surjective too. ■$_{5.1}$

The authors are grateful to William B. Johnson for pointing out that Proposition 5.1 is given by the proof of the Open Mapping Theorem.

All the Banach spaces mentioned henceforth will be infinite dimensional.

The Separable Quotient Problem is perhaps the most prominent open problem in nonseparable Banach space theory. The question is whether for every nonseparable Banach space X there exist a separable Banach space Y and a surjective operator $T\colon X \to Y$. Let $T\colon X \to Y$ be a Banach space operator and consider the structure (X, Y, T) (the sorts of this structure are X and Y). In this section we use Corollary 4.7 to prove that if $T\colon X \to Y$ is surjective and has infinite dimensional kernel, then there exists an operator $\hat{T}\colon \hat{X} \to \hat{Y}$ such that

(i) $(X, Y, T) \equiv (\hat{X}, \hat{Y}, \hat{T})$,
(ii) $\hat{X}$ has density character ω_1,
(iii) $\hat{Y}$ is separable.

It follows from (i) and Proposition 5.1 that $\hat{T}$ is surjective.

Lemma 5.2. *Suppose that X is a Banach space and Y is a closed proper subspace of X. Then there exists a non zero linear functional $f\colon X \to \mathbb{R}$ whose restriction to Y is zero. Up to multiplication by a scalar we may further assume that $\|f\| = 1$.*

Proof. This is a well-known application of the Hahn-Banach theorem; the proof can be found in a textbook, e.g., [FHH$^+$01]. ■$_{5.2}$

Lemma 5.3. *If X is a Banach space of density character κ, there exists a family $(x_i)_{i<\kappa}$ in X such that $\|x_i\| = 1$ for every $i < \kappa$ and $\|x_i - x_j\| \geq 1$ for $i < j < \kappa$.*

Proof. The construction of $(x_i)_{i<\kappa}$ is inductive. Fix $j < \kappa$ and suppose that constructed x_i is defined for $i < j$. Let Y be the closed linear span of $\{x_i \mid i < j\}$, which is a proper closed subspace of X. By Lemma 5.2, take $f\colon X \to \mathbb{R}$ such that $f(x) = 0$ for every $x \in Y$ and $\|f\| = 1$. Let now x_j be an element of the unit sphere of X such that $|f(x)| = \|f\| = 1$. Then, if $y \in Y$, we have $\|x_j - y\| \geq |f(x_j) - f(y)| = 1$; in particular, $\|x_j - x_i\| \geq 1$ for $i < j$. ■$_{5.3}$

Theorem 5.4. *For every surjective operator $T\colon X \to Y$ with infinite dimensional kernel there exists an operator $\hat{T}\colon \hat{X} \to \hat{Y}$ such that*

- $(X, Y, T) \equiv (\hat{X}, \hat{Y}, \hat{T})$,
- *$\hat{X}$ has density character ω_1,*
- *$\hat{Y}$ is separable.*

Furthermore, if D is a given countable subset of X, the structure $(\hat{X}, \hat{Y}, \hat{T})$ can be chosen with the following property: there exists a separable subspace X_0 of X such that $D \subseteq X_0$ and if T_0 denotes the restriction of T to X_0,

- $(X_0, T_0(X_0), T_0) \prec (X, Y, T)$,
- $(X_0, T_0(X_0), T_0) \prec (\hat{X}, \hat{Y}, \hat{T})$.

Proof. By the Löwenheim-Skolem Theorem [HI02, page 47], there exists a separable subspace X_0 of X such that if T_0 denotes the restriction of T to X_0 and $Y_0 = T_0(X_0)$,

$$(X_0, Y_0, T_0) \prec (X, Y, T).$$

(Note that X_0 can be taken so that it contains any given countable subset of X.)

Let A be a countable dense subset of X_0 and consider the structure

$$(\, X_0, Y_0, T, a \mid a \in A \,).$$

Let $\mathfrak{L}$ be the signature that results from expanding the signature of $(\, X, Y, T, a \mid a \in A \,)$ with new constant symbols $c_0, c_1, \ldots$ and c_0^* of sort $B_X(1)$ as well as new constant symbols $d_0, d_1, \ldots$ and of sort $B_Y(1)$.

Let us introduce some temporary terminology. If φ is an $\mathfrak{L}$-sentence, an $\mathfrak{L}$-structure M *satisfies* the inequality $\varphi < r$ if $\varphi^M < r$. An inequality $\varphi < r$ will be called *finitary* if the formula φ is finitary.

Let Γ consist of the following inequalities (the variable x in (vii) is of sort $B_Y(1)$):

(i) All the finitary inequalities satisfied by the structure $(\, X_0, Y_0, T, a \mid a \in A \,)$

(ii) $\neg \|c_n\| < \epsilon$ (i.e., $\|c_n\| > 1 - \epsilon$) and $\neg \|d_n\| < \epsilon$, for every $n < \omega$ and every rational $\epsilon > 0$.

(iii) $\neg \frac{1}{2}\|c_m - c_n\| < \frac{1}{2} + \epsilon$ (i.e., $\|c_n - c_m\| > 1 - 2\epsilon$) and $\neg \frac{1}{2}\|d_m - d_n\| < \frac{1}{2} + \epsilon$, for every pair m, n with $m < n < \omega$ and every rational $\epsilon > 0$.

(iv) $\neg \|c_0^*\| < \epsilon$, for every rational $\epsilon > 0$.

(v) $\neg \frac{1}{2}\|c_0^* - c_n\| < \frac{1}{2} + \epsilon$, for every $n < \omega$ and every rational $\epsilon > 0$.

(vi) $\|T(c_\omega)\| < \epsilon$, for every rational $\epsilon > 0$.

(vii) For every rational $\epsilon > 0$, the inequality

$$\sup_x \left(\bigwedge_{\substack{r_0, \ldots, r_{n-1} \in \mathbb{Q} \cap [-1,1] \\ n < \omega}} (n+1) \cdot \frac{1}{n+1} \|T(x) - \sum_{i<n} r_i d_i\| \right) < \epsilon.$$

Here $\frac{1}{n+1}\|\cdot\|$ is just the normalized norm predicate on the sort of $T(x) - \sum_{i<n} r_i d_i$, and $(n+1) \cdot \varphi$ is defined in general as $\varphi \dotplus \cdots \dotplus \varphi$ $n+1$ times.

Lemma 5.3 ensures that the hypotheses of Corollary 4.7 are satisfied with $\Sigma = \mathfrak{L}_{\omega,\omega}$. Thus by Corollary 4.7, Γ has a separable model

$$(\, X_1, Y_1, T_1, a, a_n, b_n, a_0^* \mid a \in A, n < \omega \,),$$

where for each $n < \omega$, a_n is the interpretation of c_n, b_n is the interpretation of d_n, and a_0^* is the interpretation of c_0^*. By (i), we have

$$(\, X, Y, T) \prec (\, X_1, Y_1, T_1),$$

so, in particular, by Proposition 5.1, T_1 is surjective.

Now we iterate the preceding process to find for each ordinal α with $0 < \alpha < \omega_1$ a separable structure

$$(\, X_\alpha, Y_\alpha, T_\alpha, a_n, b_n, a_i^* \mid n < \omega, i < \alpha\,)$$

such that if $0 < \alpha < \beta < \omega_1$,

- $(\, X_\alpha, Y_\alpha, T_\alpha, a_n, b_n, a_i^* \mid n < \omega, i < \alpha\,) \prec (\, X_\beta, Y_\beta, T_\beta, a_n, b_n, a_i^* \mid n < \omega, i < \alpha\,)$
- $a_i^* \in X_\alpha$ for $i < \alpha$
- $\|a_i^*\| = 1$ and $\|a_i^* - a_j^*\| = 1$ for $i < j < \alpha$
- The linear span of $\{b_n \mid n < \omega\}$ is a dense subset of $T_\alpha(X_\alpha)$.

The theorem then follows by taking $\hat{X} = \bigcup_{\alpha<\omega_1} X_\alpha$, $\hat{Y} = \bigcup_{\alpha<\omega_1} Y_\alpha$, and $\hat{T} = \bigcup_{\alpha<\omega_1} T_\alpha$. ■$_{5.4}$

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