GROUP CONFIGURATIONS AND GERMS IN SIMPLE THEORIES

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Abstract. We develop the theory of germs of generic functions in simple theories. Starting with an algebraic quadrangle (or other similar hypotheses), we obtain an "almost" generic group chunk, where the product is defined up to a bounded number of possible values. This is the first step towards the proof of the group configuration theorem for simple theories, which is completed in [BTW].

Introduction

This paper represents the first step towards the proof of the group configuration theorem for simple theories, which is achieved in [BTW]. In its stable version, this theorem is one of the cornerstones of geometric stability theory. It has many variants, stating more or less that if some dependence/independence situation exists, then there is a non-trivial group behind it, and in a one-based theory, every non-trivial dependence/independence situation gives rise to a group (see [Pil96]). The question of generalising it to simple theories arises naturally.

In the stable case, the proof can be decomposed into two main steps:

(1) Obtain a generic group chunk whose elements are germs of generic functions, and whose product is the composition.
(2) Apply the Weil-Hrushovski generic group chunk theorem.

The second step is generalised to simple theories in [Wag01, Section 3]. This paper is concerned with the generalisation of the first step, and does so with limited success: we only obtain a generic *poly*group chunk, that is a generic group chunk where product is defined only up to a bounded set of possible values. This gap is eventually filled in [BTW], and requires the use of altogether different tools: as far as we know, if we are not ready to go beyond hyperimaginaries and into the realm of graded almost hyperimaginaries, a generic polygroup chunk is indeed the best we can construct.

In order to understand the problems arising when trying to generalise the theory of germs of generic functions to simple theories, let us first take a closer look on the stable case. There, one could define generic functions as follows: Let p be a type, q, q' be two strong types, all over the same parameters. Then p acts generically from q to q' if for some (thus any) independent realizations $f \models p$, $x \models q$ we have a definable $f(x) \models q'$ such that $f, x, f(x)$ are pairwise independent. Moreover, if p acts generically from q to q', and p' acts generically from q' to q'', and p, p' are strong types, then $p \times p'$, which is the set of independent realizations of p and p', is a complete strong type, that acts generically from q to q''. And finally, f, f' have the same germ on q if

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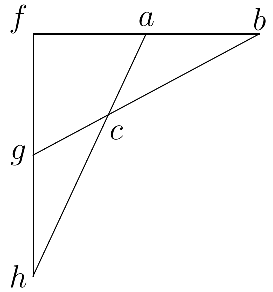
for some (thus any) x independent from both, $f(x) = f'(x)$, and this is obviously an equivalence relation.

All this makes heavy use of the stationarity of strong types, which is precisely what simple theories lack. Here is a brief description of the problems actually arising, and how we propose to overcome them:

- Assume that $p : q \mapsto q'$ and $p' : q' \mapsto q''$ are two generic actions as above: if we were to compose them, then the (parameters of the) functions would belong to $p \times p'$ (set of independent pairs of realisations), which is no longer a complete type.
  Therefore we must accept generic actions whose set of functions is a partial type. Moreover, the graph of each composed function $g \circ f$ is also a partial type for the same reason. In order to accommodate this approach, it seems useful to make a distinction between the general notion of a generic action, denoted by π, the set of (parameters of) functions of π, denoted by $\mathrm{Func}(\pi)$, and the actual functions, which we identify with their parameters $f, g, h, \ldots \in \mathrm{Func}(\pi)$. The graph of a function f is (the set of realisations of) a partial type over f.
- The passage to germs, which is essential to the theory, requires the graph of a function to be a (complete) Lascar strong type over its parameter. So partial types won't do, and we introduce the *completion* of a generic action, a procedure by which we replace each function whose graph is a partial type with the (bounded!) set of all possible extensions of the graph to a Lascar strong type over the parameter.
  Unfortunately, each function has several possible completions, and a function that is complete in this sense cannot in general be total.
- After the completion, we can pass to germs. This procedure, called *reduction*, is essentially the same as in the stable case, and results in replacing each function with the canonical base for its graph (note that unlike [HKP00], this is done uniformly for Lascar strong types which are not all conjugates one of the other). The reduction of π is denoted by $\bar{\pi}$, and the set of germs of π is $\mathrm{Germ}(\pi) = \mathrm{Func}(\bar{\pi})$.
  It should be noted, however, that due to lack of stationarity the germs are multi-functions rather than functions: for arguments on which they are defined, they give boundedly many possible values. We just accept this, as this does not introduce any new difficulties elsewhere in the construction, and generalise our notion of "generic function" accordingly.
- In order to get a set of germs where composition is a generic product, we need a generic action $\hat{\pi}$ such that a germ of the composition $\hat{\pi} \circ \hat{\pi}$ is also a germ of $\hat{\pi}$. In the stable case this is done (more or less) by defining $\hat{\pi} = \pi^{-1} \circ \pi$ for a suitable invertible generic action π, and then proving that as far as germs are concerned, the two middle terms can be eliminated from the composition $\pi^{-1} \circ \pi \circ \pi^{-1} \circ \pi$. The stable proof fails once more in the simple case, this time since a non-forking extension of a Lascar strong type is not necessarily Lascar strong: we introduce the technical notion of a generic action being *strong* on the left or on the right, and prove the required elimination under some mild strength assumption.

So far we only discussed machinery. The particular group configuration to which we apply this machinery in this paper is the algebraic quadrangle, although it can be used in other cases as well (it even plays some role in the construction of a binding group in [BW]). In a stable (or simple) theory, an algebraic quadrangle is a diagram of the form:



Where f, g, h, a, b and c are imaginary elements, such that every non-collinear triplet is independent (so in particular, every pair is), and f, g and h behave as canonical parameters for definable invertible functions sending $a \mapsto b$, $b \mapsto c$ and $a \mapsto c$, respectively: thus, the entire diagram explains in a sense how $h = g \circ f$ (we cheat a little; for the precise definition see Definition 4.1).

The theorem (for stable theories) stipulates the existence of a type-definable group G, acting faithfully and transitively on a set X, and of generic elements $f', g', h' \in G$ and $a', b', c' \in X$, such that each primed element is interalgebraic with its non-primed counterpart and: $h' = g' \cdot f'$, $b' = f' \cdot a'$ and $c' = g' \cdot b' = h' \cdot a'$. In particular, (f', g', h', a', b', c') form an algebraic quadrangle *algebraically equivalent* to the original, but this time coming directly from a group action.

In the last section we show how to obtain from an algebraic quadrangle a generic action π, which satisfies the strength assumption necessary for the elimination mentioned above. We define $\hat{\pi} = \pi^{-1} \circ \pi$ and $P = \mathrm{Germ}(\hat{\pi})$: for independent $f, g \in P$ we can define $f * g$ as the set of germs of completions of $g \circ f$, which belong to P by elimination. Since an incomplete generic function such as a composition $g \circ f$ can have many completions, product is only defined up to a bounded set of values. We obtain the existence of a non-trivial polygroup chunk in Corollary 4.8; with some more work, we get a better approximation of the stable theorem in Theorem 4.9.

1. Preliminaries

We assume familiarity with the basic definitions and properties regarding simple theories, as given in [Kim98a, KP97], as well as with hyperimaginaries and canonical bases, as given in [HKP00] or [Wag00].

Convention 1.1. We work in a first order simple theory T.
As we work with hyperimaginaries, by sort we mean a hyperimaginary sort. We usually associate variables to sorts. Thus, using the same variable in two places means they belong to the same sort.

It should be noted that although there is not much sense in speaking of a *formula* in a hyperimaginary sort, the notions of a complete or partial type in a hyperimaginary sort make perfect sense, and compactness applies to them as it does for partial types in real sorts. Moreover, manipulations of partial types, such as infinite conjunctions, finite disjunctions, and existential quantification, yield partial types. On its lowest technical level, this paper consists essentially of many constructions of partial types from others through such manipulation: thus, the treatment of hyperimaginaries in first order theories proposed in [HKP00] suffices for our needs.

Remark 1.2. Although we assume that we work in a first order theory, we hardly use first order logic's strength: negation is never used, since the negation of a partial type is not, in general, a partial type (and the need for negation never arises); although universal quantification does make sense for a partial type (even for one in a hyperimaginary sort, as long as the ambient theory is first order), we never need to use it either; and the only primitive building blocks we use are complete types and partial types that define indiscernibility.
It follows that the natural context for this paper is not a first order theory, but rather the much more general one of a thick compact abstract theory (see [Ben03a, Ben03b]): in a compact abstract theory there is no essential distinction between real and hyperimaginary sorts, and all the manipulations of partial types we propose to use make sense; thickness means that indiscernibility is type-definable, so all the primitives are available; finally, first order simplicity generalises fully to a simple thick compact abstract theory. Once one understand this context, it is a mere observation that all that we say in this paper holds without modification, and we will discuss it no further.

Notation 1.3. If $A = \{a_i : i < \alpha\}$ is some set or sequence, we note $a_{<i} = \{a_j : j < i\}$ and $a_{\leq i} = \{a_j : j \leq i\}$. Similarly, ab means the concatenation of a and b, and never the product of a and b (where that makes sense) or the application of a to b. Whenever we want to multiply two elements, compose functions, or apply a function to an argument, we use explicit notation.

Notation 1.4. When $\sim$ is a hyperdefinable equivalence relation on a given sort and a is in this sort, then its quotient $a/\!\sim$ is also noted $a_\sim$.

The Lascar strong type of a over b is noted $\mathrm{lstp}(a/b)$. We also write $a \equiv_b^{\mathrm{Ls}} a'$ instead of $\mathrm{lstp}(a/b) = \mathrm{lstp}(a'/b)$.

Fact 1.5. *For any two variables x and y, let $LS(x, y, x', y')$ be the partial type saying that $y = y'$ and there are y-indiscernible sequences $\langle x_i : i < \omega\rangle$ and $\langle x'_i : i < \omega\rangle$ such that $x = x_0$, $x' = x'_0$ and $x_1 = x'_1$.*
Then we have (taking LS on the right sorts):

1. *LS is a (type-definable) equivalence relation on the sort of x, y.*
2. *$a \equiv_b^{\mathrm{Ls}} a'$ if and only if $LS(a, b, a', b)$.*
3. *$\mathrm{lstp}(a/b) = \mathrm{tp}(a/\mathrm{bdd}(b)) = \mathrm{tp}(a/(a, b)_{LS}) = LS(x, b, a, b)$ (where equality is that of the sets of realisations, and not of the types as sets of formulas).*
4. *$\mathrm{dcl}((a, b)_{LS}) = \mathrm{dcl}(ab) \cap \mathrm{bdd}(b)$.*

Proof. Most of this (and in particular, $(a, b)_{LS} \in \mathrm{bdd}(b)$) appears in [Kim98b] and [KP97], so we only prove the last assertion. The inclusion $\mathrm{dcl}((a, b)_{LS}) \subseteq \mathrm{dcl}(ab) \cap \mathrm{bdd}(b)$ is clear. For the other, let f be an automorphism fixing $(a, b)_{LS}$. Then $f(b) = b$, and let $a' = f(a)$, so $a \equiv_b^{\mathrm{Ls}} a'$. Let g be an automorphism sending a to a' fixing $\mathrm{bdd}(b)$.

Then $f = (f \circ g^{-1}) \circ g$, where g fixes $\mathrm{bdd}(b)$ and $f \circ g^{-1}$ fixes ab. Therefore f fixes $\mathrm{dcl}(ab) \cap \mathrm{bdd}(b)$. QED

Remark 1.6. As $\mathrm{lstp}(a/b) = \mathrm{lstp}(a'/b)$ if and only if $(a, b)_{LS} = (a', b)_{LS}$, it makes sense to identify the hyperimaginary $(a, b)_{LS}$ and the type $\mathrm{lstp}(a/b)$.

We shall make much use of the independence theorem. We shall mostly use the following form:

Fact 1.7. *Suppose that $b_0 \downarrow_a b_1$, $c_i \downarrow_a b_i$, and $c_0 \equiv^{\mathrm{Ls}}_a c_1$. Then there is c with $c \equiv^{\mathrm{Ls}}_{ab_i} c_i$ and $b_0 \downarrow_{ac} b_1$.*

As we cannot restrict ourselves to complete types, we shall need a slight strengthening of results from [HKP00], as well as a remark on the definability of independence on (some) partial types.

Fact 1.8. *Let $\sim_1$ be a reflexive, symmetric hyperdefinable relation on the realisations of π, such that $a \sim_1 b \sim_1 c$ and $a \downarrow_b c$ imply $a \sim_1 c$. Let $\sim$ be its transitive closure. Then $\sim$ is hyperdefinable, and whenever $a_\sim = b_\sim$ and $a \downarrow_{a_\sim} b$ then $a \sim_1 b$.*

Proof. [Wag00, Lemma 3.3.1] QED

Definition 1.9. Let $\pi_i(x_i)$ for $i < \alpha$ be partial types in various hyperimaginary sorts over a fixed hyperimaginary parameter e. Their *independent product* (over e) noted $\prod^e_{i<\alpha} \pi_i(x_0, \ldots)$ is the partial type, if one exists, saying that each x_i satisfies π_i, independently over e of the others.

Definition 1.10. A partial type π over e is said to have *definable independence* (over e) if $\pi \times_e \pi'$ exists for any π' over e.

Proposition 1.11.
1. *Every complete type has definable independence over its domain.*
2. *If π_i has definable independence over e for every $i < \alpha$, then $\prod^e_{i<\alpha} \pi_i$ exists and has definable independence.*
3. *If π has definable independence over e, and $\pi' \vdash \pi$, then π' has it as well.*
4. *If π has definable independence over e, $\sim$ is an e-hyperdefinable equivalence relation on the sort of π, and π is $\sim$-invariant, then $\pi/\!\sim$ has definable independence over e.*

Proof.
1. This is a well known fact.
2. Existence is obvious. As for the definable independence, it is enough to notice that if $a_{<\alpha}$ is an independent set, then we have: $a_{<\alpha} \downarrow_e a$ if and only if $a_{<\alpha}, a$ is an independent set if and only if for all $i < \alpha$ we have $a_i \downarrow_e a_{<i}, a$.
3. Obvious.
4. We have $a_\sim \downarrow_e b$ if and only if there is $a' \sim a$ with $a' \downarrow_e b$. By $\sim$-invariance, $\pi(a) \iff \pi(a')$.

QED

2. Germs of partial generic functions

2.1. **Definitions.** The difficulties mentioned in the introduction lead us to the following definition:

Definition 2.1. Let $\pi(x, y, z)$, be a partial type in three hyperimaginary variables, over a hyperimaginary parameter, that from now on will be supposed to be $\varnothing$. We say that π defines *partial bounded generic functions*, or that it is a *generic action*, if:

(1) $\pi\restriction_x$, $\pi\restriction_y$ and $\pi\restriction_z$ have definable independence.
(2) $\pi(x, y, z)$ implies that x, y, z are pairwise independent.
(3) For any f, a there is at most boundedly many b such that $\pi(f, a, b)$, that is $\pi(f, a, z)$ is a bounded (possibly inconsistent) type. We note $f(a)$ the set of all such b.

We note $\mathrm{Func}(\pi) = \pi\restriction_x$, $\mathrm{Arg}(\pi) = \pi\restriction_y$, $\mathrm{Val}(\pi) = \pi\restriction_z$, namely the functions, arguments and values of π. If f is a function, we note $\mathrm{Gr}(f)(y, z) = \pi(f, y, z)$. We consider these interchangeably as a partial types and as sets. Note that $f \in \mathrm{Func}(\pi) \iff \mathrm{Gr}(f) \neq \varnothing$.

Remark 2.2. In fact, it is not necessary to require that $\mathrm{Val}(\pi)$ has defineable independence, as it follows from $\mathrm{Func}(\pi)$ and $\mathrm{Arg}(\pi)$ having it.

Proof. It will suffice to show that for $b \in \mathrm{Val}(\pi)$ and any e, $b \mathop{\smile\!\!\!\!\!\!\,|}\, e$ if and only if there are $af \mathop{\smile\!\!\!\!\!\!\,|}\, e$ such that $b \in f(a)$, as $\mathrm{Func}(\pi) \times \mathrm{Arg}(\pi)$ has definable independence by assumption. So suppose that $b \mathop{\smile\!\!\!\!\!\!\,|}\, e$, so (as $b \in \mathrm{Val}(\pi)$) there are f, a such that $b \in f(a)$ and we can suppose $af \mathop{\smile\!\!\!\!\!\!\,|}_b e$ which gives $af \mathop{\smile\!\!\!\!\!\!\,|}\, e$. Conversely, if $af \mathop{\smile\!\!\!\!\!\!\,|}\, e$ and $b \in f(a)$ then $b \mathop{\smile\!\!\!\!\!\!\,|}\, e$ as $b \in \mathrm{bdd}(af)$. QED

Here are a few definitions that will be used later on:

Definition 2.3. Let π be a generic action.

(1) We say that π is *trivial* if $\pi(x, y, z)$ implies that x, y, z are an independent triplet.
(2) We say that π is *invertible* if every function sends at most boundedly many arguments to any given value.
(3) We say the π is *well defined* if a function assigns at most one value to any argument.
(4) If π is well defined, we say that it is *well invertible* if every function it induces is injective.
(5) We say that π is *complete* if for any $f \in \mathrm{Func}(\pi)$, $\mathrm{Gr}(f)$ is a Lascar strong type (i.e. an amalgamation base) over f.
(6) We say that π is *reduced* if it is complete, and whenever $\mathrm{Gr}(f)$ and $\mathrm{Gr}(g)$ have a common non forking extension then $f = g$.
(7) We say that $\pi(x, y, z)$ and $\pi'(x', y, z)$ are *isomorphic* if there is a (hyper)definable bijection $\varphi : \mathrm{Func}(\pi) \to \mathrm{Func}(\pi')$ such that $\mathrm{Gr}(f) = \mathrm{Gr}(\varphi(f))$ for every $f \in \mathrm{Func}(\pi)$.

A remark on triviality and ranks:

Proposition 2.4. *Let π be a generic action. Then* $\mathrm{SU}(\mathrm{Arg}(\pi)) \geq \mathrm{SU}(\mathrm{Val}(\pi))$ *and* $\mathrm{SU}(\mathrm{Func}(\pi)) \geq \mathrm{SU}(\mathrm{Val}(\pi))$.
π is trivial if and only if $\mathrm{SU}(\mathrm{Val}(\pi)) = 0$ *(i.e., it is a bounded type).*

Proof. Suppose $\mathrm{SU}(\mathrm{Val}(\pi)) \geq \alpha$, so there is $b \in \mathrm{Val}(\pi)$ with $\mathrm{SU}(b) \geq \alpha$, and there are a, f such that $b \in f(a)$. As $b \in \mathrm{bdd}(af)$:

$$\mathrm{SU}(\mathrm{Arg}(\pi)) \geq \mathrm{SU}(a) = \mathrm{SU}(a/f) = \mathrm{SU}(ab/f) \geq \mathrm{SU}(b/f) = \mathrm{SU}(b) \geq \alpha$$

And similarly:

$$\mathrm{SU}(\mathrm{Func}(\pi)) \geq \mathrm{SU}(f) = \mathrm{SU}(f/a) = \mathrm{SU}(fb/a) \geq \mathrm{SU}(b/a) = \mathrm{SU}(b) \geq \alpha$$

As for the second assertion, if π is trivial then for every $fab \models \pi$ we have $b \perp fa \Longrightarrow b \perp b \Longrightarrow \mathrm{SU}(b) = 0$. On the other hand, if $\mathrm{SU}(b) = 0$ then the independence of a, f implies that of a, b, f. QED

We start by composing generic functions:

Definition 2.5. Suppose that $\pi(x, y, z)$, $\pi'(t, z, w)$ are generic actions. Then we define $\pi' \circ \pi(xt, y, w)$ to be the partial type such that $\pi' \circ \pi(fg, a, c)$ if and only if:

(1) f, g, a are independent.
(2) $c \in g \circ f(a)$, that is, there is b such that $b \in f(a)$ and $c \in g(b)$.

Proposition 2.6. *$\pi' \circ \pi$ always exists (provided that the sorts match) and it is a generic action.*

Proof. As $\mathrm{Func}(\pi)$ and $\mathrm{Func}(\pi')$ have definable independence, so does $\mathrm{Func}(\pi) \times \mathrm{Func}(\pi')$. Thus the independence of f, g, a is definable, and $\pi' \circ \pi$ exists. Now as $\mathrm{Func}(\pi' \circ \pi) \vdash \mathrm{Func}(\pi) \times \mathrm{Func}(\pi')$, it also has definable independence. The definable independence of $\mathrm{Arg}(\pi' \circ \pi)$ is easy. Boundedness is also evident. We verify the pairwise independence property: Suppose $\pi' \circ \pi(fg, a, c)$, so we have $b \in f(a)$, $c \in g(b)$. We know that $\{f, g, a\}$ are independent. So $a \perp_f g$ and thus $b \perp_f g$, as $b \in \mathrm{bdd}(af)$. Now as $f \perp b$ we have $b \perp fg$, so $\{f, g, b\}$ are independent. Similarly, we conclude that the following set are independent: $\{f, g, c\}$, $\{g, a, b\}$, $\{a, b, c\}$. In particular: fg, a, c are pairwise independent. QED

And we get immediately:

Proposition 2.7. (1) *Suppose π is a generic action, and note $\pi^{-1}(x, y, z) = \pi(x, z, y)$. Then π is invertible if and only if π^{-1} is a generic action.*
(2) *Suppose π is well defined. Then π is well invertible if and only if π^{-1} is a well defined generic action.*
(3) *Any composition of two invertible (well defined, well invertible) functions is invertible (well defined, well invertible), and $(\pi' \circ \pi)^{-1} = \pi^{-1} \circ \pi'^{-1}$.*
(4) *The inverse of an invertible complete action is complete, and* $\mathrm{Gr}(f)(x, y) = \mathrm{Gr}(f^{-1})(y, x)$.

We shall also need:

Proposition 2.8. *Let π, π' be generic actions on sorts such that $\pi' \circ \pi$ is defined. Suppose furthermore that* $\mathrm{Arg}(\pi') = \mathrm{Val}(\pi)$ *and these are Lascar strong types. Then for any independent* $f \in \mathrm{Func}(\pi)$ *and* $g \in \mathrm{Func}(\pi')$*, we have* $g \circ f \in \mathrm{Func}(\pi' \circ \pi)$*.*
If furthermore π' is non-trivial, then so is the composition.

Proof. Take any $f \mathop{\smile\!\!\!\!|}\, g$. Consider a value of f and an argument for g. Then they have the same Lascar strong type, thus by the independence theorem there is $b \mathop{\smile\!\!\!\!|}\, fg$ which is both. Thus there are a, c such that $b \in f(a)$, $c \in g(b)$. We may suppose that $a \mathop{\smile\!\!\!\!|}_{bf} g$, So $abf \mathop{\smile\!\!\!\!|}\, g \Longrightarrow a \mathop{\smile\!\!\!\!|}\, fg$. Thus $g \circ f$ is defined on a.
In such a situation we see that $\mathrm{Val}(\pi' \circ \pi) = \mathrm{Val}(\pi')$, so if π' is non-trivial, so is the composition. QED

2.2. **Completion.** Note that unlike the stable case, the composition of two complete actions need not be complete, as over a composition two arguments need not even have the same type, let alone Lascar strong type. However, the passage to germs requires that the action be complete. In case it is not, we construct its completion:

Construction 2.9. Let $\pi(x, y, z)$ be a generic action. Consider the hyperdefinable equivalence relation $LS(yz, x, y'z', x')$ from Fact 1.5 on the sort of yz, x. Note (a typical variable in) the quotient sort by $\underline{x} = (yz, x)/LS$. An element of this sort can be viewed as a pair that we shall note f_p, where f is an element in the sort of x, and p is a Lascar strong type over f in the variables yz. Now let $\underline{\pi}(\underline{x}, y, z)$ be defined as:

$$\underline{\pi}((y'z', x)/LS, y, z) = \pi(x, y, z) \wedge LS(yz, x, y'z', x)$$

So $\models \underline{\pi}(f_p, a, b)$ if and only if $b \in f(a)$ and $\mathrm{lstp}(ab/f) = p$.

Definition 2.10. We call $\underline{\pi}$ the *completion* of π. For a function f, we write $\underline{f} = \{(ab, f)_{LS} : b \in f(a)\} = \{f_{\mathrm{lstp}(ab/f)} : b \in f(a)\}$, that is the set of consistent completions of f, or the set of extensions of $\mathrm{Gr}(f)$ to a complete Lascar strong type over f. Note that this this is a bounded hyperdefinable set, non-empty if and only if $f \in \mathrm{Func}(\pi)$.

Remark 2.11. The sort of x is a quotient of that of $\underline{x}$, and every element f_p of the second is bounded over f of the first.

This is rather straightforward to verify:

Proposition 2.12.
(1) *Let π be a generic action. Then $\underline{\pi}$ is a complete generic action.*
(2) *If π is complete, then π and $\underline{\pi}$ are isomorphic.*
(3) *If π is invertible (well defined, well invertible) then so is its completion $\underline{\pi}$. If it is invertible then $\underline{\pi}^{-1}$ and $\underline{\pi^{-1}}$ are isomorphic.*
(4) *If π is non-trivial, so is $\underline{\pi}$.*

Note that as $\underline{x}$ is a bounded extension of x, all the properties of independence and definable independence are preserved.

Remark 2.13. We can improve Proposition 2.8, saying that if π and π' are complete, then the following are actually equivalent:

(1) For any independent $f \in \mathrm{Func}(\pi)$ and $g \in \mathrm{Func}(\pi')$, we have $g \circ f \in \mathrm{Func}(\pi' \circ \pi)$.
(2) $\mathrm{Arg}(\pi') = \mathrm{Val}(\pi)$ and these are complete Lascar strong types.

One direction is Proposition 2.8. For the other, suppose that $a \in \mathrm{Val}(\pi)$, $a' \in \mathrm{Arg}(\pi')$ are of different Lascar strong types. Note $p = \mathrm{lstp}(a)$, $p' = \mathrm{lstp}(p')$. Take now $f \in \mathrm{Func}(\pi)$ such that a is a possible value of f. Then by completeness $\mathrm{Gr}(f)(x,y) \models p(y)$. Similarly choose $g \in \mathrm{Func}(\pi')$ such that $g(a')$ is not empty, and $\mathrm{Gr}(g)(x,y) \models p'(x)$. We may now re-choose them such that $f \mathop{\smile\!\!\!\!\!\mid} g$. Then as $p \neq p'$, the composition $g \circ f$ is nowhere defined, so $g \circ f \notin \mathrm{Func}(\pi' \circ \pi)$.

2.3. **Reduction.** For the construction of the reduction, we suppose that π is a complete generic action.

What follows now is a variant of the construction of canonical bases, as given in [HKP00], the aim being the canonical bases of the graphs $\mathrm{Gr}(f)$. Unfortunately, the domain can be any $f \in \mathrm{Func}(\pi)$, so its type is not fixed (unlike what happens in [HKP00]). On the other hand, we can use the dependence/independence relations between the function, argument and value.

Definition 2.14. We say that two functions f, g are *primitively equivalent* if there is a $a \mathop{\smile\!\!\!\!\!\mid} fg$ such that $f(a) \cap g(a) \neq \varnothing$, and we note it $f \sim_1 g$. If $b \in f(a) \cap g(a)$, we say that ab witness $f \sim_1 g$.

First we show that the definition we gave for $\sim_1$ is essentially the same as that that given in [HKP00]:

Lemma 2.15. *$f \sim_1 g$ if and only if $\mathrm{Gr}(f)$ and $\mathrm{Gr}(g)$ have a common non-forking extension. More precisely, ab witness $f \sim_1 g$ if and only if they satisfy such an extension.*

Proof. Suppose that $f \sim_1 g$, witnessed by ab. Then $a \mathop{\smile\!\!\!\!\!\mid} fg$ gives $ab \mathop{\smile\!\!\!\!\!\mid}_g f$ and $ab \mathop{\smile\!\!\!\!\!\mid}_f g$ as $b \in \mathrm{bdd}(af) \cap \mathrm{bdd}(ag)$, and ab realize a common non-forking extension of $\mathrm{Gr}(f)$ and $\mathrm{Gr}(g)$. Conversely, suppose that ab realize such an extension, then $ab \mathop{\smile\!\!\!\!\!\mid}_g f$ and $ab \mathop{\smile\!\!\!\!\!\mid}_f g$, from each of which we can deduce $a \mathop{\smile\!\!\!\!\!\mid} fg$. Obviously, $b \in f(a) \cap g(a)$. QED

Lemma 2.16. *The relation $\sim_1$ satisfies the assumptions of Fact 1.8.*

Proof. Hyperdefinability is due to the definable independence of $\mathrm{Arg}(\pi)$ (note that this would be less evident for $\sim_1$ as defined in [HKP00], since the types of functions are not fixed). Reflexivity and symmetry are immediate. Finally, suppose that $f \sim_1 g \sim_1 h$, and $f \mathop{\smile\!\!\!\!\!\mid}_g h$. Then as $\mathrm{Gr}(g)$ is a Lascar strong type, two common non-forking extensions of it with $\mathrm{Gr}(f)$ and with $\mathrm{Gr}(h)$ can be extended to a common non-forking extension of all three, by the independence theorem, whence $f \sim_1 h$. QED

Thus, using Fact 1.8, the transitive closure of $\sim_1$ is a hyperdefinable equivalence relation $\sim$ on the sort of functions. Now, let $f \in \mathrm{Func}(\pi)$, and let $\sim'$, $\sim_1'$ be the restrictions of $\sim$, $\sim_1$ to $\mathrm{tp}(f)$. One verifies easily (compare with [Wag00, Lemma 3.3.4]) that $\sim'$ is the transitive closure of $\sim_1'$. Thus $f/\!\sim\ = \mathrm{Cb}(\mathrm{Gr}(f))$.

Definition 2.17. We call the class of a function f modulo $\sim$ the *germ* of that function, noted $\bar{f}$. The sort of germs of π is $x/\!\sim$, noted also $\bar{x}$. We define $\bar{\pi}(\bar{x}, y, z)$ as the partial type such that $\bar{\pi}(\bar{f}, a, b)$ if and only if there is $f \in \bar{f}$ such that $b \in f(a)$. If π is not complete, $\hat{f}$ is the set of germs of all the completions of f. We note the set of all germs of π by $\mathrm{Germ}(\pi)$.

Proposition 2.18. *Let π be a complete generic action. Then $\bar{\pi}$ is a reduced generic action. In particular, for any $f \in \mathrm{Func}(\pi)$, $\mathrm{Gr}_{\bar{\pi}}(\bar{f}) = \mathrm{Gr}_{\pi}(f)\restriction_{\bar{f}}$, that is the canonical restriction of $\mathrm{Gr}(f)$ (we recall that if $\rho(x,a)$ is a partial type and a_E is the quotient of a by a type-definable equivalence relation E, then $\rho(x,a)\restriction_{a_E}$ is equivalent to $\exists y\,[\rho(x,y) \wedge E(y,a)]$).*

Proof. We begin by noting that if $f \sim g$, then $\mathrm{Gr}(f)\restriction_{\bar{f}} = \mathrm{Gr}(g)\restriction_{\bar{f}}$. Now this entails that for any $f \in \mathrm{Func}(\pi)$: $\mathrm{Gr}(f)\restriction_{\bar{f}} = \mathrm{Gr}(\bar{f}) = \bar{\pi}(\bar{f}, y, z)$.
Now, let us verify that this is a generic action: $\mathrm{Func}(\bar{\pi})$ had definable independence as a quotient of $\mathrm{Func}(\pi)$. Suppose $ab \models \mathrm{Gr}(\bar{f})$. Then there is $f \in \bar{f}$ such that $b \in f(a)$, so in particular f, a, b are pairwise independent. A fortiori, $\bar{f}, a, b$ are. Finally, as $\mathrm{Gr}(\bar{f}) = \mathrm{Gr}(f)\restriction_{\mathrm{Cb}(\mathrm{Gr}(f))}$, we have $ab \mathop{\smile\!\!\!\!\!\!\;|}_{\bar{f}} f$. Thus $b \mathop{\smile\!\!\!\!\!\!\;|}_{a\bar{f}} af$, and as $b \in \mathrm{bdd}(af)$ we get $b \mathop{\smile\!\!\!\!\!\!\;|}_{a\bar{f}} b \implies b \in \mathrm{bdd}(a\bar{f})$. Thus all the elements of $\bar{f}(a)$ are bounded over af, so necessarily $\bar{f}(a)$ is bounded. We have shown that $\bar{\pi}$ is a generic action.
To show that $\bar{\pi}$ is reduced, we note at first that each $\mathrm{Gr}(\bar{f})$ is the canonical restriction of $\mathrm{Gr}(f)$, thus a canonical type. So suppose now that $\mathrm{Gr}(\bar{f})$ and $\mathrm{Gr}(\bar{g})$ have a common non-forking extension. Then, as they are canonical types, $\bar{f}$ is interdefinable with $\bar{g}$, and $\mathrm{Gr}(\bar{f}) = \mathrm{Gr}(\bar{g})$. So take any $f \in \bar{f}$, $g \in \bar{g}$, such that $f \mathop{\smile\!\!\!\!\!\!\;|}_{\bar{f}} g$. Now as $\mathrm{Gr}(f)$ and $\mathrm{Gr}(g)$ are non-forking extensions of $\mathrm{Gr}(\bar{f})$, $\mathrm{Gr}(f) \cup \mathrm{Gr}(g)$ does not fork over $\bar{f}$ by the independence theorem. Thus $f \sim_1 g$ by Lemma 2.15, and $\bar{f} = \bar{g}$. QED

Thus every complete generic action π admits a reduction $\bar{\pi}$. In case π is not complete, we take the reduction of its completion. We have: $\mathrm{Germ}(\pi) = \mathrm{Func}(\bar{\pi})$.

Lemma 2.19. *Let π be a reduced generic action. Then the following are equivalent:*
1. $f = g$
2. $\mathrm{Gr}(f) = \mathrm{Gr}(g)$
3. *There are $ab \models \mathrm{Gr}(f) \cup \mathrm{Gr}(g)$ with $ab \mathop{\smile\!\!\!\!\!\!\;|}_{g} f$ and $ab \mathop{\smile\!\!\!\!\!\!\;|}_{f} g$.*

Proof. $1 \implies 2 \implies 3$: immediate.
$3 \implies 1$: By the definition of a reduced action. QED

Corollary 2.20. (1) *If two reduced actions are isomorphic, then the isomorphism is unique, given by: $\varphi(f) = g \iff \exists xy\, x \mathop{\smile\!\!\!\!\!\!\;|} fg \wedge y \in f(x) \cap g(x)$.*
(2) *Two reduced actions are isomorphic if and only if they have the same graphs (that is, for every function of one there is a function of the other with the same graph).*

Thus we may identify any two reduced functions having the same graph. The (canonical) identification of any two isomorphic reduced actions ensues.

Definition 2.21. We say that two generic actions are *equivalent*, if their reductions are isomorphic. We note it $\pi \approx \pi'$.

Lemma 2.22. *If π is reduced, it is isomorphic to $\bar{\pi}$. In particular, for every $f \in \mathrm{Func}(\pi)$, $\mathrm{Gr}(f)$ is canonical.*

Proof. By the definition of a reduced type and Lemma 2.15, $\sim$ will be trivial on $\mathrm{Func}(\pi)$. QED

Lemma 2.23. *If π is invertible then so is its reduction $\bar{\pi}$, and $\bar{\pi}^{-1}$ and $\overline{\pi^{-1}}$ are isomorphic. In other words, $\bar{\pi}^{-1} \approx \pi^{-1}$. Moreover, for any $f \in \mathrm{Func}(\pi)$: $\hat{f}^{-1} = \widehat{f^{-1}}$ (where $\hat{f}^{-1}$ is the set of inverses of germs in $\hat{f}$).*

Proof. By Lemma 2.15, $\sim_{\pi^{-1}}$ is the same as $\sim_{\pi}$. The isomorphism ensues, whence $\bar{\pi}$ is invertible. QED

Corollary 2.24. *The inverse of an invertible reduced action is reduced.*

We pass to considering compositions. Here we suppose that π and π' are generic actions, on sorts such that $\pi' \circ \pi$ exists.

Lemma 2.25. *Let $f \in \mathrm{Func}(\pi)$, $g \in \mathrm{Func}(\pi')$, $f \mathop{\smile\!\!\!\!\!\!|}\ g$, and $h \in \mathrm{Germ}(\pi' \circ \pi)$. Suppose furthermore that $abc \models \mathrm{Gr}(f)(x,y) \cup \mathrm{Gr}(g)(y,z) \cup \mathrm{Gr}(h)(x,z)$. Then the following are equivalent:*

1. *$h \in \mathrm{bdd}(fg)$ and $af \mathop{\smile\!\!\!\!\!\!|}_b cg$.*
2. *$h \in \mathrm{bdd}(fg)$ and $ac \mathop{\smile\!\!\!\!\!\!|}_h fg$.*
3. *$a \mathop{\smile\!\!\!\!\!\!|}\ fgh$.*

Proof. $1 \Longrightarrow 3$: $af \mathop{\smile\!\!\!\!\!\!|}_b cg$ gives that a, f, g are independent, thus as $h \in \mathrm{bdd}(fg)$ we get $a \mathop{\smile\!\!\!\!\!\!|}\ fgh$.
$3 \Longrightarrow 2$: $a \mathop{\smile\!\!\!\!\!\!|}\ fgh$ gives $ac \mathop{\smile\!\!\!\!\!\!|}_h fg$ and $ac \mathop{\smile\!\!\!\!\!\!|}_{fg} h$ by boundedness. Thus $\mathrm{lstp}(ac/fg)$ has a common non-forking extension with $\mathrm{Gr}(h)$. But as the latter is a canonical type, this means that $h \in \mathrm{bdd}(fg)$.
$2 \Longrightarrow 1$: We get $a \mathop{\smile\!\!\!\!\!\!|}_h fg \Longrightarrow a \mathop{\smile\!\!\!\!\!\!|}\ fg \Longrightarrow abf \mathop{\smile\!\!\!\!\!\!|}\ g \Longrightarrow af \mathop{\smile\!\!\!\!\!\!|}_b cg$ as required. QED

When abc satisfy the conditions of this lemma (that is, all of the initial assumptions, as well as any of the equivalent conditions $1-3$), we say that they witness that $h \in \widehat{g \circ f}$.

Lemma 2.26. *Under the hypotheses of Lemma 2.25, $h \in \widehat{g \circ f}$ if and only if there are witnesses to it.*

Proof. First, suppose $h \in \widehat{g \circ f}$. Then h is the germ of some completion of $g \circ f$, so there are abc satisfying the appropriate graphs with a independent of fg. As $h \in \mathrm{bdd}(fg)$, $a \mathop{\smile\!\!\!\!\!\!|}\ fgh$.
Conversely, if abc are witnesses, then $\mathrm{lstp}(ac/fg)$ is a non-forking extension of $\mathrm{Gr}(h)$. Thus h is the germ of the completion of $g \circ f$ to $\mathrm{lstp}(ac/fg)$, and $h \in \widehat{g \circ f}$. QED

Lemma 2.27. *Suppose $f \in \mathrm{Func}(\pi)$, $g \in \mathrm{Func}(\pi')$, $f \mathop{\smile\!\!\!\!\!\!|}\ g$. Then $\widehat{g \circ f} = \bigcup_{\bar{f} \in \hat{f}, \bar{g} \in \hat{g}} \widehat{\bar{g} \circ \bar{f}}$, when we identify germs with equal graphs. In other words:*

$$\{\mathrm{Gr}_{\overline{\pi' \circ \pi}}(h) : h \in \widehat{g \circ f}\} = \{\mathrm{Gr}_{\overline{\bar{\pi}' \circ \bar{\pi}}}(h) : \bar{f} \in \hat{f}, \bar{g} \in \hat{g}, h \in \widehat{\bar{g} \circ \bar{f}}\}$$

Proof. Let $h \in \widehat{g \circ f}$, so there are witnesses $abc \models \mathrm{Gr}(f)(x,y) \cup \mathrm{Gr}(g)(y,z) \cup \mathrm{Gr}(h)(x,z)$, and $af \mathop{\smile\!\!\!\!\!\!|}_b cg$. Fix $\bar{f}$ to be the germ of the completion of f to $\mathrm{lstp}(ab/f)$, and $\bar{g}$ the germ of the completion of g to $\mathrm{lstp}(bc/g)$. Then we get at first: $f \mathop{\smile\!\!\!\!\!\!|}_b cg \Longrightarrow f \mathop{\smile\!\!\!\!\!\!|}\ bcg \Longrightarrow fg \mathop{\smile\!\!\!\!\!\!|}_{\bar{g}} bc$, and using it: $a \mathop{\smile\!\!\!\!\!\!|}_{b\bar{f}} cfg \Longrightarrow abc \mathop{\smile\!\!\!\!\!\!|}_{\bar{f}\bar{g}} fg$. Thus, $\mathrm{lstp}(ac/fg)$ is a non-forking

extension of $\operatorname{lstp}(ac/\bar{f}\bar{g})$, so they have the same canonical restriction, $\operatorname{Gr}(h)$. Thus identifying germs with equal graphs we get $h \in \widehat{\bar{g} \circ \bar{f}}$.
Conversely, suppose that $h \in \widehat{\bar{g} \circ \bar{f}}$, and take abc to witness that, so $a\bar{f} \downarrow_b c\bar{g}$ and $abc \models \operatorname{Gr}(\bar{f})(x,y) \cup \operatorname{Gr}(\bar{g})(y,z) \cup \operatorname{Gr}(h)(x,z)$.
We may choose abc such that in addition $f \downarrow_{\bar{f}\bar{g}h} abc$, so actually $f \downarrow_{\bar{f}} abc\bar{g}h$ (remember that $h \in \operatorname{bdd}(\bar{f}\bar{g})$). Let $f_p \in \underline{f}$ be the completion of f whose germ is $\bar{f}$: then $\operatorname{Gr}(f_p)$ is a non-forking extension of $\operatorname{Gr}(\bar{f})$ (and of $\operatorname{Gr}(f)$). Recall that $f_p \in \operatorname{bdd}(f)$, and that we also have $ab \downarrow_{\bar{f}} \bar{g}h$, so by the independence theorem we may re-choose ab such that $ab \downarrow_{\bar{f}} f_p\bar{g}h$ and $ab \models \operatorname{Gr}(f_p)$, and then we can re-choose c such that abc witness $h \in \widehat{\bar{g} \circ \bar{f}}$. We may further choose it such that $f \downarrow_{ab\bar{f}\bar{g}} c$, so $af_p \downarrow_{b\bar{g}} c$. From $ab \downarrow_{\bar{f}} f_p\bar{g}h$ we get $abf \downarrow \bar{g}$ and then $af_p \downarrow_b c\bar{g}$. To sum up, we did not change the type of abc over $\bar{f}\bar{g}h$, but now they witness $h \in \widehat{\bar{g} \circ f_p}$.
Take $g_q \in \underline{g}$ such that $\bar{g} = \bar{g}_q$. The same argument, with $g, \bar{g}, g_q$ instead of $f, \bar{f}, f_p$, and f_p instead of $\bar{g}$, shows that we can once again re-choose abc, without modifying their type over $f_p\bar{g}h$, such that they witness $h \in \widehat{g_q \circ f_p}$. In particular, they witness $h \in \widehat{g \circ f}$. QED

Corollary 2.28. $\pi' \circ \pi \approx \bar{\pi}' \circ \bar{\pi}$. *Moreover, composition is compatible with equivalence (that is, if* $\pi_i \approx \pi'_i$ *then* $\pi_1 \circ \pi_0 \approx \pi'_1 \circ \pi'_0$*).*

Proof. For the first assertion, we have just seen that the germs of $\pi' \circ \pi$ and $\bar{\pi}' \circ \bar{\pi}$ have exactly the same graphs. The second assertion is an immediate corollary of the first. QED

Finally, if we add the hypothesis that T is one-based, we get a few significant simplifications:

Proposition 2.29. *Suppose T is one-based.*

1. *If f is a germ and $ab \models \operatorname{Gr}(\underline{f})$ then $f \in \operatorname{bdd}(ab)$.*
2. *If f, g are germs and $h \in \widehat{g \circ f}$ then $h \downarrow f$ and $h \downarrow g$.*

Proof.
1. If f is a germ, then $f = \operatorname{Cb}(\operatorname{Gr}(f))$. One-basedness implies that $ab \models \operatorname{Gr}(f) \Longrightarrow \operatorname{Cb}(\operatorname{Gr}(f)) \in \operatorname{bdd}(ab)$.
2. Take abc witnessing that $h \in \widehat{g \circ f}$. Then a, b, c are independent, so in particular $c \downarrow_a b$ whence $h \downarrow_a f$ by the previous claim, and thus $h \downarrow f$. We get $h \downarrow g$ similarly.

QED

3. Germs of invertible functions

We start with a few more definitions:

Definition 3.1.
1. We say that a generic action π is *connected* if $\pi(x,y,z)$ is a complete type.
2. We say that a generic action π is *strong on the left (resp. on the right)* if $\pi(x,y,z)$ implies that $\operatorname{tp}(xz/y)$ (resp. $\operatorname{tp}(xy/z)$) is a Lascar strong type.

(3) We say that a composition $\pi' \circ \pi$ is *generic* if for every independent $f \in \mathrm{Func}(\pi)$, $g \in \mathrm{Func}(\pi')$, and for every $h \in \widehat{g \circ f}$, h is independent both of f and of g.

Remark 3.2. (1) Being strong on either side, being connected, or having a generic composition (for a pair of actions) is preserved by completion and reduction. Being connected is also preserved by taking the inverse action when this exists.
(2) By Proposition 2.29, in a one-based theory, every composition is generic.

Proof. Connectedness is preserved under completion since all Lascar strong extensions of a complete type are conjugate.
We prove that if π is strong on the left then its completion $\underline{\pi}$ is strong on the left: By Fact 1.5, π is strong on the left if and only if $\pi(x, y, z)$ implies that $\mathrm{dcl}(y) = \mathrm{dcl}(xyz) \cap \mathrm{bdd}(y)$. Similarly, if $\underline{\pi}(\underline{x}, y, z)$ is its completion, then by definition it implies that $\mathrm{dcl}(\underline{x}) = \mathrm{dcl}(xyz) \cap \mathrm{bdd}(x)$, thus in fact $\mathrm{dcl}(\underline{x}yz) = \mathrm{dcl}(xyz)$. Intersecting with $\mathrm{bdd}(y)$ we see that $\underline{\pi}$ is strong on the left.
The rest is easy. QED

Lemma 3.3. *Consider a composition $\pi' \circ \pi$, and $f \in \mathrm{Func}(\pi)$, $g \in \mathrm{Func}(\pi')$, $h \in \mathrm{Germ}(\pi' \circ \pi)$.*

(1) *If π is invertible, and $f \perp g$, $f \perp h$, then: $h \in \widehat{g \circ f} \iff g \in \widehat{h \circ f^{-1}}$, and abc witness the first if and only if bac witness the second.*
(2) *If π' is invertible, and $f \perp g$, $h \perp g$, then: $h \in \widehat{g \circ f} \iff f \in \widehat{g^{-1} \circ h}$, and abc witness the first if and only if acb witness the second.*

Proof. (1) Given abc satisfying the graphs, and given the independencies, the first statement is equivalent to $a \perp fgh$, while the second to $b \perp fgh$. We have

$$a \perp fgh \iff a \perp_f gh \iff ab \perp_f gh \iff b \perp_f gh \iff b \perp fgh$$

(2) Given abc satisfying the graphs, and given the independencies, both statements are equivalent to $a \perp fgh$.

QED

3.1. **Elimination.** We now wish to study when a generic action and its inverse can be eliminated from a composition.

Definition 3.4. (1) Let π, π', π'' be generic actions, π' invertible, on sorts such that the compositions ${\pi'}^{-1} \circ \pi$ and $\pi'' \circ \pi'$ exist and are generic. Then we say that they form an *elimination context*.
(2) Let π, π', π'' form an elimination context. Suppose $f \in \mathrm{Germ}(\pi')$, $g_0 \in \mathrm{Germ}(\pi'')$, $g_1 \in \mathrm{Germ}(\pi)$, $h_0 \in \mathrm{Germ}({\pi'}^{-1} \circ \pi)$, $h_1 \in \mathrm{Germ}(\pi'' \circ \pi')$. Suppose that adb witness $h_0 \in \widehat{f^{-1} \circ g_0}$, and bdc witness $h_1 \in \widehat{g_1 \circ f}$. Suppose furthermore that $ag_0h_0 \perp_{bdf} cg_1h_1$. Then $abcdfg_0g_1h_0h_1$ form an *elimination diagram.*
(3) Let π, π', π'' form an elimination context. For independent $h_0 \in \mathrm{Germ}({\pi'}^{-1} \circ \pi)$, $h_1 \in \mathrm{Germ}(\pi'' \circ \pi')$, $\mathrm{Elim}(h_0, h_1)$ is the set of all the germs of $h_1 \circ h_0$ obtained by elimination diagrams, i.e. germs of $lstp(ac/h_0h_1)$ taken from an elimination diagram $abcdfg_0g_1h_0h_1$.

As a first step, we study what happens within a single elimination diagram:

Lemma 3.5. *Let π, π', π'' form an elimination context, and $abcdfg_0g_1h_0h_1$ be an elimination diagram for it. Then:*

(1) *abd witness $g_0 \in \widehat{f \circ h_0}$, and dbc witness $g_1 \in \widehat{h_1 \circ f^{-1}}$.*
(2) *$g_0 \downfree g_1$, $h_0 \downfree h_1$, and $a \downfree fg_0g_1h_0h_1$.*
(3) *The germs of $g_1 \circ g_0$ and of $h_1 \circ h_0$ given by ac are the same.*
(4) *Note this common germ h. Then abc witness $h \in \widehat{h_1 \circ h_0}$, and adc witness $h \in \widehat{g_1 \circ g_0}$.*
(5) *h is independent of each of g_0, g_1, h_0, h_1.*
(6) *$\mathrm{Elim}(h_0, h_1) \subseteq \widehat{h_1 \circ h_0} \cap \mathrm{Germ}(\pi'' \circ \pi)$.*

Proof. (1) By Lemma 3.3 and the genericity of ${\pi'}^{-1} \circ \pi$ and $\pi'' \circ \pi'$.

(2) We have $ag_0 \downfree_{bdf} g_1$ and $ag_0 \downfree_d bf$, thus

$$ag_0 \underset{d}{\downfree} fg_1 \Longrightarrow ag_0 \downfree fg_1 \Longrightarrow a \downfree fg_0g_1 \Longrightarrow a \downfree fg_0g_1h_0h_1$$

We can also continue:

$$ag_0 \downfree fg_1 \Longrightarrow g_0 \underset{f}{\downfree} g_1 \Longrightarrow h_0 \underset{f}{\downfree} h_1$$

which give $g_0 \downfree g_1$ and $h_0 \downfree h_1$, by the genericity of, say, ${\pi'}^{-1} \circ \pi$.

(3) $a \downfree fg_0g_1h_0h_1$ gives $ac \downfree_{h_0h_1} g_0g_1$ and $ac \downfree_{g_0g_1} h_0h_1$.

(4) Clear by the above.

(5) We have $h_0 \downfree g_0$ by the genericity hypothesis, thus

$$g_0h_0 \underset{bdf}{\downfree} g_1 \Longrightarrow g_0h_0 \downfree g_1 \Longrightarrow h_0 \downfree g_0g_1 \Longrightarrow h_0 \downfree h$$

One obtains $h_1 \downfree h$ similarly. Exchanging the roles of g_i and h_i:

$$g_0h_0 \underset{bdf}{\downfree} h_1 \Longrightarrow g_0h_0 \downfree h_1 \Longrightarrow g_0 \downfree h_0h_1 \Longrightarrow g_0 \downfree h$$

and $g_1 \downfree h$ is similar.

(6) Clear by the above.

QED

We can now prove:

Theorem 3.6. *Let π, π', π'' form an elimination context, and suppose furthermore that π' is connected and strong on the left.*

(1) *For independent $h_0 \in \mathrm{Germ}({\pi'}^{-1} \circ \pi)$, $h_1 \in \mathrm{Germ}(\pi'' \circ \pi')$:*

$$\mathrm{Elim}(h_0, h_1) = \widehat{h_1 \circ h_0}$$

(2) *The composition $(\pi'' \circ \pi') \circ ({\pi'}^{-1} \circ \pi)$ is generic, and:*

$$\mathrm{Germ}(\pi'' \circ \pi' \circ {\pi'}^{-1} \circ \pi) \subseteq \mathrm{Germ}(\pi'' \circ \pi)$$

(3) *If we further suppose that $\mathrm{Val}(\pi) \wedge \mathrm{Arg}(\pi'') \vdash \mathrm{Val}(\pi')$, we have equality, that is:*

$$\pi'' \circ \pi' \circ {\pi'}^{-1} \circ \pi \approx \pi'' \circ \pi$$

Proof. (1) Suppose that abc witness $h \in \widehat{h_1 \circ h_0}$, where $h_0 \in \mathrm{Germ}(\pi'^{-1} \circ \pi)$ and $h_1 \in \mathrm{Germ}(\pi'' \circ \pi')$. As $ab \models \mathrm{Gr}(h_0)$, we can find $f \in \mathrm{Func}(\pi')$, $g_0 \in \mathrm{Func}(\pi)$ and d such that adb witness $h_0 \in \widehat{f^{-1} \circ g_0}$. Similarly we find $f' \in \mathrm{Func}(\pi')$, $g_1 \in \mathrm{Func}(\pi)$ and d' such that $bd'c$ witness $h_1 \in \widehat{g_1 \circ f'}$.
As $\pi'^{-1} \circ \pi$ is generic, we have $h_0 \perp f$, wherefore by Lemma 3.3 $g_0 \in \widehat{f \circ h_0}$ is witnessed by abd, and in particular $ah_0 \perp_b df$. Similarly, dbc witness $g_1 \in \widehat{h_1 \circ f'^{-1}}$ and $ch_1 \perp_b d'f'$. Of course, $ah_0 \perp_b ch_1$. Since π' is connected we get $df \equiv_b d'f'$. Since π' is strong on the left, we have in fact $df \equiv_b^{\mathrm{Ls}} d'f'$. Summing up, we may apply the independence theorem. Thus, fixing $abchh_0h_1$, we may assume that $d = d'$, $f = f'$, and $ah_0 \perp_{bdf} ch_1$. We may also re-choose g_0, g_1 such that $ag_0h_0 \perp_{bdf} cg_1h_1$. We have obtained an elimination diagram for π, π', π'', and $h \in \mathrm{Elim}(h_0, h_1)$ as required. The other inclusion is given by Lemma 3.5.

(2) Use the previous item, Corollary 2.28 and Lemma 3.5.

(3) Suppose now that $h \in \widehat{g_1 \circ g_0}$, where $g_0 \in \mathrm{Func}(\pi)$ and $g_1 \in \mathrm{Func}(\pi'')$ are independent, and let adc witness this. Then $d \models \mathrm{Val}(\pi) \wedge \mathrm{Arg}(\pi'') \implies d \models \mathrm{Val}(\pi')$, and we can find some $f \in \mathrm{Func}(\pi')$ and b such that $bd \models \mathrm{Gr}(f)$. We may take $bf \perp_d acg_0g_1h$, so ag_0, cg_1, bf are d-independent, and $ag_0 \perp_{bdf} cg_1$. Now $f \perp g_0g_1$, $a \perp fg_0$ and $b \perp fg_1$, so we may take $h_0 = \mathrm{Cb}(ab/fg_0)$, $h_1 = \mathrm{Cb}(bc/fg_1)$. Thus $ag_0h_0 \perp_{bdf} cg_1h_1$, and once again we have an elimination diagram. Then abc witness that $h \in \widehat{h_1 \circ h_0}$ and we are done.

QED

3.2. **Generic multi-chunks.**

Definition 3.7. We say that a generic action π is a *generic multi-chunk* if π is reduced, $\mathrm{Arg}(\pi)$ is Lascar strong, π is invertible satisfying $\pi = \pi^{-1}$, and the composition $\pi \circ \pi$ is generic satisfying $\pi^2 \approx \pi$.

So, Theorem 3.6 gives:

Corollary 3.8. *Let π be an invertible generic action, and let π' be possibly another generic action which is connected and strong on the left, such that $\mathrm{Arg}(\pi)$ and $\mathrm{Val}(\pi)$ are Lascar strong, and $\pi^{-1} \circ \pi' \approx \pi'^{-1} \circ \pi \approx \pi^{-1} \circ \pi$ are all generic compositions (so $\mathrm{Arg}(\pi') = \mathrm{Arg}(\pi)$). Note $\hat{\pi} = \overline{\pi^{-1} \circ \pi}$. Then π^{-1}, π', π form an elimination context, and $\hat{\pi}$ is a generic multi-chunk. If π is non-trivial, so is $\hat{\pi}$.*

Proof. The corollary is straightforward. For the consistency and non-triviality of $\hat{\pi}$, use Proposition 2.8 and that $\mathrm{Val}(\pi)$ is Lascar strong. QED

Usually we would have $\pi' = \pi$.

A generic multi-chunk almost suffices in order to apply [Wag01, Theorem 3.1]. We have:

Theorem 3.9. *Let π be a generic multi-chunk. Let $P = \mathrm{Germ}(\pi)$. Then the composition $\pi^2 \approx \pi$ induces a hyperdefinable function $* : P \times P \to P$, which is defined up to a bounded non-zero number of possible values. This function satisfies the hypotheses of the generalised Hrushovski-Weil theorem (*[Wag01]*), in the following sense:*

(1) Generic independence*: If $f \mathop{\smile\!\!\!\!\!\mid}\ g$ and $h \in f * g$ then f, g, h are pairwise independent.*
(2) Generic associativity*: suppose f, g, h are independent. Then $f * (g * h) = (f * g) * h$ (as sets).*
(3) Generic surjectivity*: For any independent f, g, there is h such that $g \in f * h$. Moreover, for any f, g, h: $g \in f * h \iff h \in f^{-1} * g$.*

If π is non-trivial, then P is not bounded, and in fact $f \in P$ implies that $\mathrm{tp}(f)$ *is not bounded.*

Proof. By $\pi^2 \approx \pi$, P is canonically definably isomorphic with $\mathrm{Germ}(\pi^2)$, and we may identify them. So take $f * g = \widehat{g \circ f}$.
By Proposition 2.8, $f * g \neq \varnothing$ for any $f \mathop{\smile\!\!\!\!\!\mid}\ g$, as $\mathrm{Arg}(\pi) = \mathrm{Val}(\pi)$ is a complete Lascar strong type by hypothesis. We also know this is a bounded set. So let us verify the properties:

(1) Because the composition $\pi \circ \pi$ is generic.
(2) Consider $f' \in f * (g * h)$. So there is $g' \in g * h$ and $f' \in f * g'$. Take witnesses abd to $f' \in f * g'$, and $b'cd'$ to $g' \in g * h$. Then we clearly have $b'd' \mathop{\smile\!\!\!\!\!\mid}_{g'} gh$, and also $b \mathop{\smile\!\!\!\!\!\mid}\ fg' \Longrightarrow bd \mathop{\smile\!\!\!\!\!\mid}_{g'} ff'$, and finally $f \mathop{\smile\!\!\!\!\!\mid}\ gh \Longrightarrow ff' \mathop{\smile\!\!\!\!\!\mid}_{g'} gh$. Then by the independence theorem (and $\mathrm{Gr}(g')$ being a Lascar strong type) we may suppose that $bd = b'd'$, $ff' \mathop{\smile\!\!\!\!\!\mid}_{bdg'} gh$. We may further assume that $aff' \mathop{\smile\!\!\!\!\!\mid}_{bdg'} cgh$. Thus we obtain $af \mathop{\smile\!\!\!\!\!\mid}_{bdg'} gh \Longrightarrow af \mathop{\smile\!\!\!\!\!\mid}_{b} dg'gh \Longrightarrow abf \mathop{\smile\!\!\!\!\!\mid}\ gh \Longrightarrow a \mathop{\smile\!\!\!\!\!\mid}\ fgh$. Take h' to be the germ of $\mathrm{lstp}(ac/fg)$. Then abc witness $h' \in f * g$ and acd witness $f' \in h' * h$. The other direction is similar.
(3) The moreover part is by Lemma 3.3. As $f^{-1} * g \neq \varnothing$, it gives the surjectivity.

Finally, suppose that there is some bounded $f \in P$. Then, for any $g \in P$, we have $f \mathop{\smile\!\!\!\!\!\mid}\ g$. Let abc witness $h \in f * g$. Then by the boundedness of f we have: $af \mathop{\smile\!\!\!\!\!\mid}_{b} cg \Longrightarrow a \mathop{\smile\!\!\!\!\!\mid}\ bcg \Longrightarrow ac \mathop{\smile\!\!\!\!\!\mid}\ g \Longrightarrow ac \mathop{\smile\!\!\!\!\!\mid}\ gf \Longrightarrow ac \mathop{\smile\!\!\!\!\!\mid}\ h$. But $\mathrm{lstp}(ac/h)$ is a canonical type, so h must be bounded as well. But we also have: $g \in f^{-1} * h$, so g is bounded. Thus every element of $P = \mathrm{Func}(\pi)$ is bounded, so π is trivial. QED

Remark 3.10. Suppose that T is one-based.
Let a, b, c be pairwise independent, non-independent elements. Using the hypothesis of one-basedness, we obtain a pairwise independent non-independent triplet a', b', c' such that each one is bounded over the other two, as in [BH94]. Take:

$$b'' = \mathrm{bdd}(b') \cap \mathrm{dcl}(a'b'c')$$

And:

$$\pi(x, y, z) = \mathrm{lstp}(a'b''c')$$

As in a one-based theory all compositions are generic, we see that π satisfies the hypotheses of Corollary 3.8, with $\pi = \pi'$.
Moreover, if we just seek to witness the non-triviality of T, we may assume that a, b and c are finitary hyperimaginaries (that is, quotients of finite tuples; this was pointed

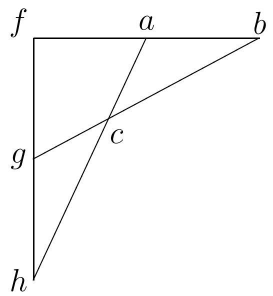
out by Ivan Tomašić). In this case, so would be a', b'' and c'. If in addition T is ω-categorical, then a finitary hyperimaginary is in fact an imaginary, and ω-categoricity applies to it. This will enable us to construct a definable group in [BTW].

4. Algebraic Quadrangles

A great deal of this section is an adaptation of passages from [Pil96, Chapter 5.4] to the simple context.

We recall the definition. We wish to consider hyperimaginaries, so we replace algebraic by bounded closure:

Definition 4.1. Let e be some hyperimaginary parameter, and (f, g, h, a, b, c) a tuple whose elements we put on a diagram as follows:



Then (f, g, h, a, b, c) is a *algebraic quadrangle* over e if it satisfies the following conditions:

1. Every non-collinear triplet is e-independent.
2. $\mathrm{bdd}(fge) = \mathrm{bdd}(fhe) = \mathrm{bdd}(ghe)$ (i.e., any two of f, g, h are e-interbounded over the third).
3. a, b are fe-interbounded, b, c are ge-interbounded, a, c are he-interbounded.
4. f is e-interbounded with $\mathrm{Cb}(ab/fe)$ [$= \mathrm{Cb}(\mathrm{lstp}(ab/fe))$], g is e-interbounded with $\mathrm{Cb}(bc/ge)$, h is e-interbounded with $\mathrm{Cb}(ac/he)$.

These can be easily verified:

Fact 4.2. *If (f, g, h, a, b, c) is an algebraic quadrangle over e as above, and (f', g', h', a', b', c') is such that each primed element is interbounded over e with the corresponding unprimed element, then (f', g', h', a', b', c') is also an algebraic quadrangle over e. In such a case we say that these quadrangles are* algebraically equivalent *(over e).*

Fact 4.3. *If (f, g, h, a, b, c) is an algebraic quadrangles over e, independent over e from $e' \supseteq e$, then it is an algebraic quadrangle over e' as well.*

We now prove a variant of [Pil96, Lemma 5.4.6] for a simple theory. This is done solely for the sake of completeness: since in any case we have to accept multi-valued generic functions, it is not of much use for us. The reader may feel free to skip directly to Proposition 4.7.

Fact 4.4. *Suppose $a \mathop{\smile\!\!\!\!\!\!\mid}_c b$ with $c \in \mathrm{bdd}(a) \cap \mathrm{bdd}(b)$. Then c is interbounded with $\mathrm{Cb}(a/b)$.*

Proof. As $c \in \mathrm{bdd}(b)$: $\mathrm{Cb}(a/bc) = \mathrm{Cb}(a/b)$. Thus $a \mathop{\smile\!\!\!\!\!\!\mid}_c b \Longrightarrow \mathrm{Cb}(a/b) \in \mathrm{bdd}(c)$. We also have:

$$a \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{Cb}(a/b)} b \Longrightarrow c \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{Cb}(a/b)} c \Longrightarrow c \in \mathrm{bdd}(\mathrm{Cb}(a/b))$$

QED

Lemma 4.5. *Let (f, g, h, a, b, c) be an algebraic quadrangle over e. Then there is $e' \supseteq e$ such that $fghabc \mathop{\smile\!\!\!\!\!\!\mid}_e e'$ (so it is an algebraic quadrangle over e' as well), and $f' \supseteq f$, $b' \supseteq b$, and a', such that (f', g, h, a', b', c) is algebraically equivalent to the original quadrangle over e', and $a' \in \mathrm{dcl}(f'b'e')$.*
Furthermore, if we had originally $b \in \mathrm{dcl}(fae)$, $c \in \mathrm{dcl}(hae)$, then we still have $b \in \mathrm{dcl}(f'a'e')$.

Proof. Realize $\mathrm{tp}(ghc/fabe)$ by $g'h'c'$, such that $ghc \mathop{\smile\!\!\!\!\!\!\mid}_{fabe} g'h'c'$. Write $e' = eg'$, so indeed $fghabc \mathop{\smile\!\!\!\!\!\!\mid}_e e'$. Write $f' = fh'$, $b' = bc'$, so over e' they are interbounded with f and b, respectively. Take now a' to be the (unordered) set of $f'b'e'$-conjugates of a (as a is bounded over $f'b'e'$, this exists, see in [BPW01]). Then clearly $a \in \mathrm{bdd}(a')$. For the other direction, notice that $f \mathop{\smile\!\!\!\!\!\!\mid}_{ae} h' \Longrightarrow fb \mathop{\smile\!\!\!\!\!\!\mid}_{ae} h'c'$, and as $a \in \mathrm{bdd}(fbe) \cap \mathrm{bdd}(h'c'e)$ we conclude that a is interbounded over e with $\mathrm{Cb}(h'c'/fbe)$. Thus, if $\hat{a} \models \mathrm{tp}(a/f'b'e')$, then $a, \hat{a}$ are e'-interbounded, whence $a' \in \mathrm{bdd}(ae')$. Thus we have obtained an algebraically equivalent quadrangle, and $a' \in \mathrm{dcl}(f'b'e')$.
For the second assertion, suppose that indeed $b \in \mathrm{dcl}(fae)$, $c \in \mathrm{dcl}(hae)$. Then $b' = bc' \in \mathrm{dcl}(fh'ae) = \mathrm{dcl}(f'ae)$. Then, if $\hat{a} \models \mathrm{tp}(a/f'b'e')$, we also have $b' \in \mathrm{dcl}(f'\hat{a}e)$. So $b' \in \mathrm{dcl}(f'a'e')$. QED

Proposition 4.6. *Let (f, g, h, a, b, c) be an algebraic quadrangle over e. Then there is $e' \supseteq e$ such that $fghabc \mathop{\smile\!\!\!\!\!\!\mid}_e e'$, and $(f'g'h', a', b', c')$ algebraically equivalent to it over e', such that a', b' are $f'e'$-interdefinable.*

Proof. Apply the lemma to obtain (f', g, h, a', b', c) over e', with $a' \in \mathrm{dcl}(f'b'e')$. Apply it again, this time with c in place of a, to obtain (f', g', h, a', b'', c') over e'', with $c' \in \mathrm{dcl}(g'b''e'')$. As we only increased b', e' to obtain b'', e'', we still have $a' \in \mathrm{dcl}(f'b''e'')$. Finally, apply it to obtain $(f'', g', h, a'', b''', c')$ over e''', with $b''' \in \mathrm{dcl}(f''a''e''')$. By the furthermore part of the lemma, we still have $a'' \in \mathrm{dcl}(f''b'''e''')$. Now we have: $f'g'ha'b''c' \mathop{\smile\!\!\!\!\!\!\mid}_{e''} e''' \Longrightarrow f'gha'b'c \mathop{\smile\!\!\!\!\!\!\mid}_{e''} e''' \Longrightarrow f'gha'b'c \mathop{\smile\!\!\!\!\!\!\mid}_{e'} e''' \Longrightarrow fghabc \mathop{\smile\!\!\!\!\!\!\mid}_{e'} e''' \Longrightarrow fghabc \mathop{\smile\!\!\!\!\!\!\mid}_e e'''$, which concludes the proof. QED

We also have:

Proposition 4.7. *Let (f, g, h, a, b, c) be an algebraic quadrangle, say over $\varnothing$, and take $\pi(x, y, z) = \mathrm{tp}(fab)$. Then π is an invertible generic action. Moreover, if π is strong on the right, then $\pi^{-1} \circ \pi$ is a generic composition.*

Proof. The only part that requires proof is the generic composition, assuming that π is strong on the right. So suppose that a, f, f' are independent, and $a' \in f'^{-1} \circ f(a)$, and we need to show that the germ of $\mathrm{lstp}(aa'/ff')$ is independent of each of f, f'. There is $b \in f(a) \cap f'(a')$, and we have $f \mathop{\smile\!\!\!\!\!\!\mid}_b f' \Longrightarrow fa \mathop{\smile\!\!\!\!\!\!\mid}_b f'a'$.
We may assume in fact that f, a, b are those of the original quadrangle. Since π is

strong on the right we have $fa \equiv_b^{\mathrm{Ls}} f'a'$, so we can find $g'c'$ such that $fgac \equiv_b^{\mathrm{Ls}} f'g'a'c'$. In the original quadrangle, we have $g \mathop{\smile\!\!\!\!\!\mid}_b f \implies gc \mathop{\smile\!\!\!\!\!\mid}_b fa$, and thus $g'c' \mathop{\smile\!\!\!\!\!\mid}_b f'a'$. By the independence theorem we may assume that $gc = g'c'$ and $fa \mathop{\smile\!\!\!\!\!\mid}_{gbc} f'a'$. We may then find h' such that $fha \mathop{\smile\!\!\!\!\!\mid}_{gbc} f'h'a'$ and $fha \equiv_{gbc} f'h'a'$. We see that:

$$fa \mathop{\smile\!\!\!\!\!\mid}_{gbc} f' \Longrightarrow fa \mathop{\smile\!\!\!\!\!\mid}_{g} f' \Longrightarrow a \mathop{\smile\!\!\!\!\!\mid} ff'g \Longrightarrow a \mathop{\smile\!\!\!\!\!\mid} ff'hh'$$

$$fh \mathop{\smile\!\!\!\!\!\mid}_{gbc} h' \Longrightarrow fh \mathop{\smile\!\!\!\!\!\mid} h' \Longrightarrow f \mathop{\smile\!\!\!\!\!\mid} hh'$$

And similarly $f' \mathop{\smile\!\!\!\!\!\mid} hh'$. Also, $a' \in \mathrm{bdd}(ahh')$ as we can pass through c instead of b. So we have that $aa' \mathop{\smile\!\!\!\!\!\mid}_{hh'} ff'$ and $aa' \mathop{\smile\!\!\!\!\!\mid}_{ff'} hh'$, thus $\mathrm{Cb}(aa'/ff') = \mathrm{Cb}(aa'/hh') \in \mathrm{bdd}(hh')$, and is therefore independent of each of f, f'. Thus the composition is generic.

QED

We obtain more or less immediately:

Corollary 4.8. *Let (f, g, h, a, b, c) be an algebraic quadrangle over e. Let $a' = \mathrm{dcl}(fabe) \cap \mathrm{bdd}(ae)$ and $b' = \mathrm{dcl}(fabe) \cap \mathrm{bdd}(be)$. Then (f, g, h, a', b', c) is algebraically equivalent over e to the original quadrangle. Take $\pi = \mathrm{lstp}(fa'b'/e)$. Then, π is strong on both sides, and it satisfies the assumptions of Corollary 3.8, with $\pi = \pi'$, yielding a generic multi-chunk $\hat{\pi} = \overline{\pi^{-1} \circ \pi}$ (over $\mathrm{bdd}(e)$).*

And with some more work:

Theorem 4.9. *Let (f, g, h, a, b, c) be an algebraic quadrangle over e. Then there are an algebraic quadrangle (f', g', h', a', b', c') also over e, a generic multi-chunk $\hat{\pi}$ over $\mathrm{bdd}(e)$, and a hyperimaginary e' such that:*

- *$e' \mathop{\smile\!\!\!\!\!\mid}_e fghabc$ and $e' \mathop{\smile\!\!\!\!\!\mid}_e f'g'h'a'b'c'$.*
- *(f, g, h, a, b, c) and (f', g', h', a', b', c') are algebraically equivalent over e'.*
- *$f', g', h' \in \mathrm{Func}(\hat{\pi})$, $a', b', c' \in \mathrm{Arg}(\hat{\pi}) = \mathrm{Val}(\hat{\pi})$, and $a'b'c'$ witness $h' \in \widehat{g' \circ f'}$.*

Moreover, if all of (f, g, h, a, b, c) and e are finitary hyperimaginaries, then we can take $\hat{\pi}$ to be over a finitary hyperimaginary $\tilde{e}$ such that $e \subseteq \tilde{e} \subseteq \mathrm{bdd}(e)$, and then $P = \mathrm{Germ}(\hat{\pi})$ of Theorem 3.9 is in a finitary hyperimaginary sort.

Proof. First, set $\tilde{e} = \mathrm{dcl}(fghabce) \cap \mathrm{bdd}(e)$. Then (f, g, h, a, b, c) is also an algebraic quadrangle over $\tilde{e}$. Working over $\tilde{e}$, we will assume that $\tilde{e} = \varnothing$.
Let $\tilde{a} = \mathrm{dcl}(fghabc) \cap \mathrm{bdd}(a)$, $\tilde{b} = \mathrm{dcl}(fghabc) \cap \mathrm{bdd}(b)$ and $\tilde{c} = \mathrm{dcl}(fghabc) \cap \mathrm{bdd}(c)$, and take $\pi = \mathrm{tp}(f\tilde{a}\tilde{b})$, $\pi' = \mathrm{tp}(h\tilde{a}\tilde{c})$, and $\hat{\pi} = \overline{\pi^{-1} \circ \pi}$. Since we work over $\tilde{e}$, π and π' are both Lascar strong types, and $\hat{\pi}$ is a generic multi-chunk by Corollary 3.8. By looking at the proof of Proposition 4.7, we see that every germ of $\pi^{-1} \circ \pi$ is a germ of $\pi'^{-1} \circ \pi'$ and vice versa, so $\hat{\pi} = \overline{\pi'^{-1} \circ \pi'}$ as well.
Write $a' = \tilde{a}$. Choose $\tilde{f} \equiv_{\tilde{b}} f$ such that $\tilde{f} \mathop{\smile\!\!\!\!\!\mid} fgh\tilde{a}\tilde{b}\tilde{c}$, and take $b' \in \tilde{f}^{-1}(\tilde{b})$. Similarly, choose $\tilde{h} \equiv_{\tilde{c}} h$ such that $\tilde{h} \mathop{\smile\!\!\!\!\!\mid} f\tilde{f}gh\tilde{a}\tilde{b}b'\tilde{c}$, and choose $c' \in \tilde{h}^{-1}(\tilde{c})$.

Then $\tilde{f}^{-1} \circ f \in \mathrm{Func}(\pi^{-1} \circ \pi)$ and $b' \in \tilde{f}^{-1} \circ f(a')$, so we can take $f' \in \widehat{\tilde{f}^{-1} \circ f} \subseteq \mathrm{Germ}(\hat{\pi})$ to be $\mathrm{Cb}(a'b'/f\tilde{f})$. Similarly, take $h' \in \widehat{\tilde{h}^{-1} \circ h} \subseteq \mathrm{Germ}(\hat{\pi})$ to be $\mathrm{Cb}(a'c'/h\tilde{h})$. Finally, take $g' \in \widehat{h' \circ f'^{-1}} \subseteq \mathrm{Germ}(\hat{\pi}^2) = \mathrm{Func}(\hat{\pi})$ to be $\mathrm{Cb}(b'c'/f'h')$.

Let $e' = \tilde f \tilde h$. Then $\tilde h \mathop{\smile\hskip -0.9em ^| \ } f \tilde f g h \tilde a \tilde b b' \tilde c$ implies $e' \mathop{\smile\hskip -0.9em ^| \ } fghabc$. Is also implies $\tilde h \tilde f \mathop{\smile\hskip -0.9em ^| \ } fha'$, whereby $\{\tilde f, \tilde h, f, h, a'\}$ are an independent set, whereby $\{\tilde f, \tilde h, f', h', a'\}$ are an independent set, and $e' \mathop{\smile\hskip -0.9em ^| \ } f'g'h'a'b'c'$ as promised.

$b'a'c'$ witness that $g' \in \widehat{h' \circ f'}^{-1}$, so $a'b'c'$ witness $h' \in \widehat{g' \circ f'}$.

By their choice, each of a', b', c', f' and h' is interbounded over e' with the unprimed element. For g and g', we need to work slightly more. Let $\pi'' = \operatorname{lstp}(g\tilde b \tilde c)$, and let $\tilde g \in \operatorname{Germ}(\pi'')$ be $\operatorname{Cb}(\tilde b \tilde c / g)$. Then g and $\tilde g$ are interbounded, and it suffices to show that $\tilde g$ and g' are interbounded over e'. But then one sees easily that $\tilde b \mathop{\smile\hskip -0.9em ^| \ } \tilde f \tilde g g' \tilde h$ and $b' \mathop{\smile\hskip -0.9em ^| \ } \tilde f \tilde g g' \tilde h$, so $b' \tilde b \tilde c c'$ witness that $g' \in \widehat{\tilde h^{-1} \circ \tilde g} \circ \tilde f$ and $\tilde b b' c' \tilde c$ witness that $\tilde g \in \tilde h \circ \widehat{g' \circ \tilde f}^{-1}$, and we are done.

For the moreover part, assume that (f, g, h, a, b, c) and e are finitary hyperimaginaries. Then so are $\tilde a$ and $\tilde b$, and $\tilde e$, and $\hat \pi$ is over $\tilde e$. In the rest of the construction, if the input is in finitary hyperimaginary sorts, then so is the output. QED

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